

22

Linear Programming and Marginal Analysis

This chapter provides a basic overview of linear programming, and discusses its relationship to the maximization and minimization techniques used for the factor-factor and product-product models. The assumptions of linear programming are given. The fixed-proportion production function, which forms the basis for linear programming, is compared with the linear production function. A simple linear programming problem is illustrated using graphics, and solved numerically using the simplex solution algorithm. An application of linear programming to a small farm resource allocation problem is presented.

Key terms and definitions:

- Classical Optimization Methods
- Operations Research
- Linear Programming
- Mathematical Programming
- Computer Programming
- Algorithm
- Nonlinearities
- Linearity
- Additivity
- Divisibility
- Nonnegativity
- Single Valued Expectations
- Fixed Proportion Production Function
- Linear Production Function
- Activities
- Resource Constraints
- Inequality Constraint
- Feasible Solution Area
- Simplex Algorithm
- Slack Variables
- Duality
- Personal Microcomputer

22.1 Introduction

This book has made extensive use of what is sometimes referred to as classical optimization methods. Classical optimization methods involve the maximization or minimization of a function subject to one or more constraints. To do this, a new variable called a Lagrangean multiplier is added for each constraint, and the maximization or minimization entails setting the partial derivatives of the Lagrangean function with respect to each variable, including the Lagrangean multipliers, equal to zero. The Lagrangean multipliers can be interpreted as the increase (or decrease) in the function to be maximized (or minimized) associated with a relaxation of the constraint by 1 unit. Lagrangean multipliers have substantive economic interpretation for a diverse array of problems.

Classical optimization methods are but one method for maximizing or minimizing a function subject to one or more constraints. The field of operations research is broadly concerned with problems of constrained maximization or minimization. Operations research has applications in economics and agricultural economics, as well as in many other areas in which problems are found that involve finding the optimal value of a function subject to one or more constraints.

Mathematical programming is another general term commonly used to describe problems that involve constrained maximization or minimization. The term *algorithm* as in mathematical programming algorithm is used to refer to a method or procedure for solving a mathematical programming problem.

Students sometimes confuse mathematical programming with computer programming. The two terms are entirely different. *Computer programming* refers to the process of providing a computer with a set of instructions to tell it what calculations to perform. Mathematical programming algorithms usually require many calculations, and therefore can quickly become complicated. A computer is usually used to perform the large number of complicated calculations, so computer programming is often needed to solve mathematical programming problems. However, small mathematical programming problems can be solved without the aid of a computer. Moreover, computer programs can be written that have nothing to do with maximizing or minimizing a function subject to constraints.

The problems in this text that involved maximizing or minimizing a function subject to a constraint are actually mathematical programming problems. The procedure that involved setting the partial derivatives of the Lagrangean function equal to zero could be thought of as the algorithm for solving the problems. The graphic representation could be thought of as the graphical solution to a specific mathematical programming problem. No computer was needed to find a solution.

22.2 Classical Optimization and Linear Programming

Mathematical programming can be divided into two major subcategories, nonlinear programming and linear programming. The problems that involved constrained maximization using the Lagrangean function are examples of nonlinear programming problems. In every case, either the objective function was nonlinear, or the constraint was nonlinear, or both. The production function $y = f(x_1, x_2)$ can be either a linear or a nonlinear function. The production function $y = Ax_1^a x_2^b$ is clearly nonlinear. The constraint $C = v_1 x_1 + v_2 x_2$ is clearly linear. Becoming familiar with classical optimization methods makes one familiar with a specific procedure or algorithm for solving certain nonlinear programming problems.

Linear programming involves the maximization or minimization of a linear function subject to linear constraints. Unlike classical optimization problems, in which at least one of the functions was nonlinear, with linear programming, every function is linear.

It may come as a surprise that linear programming has broad application to agricultural economics. Most of marginal analysis is dependent on relationships that involve nonlinearities. The most basic example is the law of diminishing returns. Cost and production functions are seldom linear. Isoquants and product transformation functions are usually curved, meaning that there is no limit to the number of possible solutions (combinations of inputs or outputs) as relative prices on inputs and outputs change.

The assumptions underlying linear programming in some ways are more restrictive and in other ways are less restrictive than the assumptions inherent in classical optimization techniques. Classical optimization required at least one of the functions to be nonlinear. If isoquants or product transformation functions did not have continuously turning tangents, a single unique solution to the constrained maximization problem would not exist. Isoquants and product transformation functions represented in linear programming models never have continuously turning tangents. Nonlinear functions are approximated with short or piecewise linear segments. If an isoquant or product transformation function is nonlinear, any change in the relative prices will result in a change in the quantities of the inputs used or the products produced. With certain linear programming problems, even large changes in relative prices will not lead to a change in the relative quantities of inputs used or products produced.

With classical optimization methods, corner solutions that involved the use of none of certain of the inputs, or production of none of certain of the outputs were not allowed. All constraints must hold in strict equality. This is because derivatives are defined only on open sets. Corner solutions are commonplace with linear programming. Inputs are not often fully utilized and therefore constraints need not hold in strict equality. Possible outputs are not necessarily always produced. Classical optimization methods are therefore more flexible in that all functions do not have to be linear, but less flexible in not allowing for corner solutions.

22.3 Assumptions of Linear Programming

Five basic assumptions underlie any linear programming model. These assumptions are (1) linearity (2) additivity, (3) divisibility, (4) nonnegativity, and (5) single-valued expectations. Mathematical programming techniques other than linear programming can sometimes be used for problems in which one or more of the assumptions of linear programming have been violated.

Linearity. The objective function and the constraints in a linear programming problem are linear. If the linearity assumption does not hold, one of the nonlinear programming techniques is required. Classical optimization methods are well known, but many other advanced techniques are available for solving optimization problems involving one or more nonlinear functions. A technique called *quadratic programming*, for example, can be used when the objective function is quadratic in form.

Additivity. Suppose that in order to produce a unit of y_1 , 2 units of x_1 , and 3 units of x_2 are required. Two units of output will require 4 units of x_1 and 6 units of x_2 . Five hundred units of y_1 will require 1000 units of x_1 and 1500 units of x_2 . Hence constant returns to scale exist. The additivity assumption is fundamental to the use of linear programming in production economics.

Divisibility. If 1 unit of y_1 can be produced using 1 unit of x_1 and 1 unit of x_2 , then 1/2 unit of y_1 can be produced with 1/2 input of x_1 and 1/2 unit of x_2 . One-tenth of a unit of y_1 can

be produced by using one tenth of a unit of x_1 and one-tenth of a unit of x_2 . The divisibility assumption becomes silly for certain categories of agricultural inputs. A linear programming problem might call for a solution that requires 1.457 bulls and 3.567 tractors. A technique called *integer programming* will force the solution to contain only integer values for inputs that cannot be divided, such as a tractor or a bull.

Nonnegativity The solution should not require that negative quantities of an input or resource be used. The usual solution algorithms for linear programming models do not allow for negative quantities of inputs to be used nor negative outputs to be produced. Zero quantities for both outputs and inputs are allowed.

Single-Valued Expectations. Linear programming models assume that coefficients such as input requirements and prices are known a priori with certainty. For example, if wheat, corn, and soybeans are to be included as possible enterprises within a linear programming model, the prices for which these commodities sell must be known in order to construct the model. If certain coefficients are not known with certainty, one of the stochastic programming techniques might be used.

22.4 Technical Requirements and Fixed-Proportion Production Functions

The production function underlying a linear programming model is sometime called a fixed proportion production function. The fixed proportion function is sometimes written as

$$\text{22.1} \quad y_1 = \min(a_1x_1, a_2x_2)$$

Production is determined by the most limiting input. Suppose that $a_1 = 4$ and $a_2 = 6$; and that 10 units of x_1 is available and 15 units of x_2 is available. The output of y_1 is determined by the smaller of $4 \times 10 = 40$ or $6 \times 15 = 90$. In this example, y_1 would be 40.

The fixed proportion production function is very different from the linear production function

$$\text{22.2} \quad y_1 = a_1x_1 + a_2x_2$$

The linear production function assumes that inputs x_1 and x_2 can substitute for each other. The marginal product of x_1 is a_1 and the marginal product of x_2 is a_2 . The MRS_{x_1, x_2} is a_1/a_2 .

With the fixed-proportion production function, one input does not substitute for the other, but rather, inputs must be used in fixed proportions with each other. The isoquant map for a fixed proportion production function is a series of right angles with a production surface similar to that illustrated in case 1 of Figure 12.2. The isoquant map for the linear production function consists of isoquants with a constant slope of a_1/a_2 . The surface is as illustrated in case 5 of Figure 12.2.

22.5 A Simple Constrained Maximization Problem

Suppose that the following objective function is to be maximized subject to constraints

$$\text{22.3} \quad \text{Maximize } 4y_1 + 5y_2$$

where y_1 and y_2 are two commodities. The 4 and 5 represent the price per unit of y_1 and y_2 , respectively.

The constraints are

$$\text{22.4 Resource or input } x_1: 2y_1 + 1y_2 \leq 12$$

$$\text{22.5 Resource or input } x_2: 1y_1 + 2y_2 \leq 16$$

There are 12 units of resource or input x_1 available, and 16 units of resource or input x_2 available. Units of commodity y_1 each require 2 units of x_1 and 1 unit of x_2 . Units of commodity x_2 each require 1 unit of x_1 and 2 units of x_2 .

All the available x_1 and x_2 need not be used, as indicated by the inequality signs. The matrix

$$\text{22.6} \quad \begin{array}{cc} & y_1 \quad y_2 \\ & 2 \quad 1 \quad x_1 \\ & 1 \quad 2 \quad x_2 \end{array}$$

represents the technical input requirements for x_1 and x_2 needed to produce 1 unit of y_1 and y_2 . In this example both x_1 and x_2 are needed in order to produce either y_1 or y_2 , but this need not always be the case, and some of the input requirements could be zero. The columns of matrix 22.6 are sometimes referred to as the *activities* while the rows are referred to as *resource constraints*.

This linear programming problem and its solution can be illustrated using a graph with y_1 on the horizontal axis and y_2 on the vertical axis (Figure 22.1). Suppose that only y_1 were produced. According to the first constraint, 12/2 or 6 units could be produced. Therefore, the first constraint intersects the y_1 axis at 6 units. Now suppose that only y_2 were produced. According to the first constraint, 12/1 units or 12 units could be produced. The first constraint intersects the vertical axis at 12 units. The slope of the first constraint would be 12/6 or 2:1.

If only y_1 were produced, the second constraint would intersect the y_1 axis at 16/1 or 16. If only y_2 were produced, the second constraint would intersect the y_2 axis at 16/2 or 8. The slope of the second constraint would be 8/16 or 1:2. If y_1 appears on the horizontal axis, and y_2 appears on the vertical axis, the slope of each constraint is equal to the coefficient on y_1 divided by the coefficient on y_2 .

The area inside both constraints represents the feasible solution area. The feasible solution area looks like a diagram representing a product transformation curve. The product transformation curve is made up of portions of each constraint that lie inside the other constraint. Instead of being a smooth, continuously turning curve, this product transformation curve is approximated with two linear segments, each with a different but constant downward slope. The constraints intersect at the point where the slope of the linear segments of the product transformation function change.

If less than 2 2/3 units of y_1 is produced, the slope of the product transformation function is the same as the slope of the second constraint. If more than 2 2/3 units of y_1 is produced, the slope of the product transformation function is the same as the slope of constraint 1. The slope of the product transformation function where the two constraints intersect and the output of y_1 is exactly 2 2/3 units is undefined.

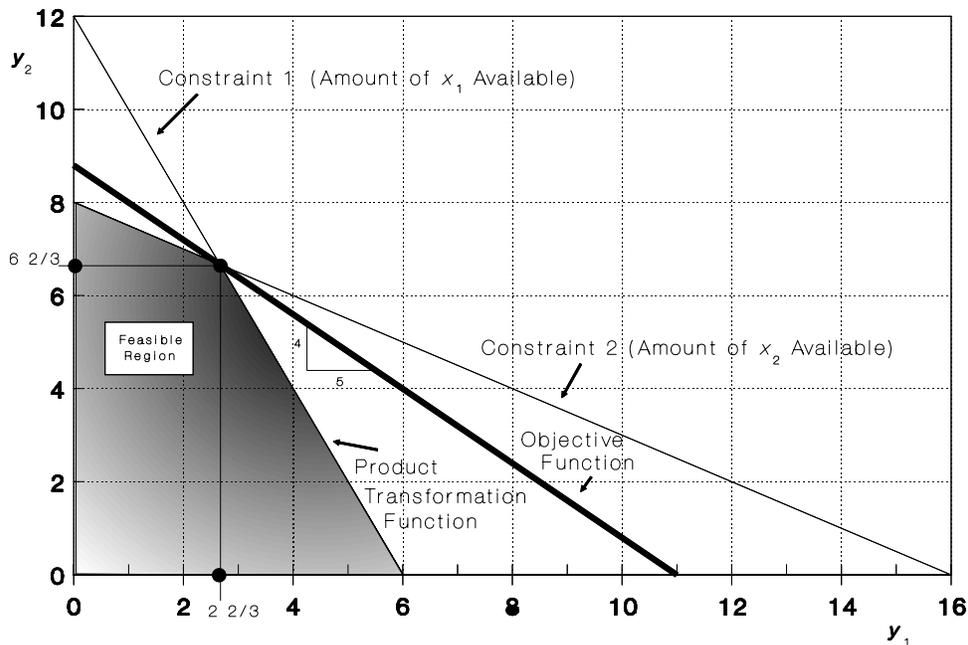


Figure 22.1 Linear Programming Solution in Product Space

The objective function is identical to the revenue function in the product-product model. The slope of the objective function will also be equal to the coefficient on y_1 divided by the coefficient on y_2 or 4:5 in this example.

A small linear programming problem can be solved with only some carefully drawn graphics. The solution represents the largest possible revenue consistent with the specified constraints. This means that the revenue function must be pushed as far as possible from the origin of the graph, but still touching the feasible solution area represented by the area inside both constraints.

For this problem, there are but three possible solutions, not counting $0y_1, 0y_2$. However, if the objective function has exactly the same slope as one of the constraints, all y_1 can be produced, all y_2 can be produced, or the combination of y_1 and y_2 that occurs at the point where the two constraints intersect can be produced. If the objective function has the same slope as one of the constraints, an infinite number of combinations of y_1 and y_2 can be found that maximize the objective function.

The first solution would be to produce all y_2 and no y_1 . Constraint number 2 (the availability of x_2) would limit the production of y_2 to only 8 units, despite the fact that there is enough x_1 to produce 12 units. The remainder of x_1 would be unused, or in disposal. For this

corner solution to occur, the relative price ratio of y_1 and y_2 (p_1/p_2) would have to exceed the slope of the second constraint, which in this problem is 1:2. This would imply that the price of y_2 would need to be more than twice the price of y_1 . If all y_1 were produced, revenue for this problem would be 5×8 or \$40.

The second solution would be to produce all y_1 and no y_2 . In this case, constraint 1 (the availability of x_1) would limit production of y_1 to only 6 units, and x_2 would be in disposal. For this solution to occur, the relative price ratio of y_1 and y_2 (p_1/p_2) would have to be less than the slope of the second constraint, which in this problem is 2:1. This would imply that the price of y_2 would have to be less than half the price of y_1 , or that y_1 is more than twice the price of y_2 . If all y_1 were produced, revenue for this problem would be 4×6 or \$24.

The third solution would be to produce the mix represented by the intersection of the two constraints. A carefully drawn graph will reveal that this mix is $2 \frac{2}{3}$ units of y_1 and $6 \frac{2}{3}$ units of y_2 . For this solution to occur, the price of y_2 would need to be less than twice the price of y_1 , but greater than one half the price of y_1 . This implies a price ratio of

$$1/2 < p_1/p_2 < 2/1$$

The price ratio for this problem of $4/5$ falls within this range. If this combination were produced, revenue would be $4 \times 2 \frac{2}{3} + 5 \times 6 \frac{2}{3} = \44 . The \$44 represents the maximum revenue possible given the price ratios and the two constraints. Both of the other possible solutions yield less revenue.

One final possibility exists. Suppose that the slope of the objective function were exactly the same as the slope of one of the two constraints. If the slope of the objective function were exactly $1/2$, as would be the case if the relative prices were $4/8$, then the solution that produced all y_2 would result in the exact same revenue as the solution that produced a combination ($8 \times 8 = 8 \times 6 \frac{2}{3} + 4 \times 2 \frac{2}{3} = 64$). In that case, the linear programming problem would maximize the function either if all y_2 or the combination of y_1 and y_2 were produced. Any combination consistent with the constraints that called for the production of more than $6 \frac{2}{3}$ units of y_2 would also result in the same revenue. In this case there is not a single solution to the linear programming problem.

If the relative prices were in the ratio $2/1$, such as $8/4$, then the solution that produced all y_1 would result in the same revenue as that obtained from the combination ($8 \times 6 = 8 \times 2 \frac{2}{3} + 4 \times 6 \frac{2}{3} = 48$). The same problem would result. If this were to occur, computer routines designed to solve linear programming problems usually warn the user that the solution is not unique.

22.6 Other Approaches for Solving Linear Programming Models

The graphical presentation of a linear programming problem is useful in establishing the linkages between linear programming and the product-product model. However, the diagrammatic approach cannot be used for problems that involve the production of more than two outputs. Diagrams become messy for problems involving a large number of constraints. Numerous algorithms are available for solving linear programming problems. The simplex algorithm is widely known. For small problems, the simplex algorithm is simple enough that the needed calculations can be performed by hand.

Computer programs based on the simplex algorithm for solving linear programming problems are available. However, most computerized algorithms used in agricultural economics make use of algorithms for solving linear programming problems that are more complicated to understand than the simplex algorithm but require fewer calculations by the computer. The problem of developing algorithms that solve various mathematical programming problems while minimizing the required computer time is a major research effort at some universities.

Algorithms for solving linear programming problems on a small, personal microcomputer are available. The size of the problem in terms of the number of columns (or activities) and the number of rows (or constraints) place limits on the size of the problem that can readily be solved on a small computer within a short period.

Currently available from several vendors are algorithms that will solve linear programming problems of up to 100×200 in matrix size, and within a few minutes on a personal computer at least 256K in size. Large university computers are able to solve very large linear programming problems within minutes or even seconds of computer time.

The solve time for a linear programming problem seems to increase exponentially, rather than linearly with the addition of rows and columns. A problem with 100 rows and 100 columns would probably take substantially more than twice as much time to solve as one with a 50×50 matrix. Large and complex mathematical programming problems still require large and fast computers for quick solutions.

22.7 The Simplex Method

The problem presented graphically in Section 22.4 will be solved using hand calculations with the aid of the simplex method. The problem was

Maximize

$$\text{22.7} \quad 4y_1 + 5y_2$$

where y_1 and y_2 are two commodities. The 4 and 5 represent the price per unit of y_1 and y_2 , respectively.

The constraints are

$$\text{22.8} \quad 2y_1 + 1y_2 \leq 12$$

$$\text{22.9} \quad 1y_1 + 2y_2 \leq 16$$

$$y_1, y_2 \geq 0$$

where the coefficients on y_1 and y_2 represent the technical requirements for x_1 and x_2 per unit of output. Input x_1 has 12 units available and x_2 has 16 units available.

The first step is to introduce two new variables, called *slack variables* (s_1 and s_2). Slack variables are used to convert the inequalities into equalities. One is required for each inequality constraint. The slack variables can each be thought of as a garbage dump for holding units of input x_1 or x_2 not being used in the solution. The coefficients on slack variables are initially zeros in the objective function. The coefficient on a slack variable that appears in an equation is 1, and the coefficient on the slack variables not appearing in an equation is zero. At the start, no y_1 or y_2 is produced, and therefore the value of the objective function is zero. The problem is then rewritten as

$$\text{22.10} \quad 2y_1 + 1y_2 + 1s_1 + 0s_2 = 12$$

$$\text{22.11} \quad 1y_1 + 2y_2 + 0s_1 + 1s_2 = 16$$

$$\text{22.12} \quad 4y_1 + 5y_2 + 0s_1 + 0s_2 = 0$$

The 12 and 16 in equations 22.10 and 22.11 are sometimes referred to as the right-hand side (*RHS*), since they appear on the right hand side. The right hand side represents the availability of inputs or resources x_1 and x_2 . The problem can be rewritten as follows:

22.13

	Column				
Row	y_1	y_2	s_1	s_2	<i>RHS</i>
x_1	2	1	1	0	12
x_2	1	2	0	1	16
Objective	4	5	0	0	0

The usual place to start is to bring in units of the output or activity with the largest price or coefficient in the objective function. This would be y_2 . However, it really makes no difference, and y_1 could be chosen. In this example, the conventional rule is followed and y_2 is chosen. Thus y_2 becomes what is called the *pivotal column*.

Since the objective function is to be maximized, the most limiting input must be determined. Each unit of y_2 requires 2 units of x_2 and 16 units of x_2 are available. Each unit of y_2 requires 1 unit of x_1 and 12 units of x_1 are available. Thus x_2 is most limiting ($16/2 = 8 < 12/1 = 12$). The row labeled x_2 becomes what is called the *pivotal row*.

Every element in the x_2 row is divided by the coefficient that appears at the intersection of the pivotal row and the pivotal column, 2 in this case. This results in a table with a new row x_2 labeled nx_2

2.14

	Column				
Row	y_1	y_2	s_1	s_2	RHS
x_1	2	1	1	0	12
nx_2	$1/2 = 0.5$	$2/2 = 1$	$0/2 = 0$	$1/2 = 0.5$	$16/2 = 8$
Objective	4	5	0	0	0

The new x_1 row (nx_1) is found by subtracting from the old x_1 row the product of the element in the nx_2 row and the column under consideration times the element at the intersection of entering column y_2 and the x_1 row. Suppose that the element to appear in column y_1 of row x_1 is to be found. That number will be $2 - 0.5 \times 1$. The number 2 appears in the old x_1 row for column y_1 , a 0.5 appears in row nx_2 for column y_1 , and 1 appears at the intersection of the entering y_2 column and the x_1 row. Following the same rule, the corresponding new element at row x_1 and column y_2 is $1 - 1 \times 1 = 0$. Similarly, the new element at row x_1 and column s_1 is $1 - 0 \times 1 = 1$, and so on. This results in the new matrix

2.15

	Column				
Row	y_1	y_2	s_1	s_2	RHS
nx_1	1.5	0	1	0.5	4
nx_2	0.5	1	0	0.5	8
Objective	4	5	0	0	0

A similar approach is used on the objective function row. The new element for the intersection of the objective row and column y_1 is $4 - 0.5 \times 5 = 1.5$. The new element for the intersection of the objective row and the right hand side is $0 - 8 \times 5 = -40$. This number is the negative of the current objective function value. Notice also that 40 was the profit in the graphical solution when only y_2 entered. The completed matrix is

2.16

	Column				
Row	y_1	y_2	s_1	s_2	RHS
x_1	1.5	0	1	0.5	4
x_2	0.5	1	0	0.5	8
nObjective	1.5	0	0	2.50	40

If all numbers appearing in the columns representing outputs are 0 or negative, the optimal solution has been found. In this example, the value at the intersection of the y_1 column and the new objective row is positive, indicating that production of y_1 will further increase profits. Following the same procedure, a new table is constructed. However, this time the entering row is y_1 . The resultant new table is

2.17

	Column				
Row	y_1	y_2	s_1	s_2	RHS
x_1	1	0	.67	.33	2.67
x_2	0	1	.33	.67	6.67
nObjective	0	0	1.00	2.00	44.00

The optimal solution has been found that maximizes revenue from the sale of y_1 and y_2 subject to the two constraints. The 2.67 and 6.67 represent the output of y_1 and y_2 , respectively. The 44.00 is the negative of the objective function value. The solution produces \$44 of revenue. All this information was available from the graphical solution.

However, a new piece of information is also available. The numbers appearing in the objective function row and the slack columns labeled s_1 and s_2 are the negatives of the imputed values of an additional unit of x_1 and x_2 , respectively. If one additional unit of x_1 were available, it would, if allocated properly, contribute 1 additional dollar to revenue. An additional unit of x_2 would contribute 2 dollars to revenue. These are the shadow prices for x_1 and x_2 . These shadow prices indicate the maximum amount that the manager would be willing to pay for the next unit of x_1 and x_2 .

The shadow prices obtained from a linear programming model can be interpreted in exactly same manner as the Lagrangean multipliers obtained using classical optimization methods. In both cases they represent the change in the objective function associated with a relaxation of the corresponding constraint by 1 unit.

If the shadow price for an additional unit of an input is \$2.00, 1000 additional units of the input are not necessarily worth \$2000 to the farmer. The shadow prices really apply only to the next unit of the input. Shadow prices usually decline in discrete steps as the availability of the input is increased. To be perfectly accurate, the incremental unit of the input should be infinitely small, or x should be dx . The same interpretation problem occurs with Lagrangean multipliers in a classical optimization model.

If the linear programming solution does not use all available units of an input, its shadow price, or implicit worth will be zero. Additional units of an input already in excess have an imputed value of zero and are worth nothing to the farmer.

22.8 Duality

In earlier chapters it was shown that any constrained maximization problem can be converted into a corresponding constrained minimization problem, and that any constrained minimization problem can be converted into a corresponding constrained maximization problem. The use of the inputs becomes the function to be minimized, the revenue function becomes the constraint.

Any linear programming problem can be converted to its corresponding dual. The primal problem might involve either the maximization or minimization of an objective function subject to constraints. If the primal is a constrained maximization problem, the dual will be a constrained minimization problem. If the primal is a constrained minimization problem, the dual will be a constrained maximization problem.

The constrained revenue maximization problem found the combination of outputs y_1 and y_2 that maximized revenue subject to the constraints and was similar to a constrained revenue maximization problem in product-product space. The corresponding dual is a constrained minimization problem. The imputed cost of inputs x_1 and x_2 is minimized subject to a revenue constraint. The problem is similar to that of finding the least cost combination of inputs in factor-factor space.

The dual of the maximization problem is

$$\text{22.18} \quad \text{minimize } 12x_1 + 16x_2$$

where x_1 and x_2 are imputed costs of inputs or resources. The constraints are

$$\text{22.19} \quad 2x_1 + 1x_2 \geq 4$$

$$\text{22.20} \quad 1x_1 + 2x_2 \geq 5$$

$$x_1, x_2 \geq 0$$

Notice that the rows of the primal are the columns of the dual. The columns of the primal are the rows of the dual. The right-hand side is made up of coefficients that formerly were prices of outputs. The objective function to be minimized has coefficients that formerly were values for input availability on the right-hand side. Less than or equal to constraints now are greater than or equal to constraints.

The dual can also be solved graphically, but the axes are now inputs x_1 and x_2 rather than outputs y_1 and y_2 . The first constraint will intersect the x_1 axis at 2, and the x_2 axis at 4. The second constraint will intersect the x_1 axis at 5 and the x_2 axis at 2.5 (Figure 22.2).

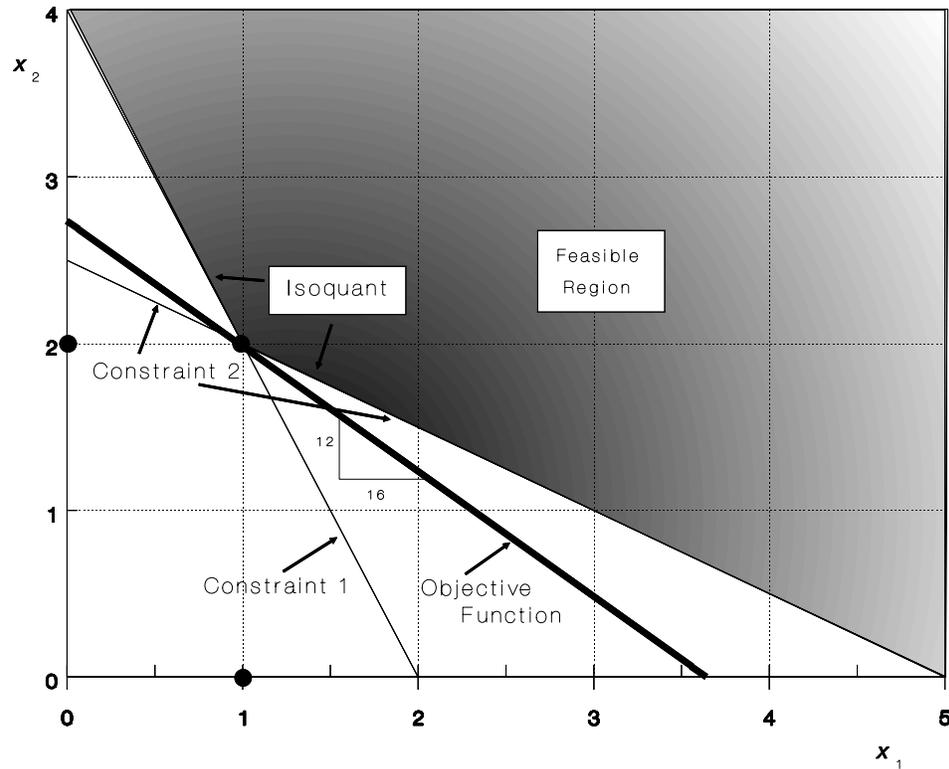


Figure 22.2 Linear Programming Solution in Factor Space

This time the feasible solution area lies outside of both constraints. The feasible solution area is again bounded by the two constraints. The line following the portion of each constraint outside the other constraint represents an isoquant constructed of two linear segments rather than a product transformation function.

The objective function is similar to an isocost line in factor-factor space. The slope of the objective function is v_1/v_2 , where $v_1 = 12$ and $v_2 = 16$. Again, three solutions are possible. All x_1 can be used, all x_2 can be used, or a combination of x_1 and x_2 can be used. A carefully drawn diagram indicates that the combination would be 1 unit of x_1 and 2 units of x_2 . These numbers look familiar.

The minimization problem can be solved via the simplex method. The procedure is the same as for the maximization problem with two exceptions. Since the constraints are greater than or equal, the slack variables must have negative rather than positive signs when they appear. In the maximization problem, the entering row was selected on the basis of the input that was most limiting or the smallest ratio of the right hand side value to the corresponding coefficient for the row in the column selected for entry. In the minimization problem, the row would be selected on the basis of the largest ratio of the right hand side value to the corresponding coefficient in the row for the column considered for entry.

However, it is not necessary to solve the dual if the primal has already been solved. The value for the objective function will be the same for both problems. The optimal levels of y_1 and y_2 in the primal are the shadow prices for x_1 and x_2 in the dual. The optimal values for x_1 and x_2 (the 1 and 2) in the dual are the shadow prices for the primal problem. The same solution will result regardless of whether the primal or dual is solved.

22.9 An Application

The use of linear programming in agricultural economics is illustrated with a simple problem. The problem is purposely kept small in order to shorten the explanation. The problem illustrates how linear programming might be used as the basis for developing a much larger farm planning model. The farmer has the choice of the enterprises listed in Table 22.1. Net revenues per unit of each enterprise over variable costs are listed.

Table 22.1 Enterprises and Net Revenues over Variable Costs

Enterprise	Units	Revenues Over Variable Costs
Corn	Acres	\$45 per acre
Wheat	Acres	\$32 per acre
Oats	Acres	\$20 per acre
Sows	10 sows	\$2400 per 10 sows
Steers	10 steers	\$1400 per 10 steers
Layers	10,000 layers	\$5400 per 10,000 layers

The farmer has 100 acres of land. Steers, sows, and layers are kept in confinement, so incremental units will not require any more land. The farm has a wheat allotment limiting wheat acreage to no more than 12 acres. Only 50 of the 100 available acres are suitable for the production of row crops. The production of grain crops will require access to no additional capital, but the production of hogs, steers, and layers will require capital for the purchase of needed animals and feed. Labor is broken into three periods, January to April, May to August and September to December. Table 22.2 lists the resource or input availability that will comprise the right hand side. Table 22.3 provides the resource or input requirements for each enterprise.

detail, breaking down items such as labor into weekly or even daily periods, and including far more possible enterprises or activities.

Linear programming models that do a comprehensive and detailed job of allocating inputs among enterprises have a tendency to quickly become very large and can require a large computer to solve and/or a substantial amount of computer time. The model presented here is easy to solve using a PC-based linear programming solver, so it lends itself to experimentation. The model provides an indication of the types of problems a larger and more detailed model would be able to solve.

Problems and Exercises

1. Does linear programming tighten or weaken the assumptions underlying classical optimization methods? Explain.
2. Solve the following linear programming problem by hand, using the simplex method outlined in the text.

$$\text{maximize } 2y_1 + 3y_2$$

subject to

$$3y_1 + 4y_2 \leq 20$$

$$1y_1 + 6y_2 \leq 24$$

Now find a computer program for solving the problem, and solve the problem on the computer. Compare the results with your hand solution. Now solve the dual with the same computer program, and compare the results.

3. What happens to the solution if the price of y_1 increases to \$10? Does the second resource become more valuable as measured by its shadow price?
4. Are a Lagrangean multiplier obtained from a classical optimization problem and a shadow price obtained from a linear programming problem the same thing? Explain.
5. Explain why the maximum number of possible solutions to a linear programming problem can be no greater than one more than the number of constraints.
6. Set up on the computer the farm planning problem contained in this chapter, and solve. Compare the results with those obtained in the text. Are the results presented in the text accurate? Now change the prices on one of the outputs and observe what happens to the optimal solution and the shadow prices on each input or resource.