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Contemporary Production Theory: The Factor Side

The purpose of this chapter is to explore some of the implications of recent work dealing with duality, elasticities of substitution, and translog specifications of production functions for agricultural research. These theoretical developments have a broad-based applicability to research in production economics and demand analysis for agricultural problems at varying levels of aggregation. The duality principles can be illustrated using simple multiplicative functions of the Cobb-Douglas type. However, the specific focus in this chapter is on the development of empirical estimates of elasticities of substitution by making use of contemporary production theory, and functional forms more complex than the Cobb-Douglas type are needed. In this publication, the highly flexible translog cost and production functions introduced within the economics literature by Christensen, Jorgenson, and Lau in the early 1970s are used to provide estimates of elasticities of substitution between major input categories for U.S. agriculture.

Key terms and definitions

- Contemporary Production Theory
- Duality
- Homothetic Production Function
- Translog Function

24.1 Introduction

Applications of contemporary thought have been made to specific problems within the agricultural economics literature. Ball and Chambers did a study for meat packing plants at the firm as the observation level of aggregation. Aoun estimated a translog cost function from time series data for all of U.S. agriculture, as a basis for obtaining elasticities of substitution between input pairs reported in this publication. Furtan and Gray conducted a similar study for a Canadian Province. Hoque and Adelaja and Grisley and Gitu used the approach in conjunction with studies conducted for dairy farms.

The approaches outlined in this chapter have applications to studies conducted for entire regions or countries, but are also applicable to studies conducted on data from farm records for individual firms. Census data on small groups of farms that have been classified according to major enterprises constitutes another possible data and aggregation level for such research. Approaches outlined in this publication are useful in situations where cost and input price data relating to agricultural enterprises are available, regardless of the aggregation level.

Fundamental duality concepts are presented. Some basic algebraic concepts relating to elasticities and logarithms are reviewed, and the concept of the elasticity of substitution between input pairs is developed in its various forms. The basic assumptions of contemporary production theory are outlined. Linkages between the translog functions and earlier functional forms are developed. Finally, a contemporary translog model designed to estimate elasticities of substitution between input pairs is introduced and empirical results for U.S. agriculture are presented.

24.2 Fundamentals of Duality

Agricultural economists are perhaps most familiar with the concept of duality as it relates to linear programming models. Within a linear programming context, duality refers to the fact that any linear programming problem can be expressed either as a maximization problem or a corresponding minimization problem subject to appropriate constraints. The primal problem may be either a maximization or a minimization problem. If the primal is a maximization problem, the corresponding dual will be a minimization problem, and, conversely, if the primal is a minimization problem, the corresponding dual will be a maximization problem.

The key characteristic of the dual relationship, as illustrated by a linear programming problem, is that all of the information about the solution to the primal can be obtained from the corresponding dual, and all of the information with respect to the solution of the dual can be obtained from the corresponding primal. Either the maximization or the minimization problem may be solved as the primal, and all information regarding the solution to the dual is obtained without resolving the problem.

Production functions have corresponding dual cost functions or perhaps correspondences. The term dual used in this context means that all of the information needed to obtain the corresponding cost function is contained in the production function, and, conversely, the cost function contains all of the information needed to derive the underlying production function. A simple example is the single input production function

$$24.1 \quad y = f(x).$$

If $f(x)$ is monotonically increasing, and the inverse function exists, the corresponding dual cost function expressed in physical terms is the inverse of the production function

$$\text{D4.2} \quad x = f^{-1}(y)$$

where f^{-1} = the inverse of f .

A simple example is the production function $y = x^b$. The corresponding dual cost function expressed in physical terms is $x = y^{1/b}$. All of the information with respect to the parameters of the production function is obtained from the corresponding dual cost function. Cost functions are usually expressed in dollar, rather than physical terms. The cost function expressed in dollar terms under the constant input price assumption is

$$\text{D4.3} \quad vx = v f^{-1}(y)$$

where v = the price of the input x .

Not all functions can be inverted. In general, a production function can be inverted to generate the corresponding dual cost function only if the original production function is monotonically increasing or decreasing. For example, if the production function is the familiar neoclassical three stage production function, the resultant dual is a correspondence, but not a function, for two values of x are assigned to at least some values for y .

Single-input cost functions are not normally thought of as arising from an optimization procedure. However, it is well known that any point on a single input production function represents a technical maximum output (y) for the specific level of input use (x) associated with the point. Each point on the inverse cost function is optimal in the sense that it represents the lowest cost method of producing the specific amount of output associated with the chosen point. (However, if the underlying production function is not always monotonically increasing, and as a result, the dual is a correspondence, a point on the dual cost correspondence is not necessarily a least cost point for the chosen level of output.)

In a multi-factor setting, the duality of the production function and the corresponding cost function becomes somewhat more complicated. Suppose that a production function for an output y is given by $y = f(\mathbf{x})$, where \mathbf{x} is a vector of inputs treated as variable. Under a specific set of conditions, the corresponding dual cost function exists (McFadden, 1978, pp. 8-9). These conditions are

- (1) *Marginal products of the inputs are non-negative.* The non-negativity implies free disposal of inputs. This assumption implies that if there is some input vector denoted as \mathbf{x}_0 which can produce some output vector called \mathbf{y}_0 , then if there exists a second bundle called \mathbf{x}_1 which is at least as large as \mathbf{x}_0 in every input, then \mathbf{x}_1 can also produce \mathbf{y}_0 . One implication of this assumption is that isoquant maps consisting of concentric rings are ruled out, and that positive slopes on isoquants are not allowed.
- (2) *Marginal rates of substitution between input pairs are non-increasing.* In the two factor case, this implies that each isoquant is weakly convex to the origin. However, regions of constant slope are allowed, and thus the isoquant need not have continuously turning tangents.

If conditions (1) and (2) are met, the production possibilities set satisfying assumptions (1) and (2) is termed input conventional (McFadden, 1978, pg. 10). Figure 24.1 illustrates some examples of isoquant maps fulfilling and violating conditions (1) and (2). Note that the ring isoquant maps sometimes used in courses in agricultural production economics are ruled out.

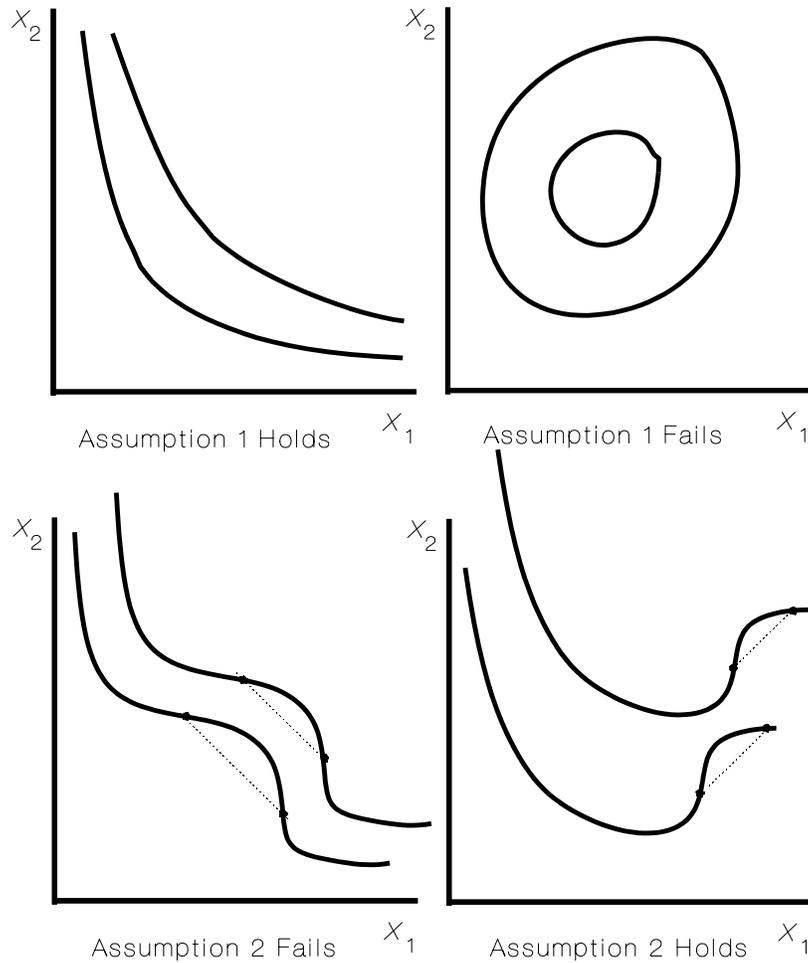


Figure 24.1 Assumptions (1) and (2) and the Isoquant Map

Source: Adapted from McFadden

The cost function that corresponds to the production function is $c(y; \mathbf{v}) = \min[\mathbf{v} \cdot \mathbf{x}; f(\mathbf{x}) = y]$. If conditions (1) and (2) are met, then this minimum cost function that corresponds to the production function:

(a) *exists*. This is true because any continuous function defined on a closed and bounded set achieves its minimum within the set.

(b) *is continuous*.

(c) *is non-decreasing* for each price in the input price vector.

(d) *is homogeneous of degree one* in all variable input prices. This implies that if all input prices double, so also will total variable cost, and

(e) *is concave* in each input price for a given level of output (y^*).

Detailed proofs of (a)-(e) can be found in McFadden, 1978, pp. 10-13. The isoquant maps needed for the existence of a corresponding dual cost function are not necessarily more plausible in an applied setting than other isoquant maps, but rather are a matter of mathematical convenience. For example, the Cobb-Douglas, CES and Translog production functions discussed in this publication all generate isoquant maps consistent with these assumptions, under the usual parameter restrictions, while the Transcendental does not.

Consider a particular class of production functions known as *homothetic* production functions, which include both homogeneous production functions and monotonic transformations of homogeneous production functions. A key characteristic of the homothetic production functions is that a line of constant slope drawn from the origin of the corresponding isoquant map will connect points of constant slope. Hence, homothetic production functions have linear expansion paths. Moreover, any isocline drawn from the origin will have a constant slope. An isocline of constant slope represents all points in which the ratio of the inputs remains fixed or constant, and can be referred to as a factor beam (Beattie and Taylor p. 42).

Now consider the factor beam for the homothetic production function representing the expansion path, or least cost combination of inputs. The production surface arising above the expansion path represents the production function for the use of the optimal bundle as defined by the least cost combination of inputs according to expansion path conditions. Therefore, every point on the production surface directly above the expansion path is optimal in that it represents the minimum cost of producing a given level of output. The production function represented by the expansion path conditions along the factor beam in an n input setting can be written as

$$\text{24.4} \quad y^* = f(x_1^*, \dots, x_n^*)$$

where

x_1^*, \dots, x_n^* the least cost quantities of x_1, \dots, x_n

y^* = output at each point associated with the expansion path conditions.

The cost function that is dual to 24.4 can be obtained by making use of the expansion path conditions.

For example, suppose that the production function is given by

$$\text{24.5} \quad y = Ax_1^{\beta_1} x_2^{\beta_2}$$

The input cost function is

$$\text{24.6} \quad C = v_1x_1 + v_2x_2$$

where A , $\$1$ and $\$2$ are parameters, x_1 and x_2 are inputs, and v_1 and v_2 are the respective prices on x_1 and x_2

The dual cost function for a Cobb-Douglas type production function is found using the following procedure. First, the equation for the expansion path is found by partially differentiating the production function with respect to x_1 and x_2 , to find the marginal products. The negative ratio of the marginal products is the MRS_{x_1, x_2} . The MRS_{x_1, x_2} is equated to the inverse input price ratio. The result can be written as

$$\text{24.7} \quad \$2v_1x_1 = \$1v_2x_2$$

Equation 24.7 defines the points of least cost combination along the expansion path.

Equation 24.7 is solved for x_1 to yield

$$\text{24.8} \quad x_1 = \$1v_2x_2\$2^{-1}v_1^{-1}$$

Equation 24.8 is inserted into equation 24.6 and x_2 is factored out

$$\text{24.9} \quad C = x_2(\$1v_2\$2^{-1} + v_2)$$

Equation 24.9 defines the quantity of x_2 that is used in terms of cost (C) and the parameters of the production function

$$\text{24.10} \quad x_2 = C/(\$1v_2\$2^{-1} + v_2)$$

Similarly, for input x_1

$$\text{24.11} \quad x_1 = C/(\$2v_1\$1^{-1} + v_1)$$

Inputs x_1 and x_2 are now defined totally in terms of cost C , the input prices (v_1 and v_2) and the parameters of the production function. Inserting equations 24.10 and 24.11 into the original production function (equation 24.5) and rearranging, results in

$$\text{24.12} \quad y = C^{(\$1+\$2)}A(\$2v_1\$1^{-1} + v_1)^{-\$1}(\$1v_2\$2^{-1} + v_2)^{-\$2}$$

Solving equation 24.12 for C in terms of y , the production function parameters and the input prices yields the optimal total cost function defined in terms of the expansion path conditions

$$\begin{aligned} \text{24.13} \quad C^* &= y^{[1/(\$1+\$2)]} A^{[1/(\$1+\$2)]} (\$1^{-1}\$2v_1 + v_1)^{[\$1/(\$1+\$2)]} (\$2^{-1}\$1v_2 + v_2)^{[\$2/(\$1+\$2)]} \\ &= A^{1/(\$1+\$2)} (\$2/\$1 + 1)^{\$1/(\$1+\$2)} (\$1/\$2 + 1)^{\$2/(\$1+\$2)} y^{1/(\$1+\$2)} v_1^{\$1/(\$1+\$2)} v_2^{\$2/(\$1+\$2)} \\ &= D_0 y^{1/(\$1+\$2)} v_1^{\$1/(\$1+\$2)} v_2^{\$2/(\$1+\$2)} \\ &= y^{[1/(\$1+\$2)]} Z \end{aligned}$$

C^* is the least cost method of producing the specific output level y as defined by the expansion path conditions.

Equation 24.13 represents the total cost function that is dual to the production function defined along the expansion path factor beam. Any point on the dual cost function representing a particular quantity of output designated as y^o is optimal in the sense that it represents the minimum cost, or least cost combination of inputs needed to produce y^o . However, at most only one point on the dual cost function represents global optimality, where the marginal cost of producing the incremental unit of output using the least cost combination of factors is exactly equal to the marginal revenue obtained from the sale of the incremental unit of y .

For the Cobb-Douglas case, y is raised to the power 1 over the degree of homogeneity of the original production function. The value of Z treated as a constant, since it is dependent only on the assumed constant prices of the inputs and the assumed constant parameters of the production function. If prices for inputs are available and constant, all of the information needed to obtain the corresponding dual cost function can be obtained from the production function. The coefficients or parameters of a Cobb-Douglas type production function uniquely define a corresponding dual cost function C^* .

Marginal cost associated with the expansion path factor beam (least cost marginal cost) is

$$24.14 \quad MC^* = dC^*/dy = [1/(\$_1 + \$_2)]y^{1/(\$_1 + \$_2) - 1}Z.$$

The slope of MC^* is positive if the sum of the individual partial production elasticities or function coefficient is less than 1. If the individual production elasticities sum to a number greater than 1, then MC^* is declining. MC^* has a zero slope when the production elasticities sum exactly to 1. The least cost supply function for a firm with a Cobb-Douglas type production function can be found by equating marginal cost (equation 24.14) with marginal revenue or the price of the product and solving the resultant equation for y .

Average cost associated with the least cost factor beam is

$$24.15 \quad AC^* = C^*/y = y^{1/(\$_1 + \$_2) - 1}Z.$$

Since Z is positive, average cost decreases when the partial production elasticities sum to a number greater than 1. Average cost increases if the partial production elasticities sum to a number less than 1. If the production function is a true Cobb-Douglas then total cost is given by

$$24.16 \quad C^* = yZ.$$

In the true Cobb-Douglas case, both marginal and average cost are given by the constant Z , and therefore both MC^* and AC^* have a zero slope. For a Cobb-Douglas type production function, MC^* and AC^* never intersect, except in the instance where the function coefficient (or the cost elasticity) is 1, in which case MC and AC are the same everywhere.

The ratio of marginal to average cost along the least cost factor beam, or the dual cost elasticity (R^*) that applies to the expansion path conditions is

$$24.17 \quad R^* = [1/(\$_1 + \$_2)].$$

$$= 1/E,$$

where E is the returns to scale parameter, or function coefficient for the underlying production function for the output arising from the least cost combination of inputs along the expansion path factor beam.

If total product along the expansion path is increasing at a decreasing rate, then costs are increasing at an increasing rate. If total product along the expansion path is increasing at an increasing rate, then costs are increasing at a decreasing rate. If total product along the expansion path is increasing at a constant rate (the true Cobb-Douglas) then costs are also increasing at a constant rate. If the product sells for a fixed price, that price is a constant marginal revenue (MR). Marginal revenue (MR) can be equated to the least cost marginal cost (MC^*) only if MC^* is increasing. With fixed input prices and elasticities of production, this can happen only if the cost elasticity is greater than one, which means that the function coefficient for the underlying production function is strictly less than 1.

The profit function representing the least cost method of generating a specific amount of profit, and corresponding to the dual cost function can be written as

$$24.18 \quad A^* = TR - C^*.$$

If output price (p) is constant

$$24.19 \quad A^* = py - Zy^{(1/E)},$$

where E is the function coefficient.

Maximum profits occur if

$$24.20 \quad dA^*/dy = p - (1/E)y^{[(1/E) - 1]} = 0$$

$$MR - MC^* = 0$$

and

$$24.21 \quad d^2A^*/dy^2 = -(1/E)(1/E)y^{[(1/E) - 2]} < 0.$$

E is positive. The only way the second derivative can be negative is for E to be smaller than 1. This implies that MC^* is increasing. If E is equal to one, the second derivative of the profit function is zero, and that MC^* is constant. If E is greater than 1, the second derivative of the profit function is positive, and MC is decreasing.

24.3 Duality Theorems

The two most famous theorems relating to duality are Hotelling's lemma and Shephard's lemma. Both are specific applications of a mathematical theorem known as the envelope theorem. The proofs of the envelope theorem, Shephard's lemma, and Hotelling's lemma are adapted from those found in Beattie and Taylor (Chapter 6). More detailed and rigorous proofs can be found in McFadden, 1978, pp. 14! 15 and appendices.

24.4 The Envelope Theorem

Consider a function z to be maximized with respect to each w_i

$$\text{24.22} \quad z = g(w_1, \dots, w_n, \theta)$$

where

z = a value to be maximized

w_i = variables

θ = a vector of parameters

First order conditions require that for each w_i

$$\text{24.23} \quad \partial g / \partial w_i = 0$$

for a maximum.

Now define the optimal value for each w_i as w_i^* in terms of the parameter vector θ . That is,

$$\text{24.23} \quad w_i^* = w_i^*(\theta)$$

for all $i = 1, \dots, n$

The optimal value for equation 24.22 is

$$\text{24.24} \quad z^* = g(w_1^*, \dots, w_n^*, \theta)$$

The envelope theorem states that the rate of change in z^* with respect to a change in θ , if all w_i are allowed to adjust, is equal to the change in g with respect to the change in the parameter θ when all w_i are assumed to be constant (Beattie and Taylor, pg 228). That is

$$\text{24.25} \quad \partial z^* / \partial \theta = \partial g / \partial \theta$$

In order to prove that equation 24.25 holds, first find the partial derivative of 24.24 with respect to the parameter vector θ

$$\text{24.26} \quad \partial z^* / \partial \theta = \sum (\partial g / \partial w_i^*) (\partial w_i^* / \partial \theta) + \partial g / \partial \theta$$

However, if the first order conditions from equation 24.23 are to hold, then $\partial g / \partial w_i^*$ must be equal to zero for all $i = 1, \dots, n$ and equation 24.25 holds.

24.5 Shephard's Lemma

Shephard's lemma (1953) is a specific application of the envelope theorem to the cost function representing the least cost way of producing a particular level of output, as in equation 24.13. Suppose that a cost function with characteristics (a)-(e) listed above exists. Then its corresponding first derivative with respect to the i th variable input is $\partial C^* / \partial x_i$. Shephard has shown that (1) this derivative is equal to the level of x_i (x_i^*) that minimizes total

cost for a given level of output, and (2) that if x_i^* exists as the minimum level of x_i for a given level of output, then M^*/M_i also exists.

Suppose the cost minimizing Lagrangian

$$\text{24.27} \quad L = \sum v_i x_i + \mathcal{G}[y^0 - f(x_1, \dots, x_n)]$$

The corresponding first order conditions are

$$\text{24.28} \quad M^*/M_i = v_i = \mathcal{G}_i$$

for all $i = 1, \dots, n$

The indirect cost function, representing the least cost method of production is

$$\text{24.29} \quad C^* = \sum v_i x_i^*$$

where the x_i^* represent the quantities of inputs defined by the expansion path factor beam.

Partially differentiating 24.29 with respect to the i th factor price yields

$$\text{24.30} \quad M^*/M_i = \sum v_i M_i^*/M_i + x_i^*$$

Substituting equation 24.28 into equation 24.30

$$\text{24.31} \quad M^*/M_i = \sum \mathcal{G}_i M_i^*/M_i + x_i^*$$

Now suppose that the original production function is defined at the cost minimizing level of input use

$$\text{24.32} \quad y = f(x_1^*, \dots, x_n^*)$$

Maximizing the production function with respect to a change in the i th input price

$$\text{24.33} \quad M/M_i = f_i M_i^*/M_i = 0$$

for all $i = 1, \dots, n$

Substituting equation 24.33 into equation 24.31 evaluated at the cost minimizing level of input use

$$\text{24.34} \quad M^*/M_i = \mathcal{G}(0) + x_i^* = x_i^* \text{ used in the least cost combination solution}$$

for all $i = 1, \dots, n$

Equation 24.34 is Shephard's lemma. Shephard's lemma thus states that the change in cost for the cost function arising from the expansion path conditions with respect to the change in the price of the i th factor, evaluated at any particular point (output level) on the least cost total cost function, is equal to the least cost quantity of the i th factor that is used.

24.6 Hotelling's Lemma

Hotelling's lemma makes use of the envelope theorem with respect to profit, rather than cost functions. Consider the case of a firm using n different inputs in order to produce m different outputs. Total revenue (R) is defined as

$$\text{24.35} \quad R = \sum p_j y_j$$

where $y_1, \dots, y_m =$ outputs

$s_j =$ the price of the j th output

Total cost is given as

$$\text{24.36} \quad C = \sum v_i x_i.$$

The output expansion path defines the revenue maximizing combination of outputs for the firm, in much the same manner as the expansion path defines the least cost combination of inputs. The indirect revenue function represents the optimal allocation of outputs to maximize revenue, and can be specified as

$$\text{24.38} \quad R^* = \sum s_j y_j^*.$$

The corresponding indirect cost function is

$$\text{24.39} \quad C^* = \sum v_i x_i^*.$$

Indirect profit is the difference between revenue and cost according to the output and input expansion path conditions given as

$$\begin{aligned} \text{24.40} \quad A^* &= R^* - C^* \\ &= \sum s_j y_j^* - \sum v_i x_i^* \end{aligned}$$

The profit-maximizing production function transforming inputs into outputs is written in its implicit form as

$$\text{24.41} \quad F(y_1^*, \dots, y_m^*; x_1^*, \dots, x_n^*) = 0.$$

The Lagrangian for maximizing profit subject to the constraint imposed by the production function is

$$\text{24.42} \quad L = \sum s_j y_j - \sum v_i x_i + \lambda [F(y_1, \dots, y_m; x_1, \dots, x_n) - 0].$$

First-order conditions on the product side require that

$$\begin{aligned} \text{24.43} \quad \partial L / \partial y_j &= s_j - \lambda \partial F / \partial y_j = 0 \\ &\text{for all } j = 1, \dots, m. \text{ The optimal } y_j \text{ is } y_j^*. \end{aligned}$$

First order conditions on the factor side require that

$$\text{24.44} \quad M/M_i = v_i \quad \text{and} \quad nM/M_i = 0$$

for all $i = 1, \dots, n$. The optimal x_i is x_i^* .

Now differentiate equation 24.40 with respect to the k th product price

$$\text{24.45} \quad \Delta M^*/M_k = y_k^* + E_{s_j}(M_j^*/M_k) - E_{v_i}(M_i^*/M_k)$$

Equations 24.43 and 24.44 are then substituted into 24.45 for the product and factor prices to yield

$$\text{24.46} \quad \Delta M^*/M_k = y_k^* + n \{ (E_{M^*/M_j^*})(M_j^*/M_k) - (E_{M^*/M_i^*})(M_i^*/M_k) \}$$

Differentiate equation 24.41 with respect to the k th product price

$$\text{24.47} \quad \Delta \Pi/M_k = 0 = E_{(M^*/M_j^*)}(M_j^*/M_k) + E_{(M^*/M_i^*)}(M_i^*/M_k)$$

Substitute 24.47 into 24.46

$$\text{24.48} \quad \Delta M^*/M_k = y_k^*$$

Equation 24.48 is Hotelling's lemma as applied to product supply. The lemma states that the change in the indirect profit function arising from the output expansion path with respect to the k th product price is equal to the optimal quantity of the k th output that is produced.

Hotelling's lemma can also be applied to the factor side. Differentiate the indirect profit function with respect to the k th input price

$$\text{24.49} \quad \Delta M^*/M_k = E_{s_j}(M_j^*/M_k) - E_{v_i}(M_i/M_k) - x_k$$

Again substitute equations 24.43 and 24.44 for the product and input prices

$$\text{24.50} \quad \Delta M^*/M_k = n \{ E_{(M^*/M_j^*)}(M_j^*/M_k) - E_{(M^*/M_i^*)}(M_i^*/M_k) \} - x_k^*$$

Differentiate equation 24.41 with respect to the k th input price

$$\text{24.51} \quad \Delta \Pi/M_k = 0 = E_{(M^*/M_j^*)}(M_j^*/M_k) + E_{(M^*/M_i^*)}(M_i^*/M_k)$$

Substitute 24.51 into 24.50

$$\text{24.52} \quad \Delta M^*/M_k = -x_k^*$$

Equation 24.52 is Hotelling's lemma applied to the factor demand side. The lemma states that the change in the indirect profit function with respect to a change in the k th factor price is equal to the negative of the optimal quantity of the k th input as indicated by the expansion path conditions.

Hotelling's and Shephard's lemmas are of considerable importance for empirical research. If the firm is operating according to the assumptions embodied in the expansion path

conditions on both the factor and product sides, then product supply and factor demand equations can be obtained without any need for estimating the production function from physical input data. For example, equation 24.13 is the indirect (minimum) cost function arising from a two input Cobb-Douglas type production function. The conditional factor demand function for input x_j can be found by partially differentiating 24.13 with respect to v_j , treating y as constant, and setting the partial derivative equal to x_j^* from Shephard's lemma.

Rewriting equation 24.13

$$24.53 \quad C^* = D_0 y^{\alpha} v_1^{\beta_1} v_2^{\beta_2}$$

The choice of a Cobb-Douglas type production function to represent a production process within agriculture is primarily one of mathematical convenience. A Cobb-Douglas type cost function may also be appropriate so long as certain assumptions with regard to the parameters are met.

Indirect cost functions should be homogeneous of degree one in all factor prices. A doubling of all factor prices should exactly double cost. Only relative prices enter the factor allocation. Since, from Shephard's lemma the factor demand function for each input is the first derivative of the indirect cost function, then the factor demand equation for each input should be homogeneous of degree zero in all factor prices. The symmetry condition follows from Young's theorem, and implies that the elasticity of demand for the i th input with respect to the j th input price should equal the elasticity of demand for the j th input with respect to the i th input price.

Indirect profit functions conforming to a Cobb-Douglas type might also be assumed. An example is

$$24.54 \quad A^* = G_0 s_1^2 v_1^2 v_2^2$$

Indirect profit functions should be homogeneous of degree one in all prices, and therefore, a doubling of all prices will double profit. The corresponding product supply and factor demand equations based on Hotelling's lemma will be homogeneous of degree zero in all prices. Restrictions regarding the indirect profit, cost, factor demand and product supply functions can be readily incorporated within the estimation procedures found in many regression packages.

24.7 Alternative Elasticity of Substitution Measures

Any elasticity might be written as the derivative of one natural log with respect to another. For example, the elasticity of demand for good q can be written as

$$24.55 \quad E_d = d \ln q_d / d \ln p$$

where

q_d = the quantity of the good demanded

p = the price of the good

This is true, because if

$$\text{24.56} \quad z = \ln q_d$$

then

$$\text{24.57} \quad dz/dq_d = 1/q_d$$

and

$$\text{24.58} \quad dz = dq_d/q_d$$

Similarly, if

$$\text{24.59} \quad r = \ln p$$

$$\text{24.60} \quad dr/dp = 1/p$$

and

$$\text{24.61} \quad dr = dp/p$$

Hence

$$\text{24.62} \quad E_d = dq_d/dp(p/q_d) = d \ln q_d / d \ln p.$$

As indicated in Chapter 12, The elasticity of substitution is a pure number that indicates the extent to which one input substitutes for another and hence indicates the shape of an isoquant according to the "usual" definition (Henderson and Quandt). The elasticity of substitution can be represented by the ratio of two percentages. Suppose that there are two inputs, x_1 and x_2 . The elasticity of substitution between x_1 and x_2 is usually defined as

$$\text{24.63} \quad F = \% \text{ change in } (x_2/x_1) / \% \text{ change in } MRS_{x_1, x_2}.$$

Many approximately equivalent expressions for the elasticity of substitution between two input pairs exist. For example, it is possible to calculate a point or an arc elasticity of substitution. The expression

$$\text{24.64} \quad F_a = [^*(x_2/x_1)/(x_2/x_1)] / [^*(MRS_{x_1, x_2}/MRS_{x_1, x_2})]$$

could be thought of as an arc elasticity of substitution in that it represents the proportionate percentage change in the input ratio (x_2/x_1) relative to the percentage change in the Marginal Rate of Substitution as one moves downward and to the right along an isoquant from point P_1 to point P_2 (Figure 24.2). As one moves along an isoquant from point P_1 to point P_2 , two things happen. First, the ratio of the inputs (x_2/x_1) changes. Second, the slope of the isoquant as measured by MRS_{x_1, x_2} at point P_2 is different from that at point P_1 . The ratio of these two changes in percentage terms is the arc elasticity of substitution.

A point elasticity of substitution can be defined by the formula

$$\text{24.65} \quad F = [d(x_2/x_1)/(x_2/x_1)] / [dMRS_{x_1, x_2}/MRS_{x_1, x_2}]$$

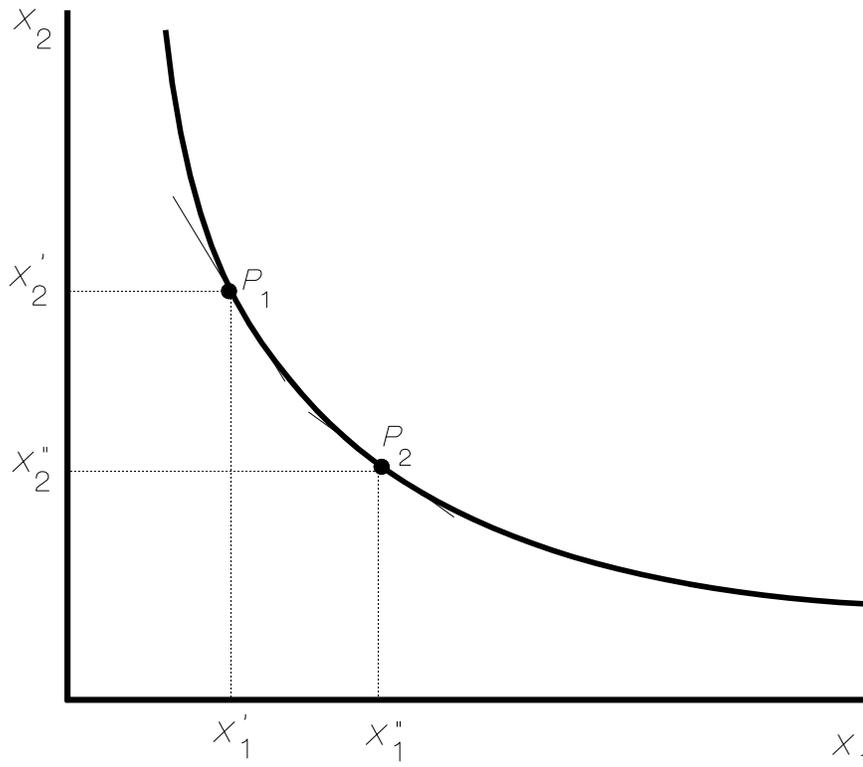


Figure 24.2 A Graphical Representation of the Elasticity of Substitution

or with the equivalent definition (Henderson and Quandt, p. 62)

$$\text{24.66} \quad F = [d(x_2/x_1)/(x_2/x_1)]/[d(f_1/f_2)/(f_1/f_2)]$$

where f_1 and f_2 are the marginal products of x_1 and x_2 , respectively. Now define the input ratio (x_2/x_1) as x . Then the elasticity of substitution F is given as

$$\begin{aligned} \text{24.67} \quad F &= [dx/x]/[dMRS_{x_1,x_2}/MRS_{x_1,x_2}] \\ &= d \ln x / d \ln MRS_{x_1,x_2} \end{aligned}$$

The elasticity of substitution is a very important parameter of a production process involving a pair of inputs. As indicated in Chapter 12, it provides an important indication of the shape of an isoquant. By this definition, isoquants forming right angles (the classic example is tractors and tractor drivers) have zero elasticities of substitution, while diagonal isoquants have an elasticity of substitution approaching infinity. Of course, if there is truly no change in the marginal rate of substitution between points P_1 and P_2 , then the percentage change in the marginal rate of substitution is zero, and the elasticity of substitution is undefined.

The inverse factor price ratio (v_1/v_2) measures the marginal rate of substitution of x_1 for x_2 (dx_2/dx_1) at the point of least cost combination in competitive equilibrium. Therefore, if competitive equilibrium is assumed, the elasticity of substitution in the two factor case at the point of least cost combination on the isoquant may be rewritten as

$$\text{¶4.68} \quad [d(x_2/x_1)/d(v_1/v_2)]/(v_1/v_2)/(x_2/x_1)$$

or as

$$\begin{aligned} & d\ln(x_2/x_1)/d\ln(v_1/v_2) \\ &= (d\ln x_2 - d\ln x_1)/(d\ln v_1 - d\ln v_2). \end{aligned}$$

Equation ¶4.68 is the definition attributed to Hicks (See also Varian, pp. 44-45). Notice, however, that v_1/v_2 is equal to the MRS_{x_1, x_2} only in competitive equilibrium.

A large elasticity of substitution indicates that the entrepreneur (such as a farmer) has a high degree of flexibility in dealing with input price variation. If there existed a large elasticity of substitution between a pair of factors, the farmer would quickly adjust the input mix in response to changing relative prices. However, if the elasticity of substitution were small, the input mix would be hardly altered even in the face of large relative shifts in prices. The extent to which a farmer adjusts the input mix to changing relative prices thus indicates the magnitude of the elasticity of substitution between input pairs.

In the two factor case, the elasticity of substitution will lie between zero and plus infinity. However, if there are more than two inputs, some input pairs may be complements with each other, thus leading to a potential negative elasticity of substitution for some of the input pairs. The definition of an elasticity of substitution in an n factor case is further complicated because a series of specific assumptions must be made with regard to the prices and input levels for those factors of production not directly involved in the elasticity of substitution calculation, and the elasticity of substitution between inputs i and j will vary depending on these assumptions.

The definition of the elasticity of substitution attributed to Hicks can be generalized to the n factor case such that

$$\begin{aligned} \text{¶4.69} \quad F_{ij} &= [d\ln(x_j/x_i)]/[d\ln(v_i/v_j)] \\ &= (d\ln x_j - d\ln x_i)/(d\ln v_i - d\ln v_j) \end{aligned}$$

Equation ¶4.68 is sometimes referred to as the two-input, two-price or *TTES*, elasticity of substitution, or the "usual" definition of the elasticity of substitution in the n factor case (Fuss, McFadden and Mundlak, p. 241, Ball and Chambers). However, when n is greater than two, specific assumptions for the calculation need to be made with regard to prices and quantities of inputs other than i and j .

Moreover, a number of alternative definitions for the elasticity of substitution are possible. The one-input, one-price elasticity of substitution (*OOES*) is proportional to the cross price input demand elasticity evaluated at constant output

$$\text{¶4.70} \quad N_{ij} = \$(d\ln x_j)/(d\ln v_i)$$

The two input one price form (*TOES*) involves two input quantities but only one input price

$$\text{¶4.71} \quad T_{ij} = (d\ln x_j - d\ln x_i)/(d\ln v_i)$$

Furthermore, each definition can be evaluated based on constant output, cost, or marginal cost (Fuss, McFadden, and Mundlak, p. 241). Each of these alternative definitions can be evaluated assuming the prices on the remaining inputs other than i and j are held constant. The quantities of inputs other than i and j can also be held constant or allowed to vary as v_i and v_j vary which generates short and long run elasticity of substitution measures.

Allen (1938) uses the Hicks definition of the elasticity of substitution (p. 341), but Allen also develops an alternative measure of his own, which is linked to the own and cross price constant output factor demand elasticity (See also Hicks and Allen). This definition of the elasticity of substitution attributable to Allen (pg. 504) is

$$\text{¶4.72} \quad F_{ij}^A = S_j E_{ij}$$

where

S_j = the share of total cost attributable to the j th input, or $v_j x_j / C^*$

$E_{ij} = (d \ln x_i) / (d \ln v_j)$ evaluated at constant output. (This is in reality the cross price factor demand elasticity.)

This elasticity of substitution has been dubbed the Allen Elasticity of Substitution (or *AES*), and is of the *OOES* form, since only one price (i) and one input (j) are involved (Ball and Chambers). Notice, also, that an Allen own price elasticity of substitution can be defined as

$$\text{¶4.73} \quad F_{jj}^A = S_j E_{jj}$$

where

$S_j = v_j x_j / C^*$ the cost share represented by the j th input

$E_{jj} = (d \ln x_j) / d \ln v_j$

The *AES* concept forms the basis for still other elasticity of substitution concepts. For example, the Morishima elasticity of substitution (Koizumi) is an example of a *TOES* elasticity of substitution and is defined in terms of the *AES* as

$$\begin{aligned} \text{¶4.74} \quad F_{ij}^M &= S_j (F_{ij}^A - F_{jj}^A) \\ &= E_{ij} - E_{jj} \end{aligned}$$

This elasticity is the difference between the cross and own price elasticity of factor demand evaluated at constant output. This elasticity of substitution is *TOES* since

$$\text{¶4.75} \quad E_{ij} - E_{jj} = (d \ln x_i - d \ln x_j) / (d \ln v_j)$$

Notice that the Morishima elasticity of substitution is not symmetric, that is

$$\text{¶4.76} \quad (d \ln x_i - d \ln x_j) / (d \ln v_j) \neq (d \ln x_j - d \ln x_i) / (d \ln v_i)$$

and therefore $F_{ij}^M \neq F_{ji}^M$

The Shadow Elasticity of Substitution (McFadden, 1963) is an example of a *TTES*, and is therefore closer to the original Hicks definition than is the Morishima or Allen definitions. The Shadow Elasticity of Substitution allows all inputs not involved in the calculation to vary, and thus can be thought of as a long run elasticity of substitution. The shadow elasticity can be expressed in terms of the Allen measure as

$$\text{¶4.77} \quad F_{ij}^S = [(S_i S_j)/(S_i + S_j)] [(2F_{ij}^A - F_{ii}^A - F_{jj}^A)]$$

Thus, if the *AES* and input cost share data are available, the Shadow Elasticity of Substitution can be readily calculated.

24.8 Elasticities of Substitution and the Cobb-Douglas Specification

Specific production functions used by researchers in empirical analysis frequently embody assumptions that come along with the functional form. Fuss, McFadden and Mundlak refer to these assumptions as maintained hypotheses. These maintained hypotheses frequently are not explicitly recognized by the researcher, but do impose constraints on the possible outcomes that can be generated by the analysis.

An excellent example of a maintained hypothesis is the assumption with regard to the Hicksian elasticity of substitution that exists between input pairs when a Cobb-Douglas (CD) type functional form is chosen to represent the production process. Consider, for example a CD type specification with no imposition of a particular sum on $\$_1 + \$_2$.

$$\text{¶4.78} \quad y = A x_1^{\$} x_2^{\$}$$

The marginal rate of substitution of x_1 for x_2 is given by

$$\begin{aligned} \text{¶4.79} \quad MRS_{x_1, x_2} &= (\$/\$(x_2/x_1)) \\ &= MRS_{x_1, x_2} = \$x \end{aligned}$$

where

$$\$ = \$_1/\$(x_2), \text{ and } x = x_2/x_1$$

Henderson and Quandt (Chapter 3) provide a somewhat messy proof that the *TTES* elasticity of substitution for any functional form of the CD type is 1 as a maintained hypothesis. As was indicated in Chapter 12, equations (20.15) - (20.20), simple proof is

$$\text{¶4.80} \quad MRS_{x_1, x_2} = \$x$$

$$\text{¶4.81} \quad \ln MRS_{x_1, x_2} = \ln x + \ln \$$$

$$\text{¶4.82} \quad \ln x = \ln MRS_{x_1, x_2} - \ln \$$$

$$\text{¶4.83} \quad F = d \ln x / d \ln MRS_{x_1, x_2} = 1$$

Equation ¶4.83 holds even if the production function is not linearly homogeneous, and the partial production elasticities sum to a number other than 1. Moreover, it can be easily shown that the relationship holds for any factor pair if the function contains more than two inputs.

A maintained hypothesis that the elasticity of substitution between labor and capital is 1 may be tolerable in a 1928 study dealing with a production process representing the output of a society and utilizing capital and labor as inputs. As will be empirically shown, it is clearly intolerable in a study conducted in the 1980s dealing with the substitutability between energy and machinery within U.S. agriculture.

Subsequent to the Hicks and Allen publications, the maintained hypothesis regarding the elasticity of substitution between labor and capital became an issue of some discussion. Economists have devoted considerable effort aimed at remaking the original Cobb - Douglas article.

24.9 The CES, or Constant Elasticity of Substitution Specification

The CES or Constant Elasticity of Substitution production function (Arrow et al.) was an effort to remake the original CD article without the maintained hypothesis regarding the elasticity of substitution. A specification for the CES function (without linear homogeneity imposed) is

$$24.84 \quad y = A[\$x_1^{1+D} + \$x_2^{1+D}]^{1/D}$$

Suppose that the marginal rate of substitution from some unknown production function is given by

$$24.85 \quad MRS_{x_1, x_2} = \$x^{1+D}$$

where

$$\$ = \text{a constant}$$

$$x = x_2/x_1$$

Taking logs

$$24.86 \quad \ln MRS_{x_1, x_2} = \ln \$ + (1+D)\ln x$$

$$24.87 \quad \ln x = [1/(1+D)]\ln MRS_{x_1, x_2} - [1/(1+D)]\ln \$$$

$$24.88 \quad d\ln x / d\ln MRS_{x_1, x_2} = 1/(1+D) = \sigma$$

The elasticity of substitution is given by the power to which the input ratio is raised. In general, for any production function where the marginal rate of substitution is given by

$$24.89 \quad MRS = \$x^*$$

where

$$24.90 \quad x = x_2/x_1$$

The elasticity of substitution (Hicks) is given by $1/\sigma^*$. It is easily shown that the *MRS* for the CES is of this form

$$24.91 \quad f_1 = \frac{1}{D} \frac{1}{DA} (\$_1 x_1^{1-D} + \$_2 x_2^{1-D})^{1/D-1} (-D \$_1 x_1^{1-D-1})$$

$$24.92 \quad f_2 = \frac{1}{D} \frac{1}{DA} (\$_1 x_1^{1-D} + \$_2 x_2^{1-D})^{1/D-1} (-D \$_2 x_2^{1-D-1})$$

$$24.93 \quad \begin{aligned} dx_2/dx_1 &= (\$_1 x_1^{1-D-1}) / (\$_2 x_2^{1-D-1}) \\ &= (\$_1 / n \$_2) (x_2/x_1)^{1+D} \\ &= \$x^{(1+D)} \end{aligned}$$

Henderson and Quandt (Chapter 3) prove that the Cobb-Douglas production function is a special case of the CES when $D = 0$. This proof requires L'Hopital's Rule. However, it is easily seen here that when D assumes a value of 0, $1+D = 1$ and the $MRS_{x_1, x_2} = \$x$, the exact same form as occurs under the Cobb-Douglas type production function. Debertin, Pagoulatos and Bradford (1977b, pp. 10-11), Chapter 12 provides a detailed discussion of the relationship of the value of D and the shape of the isoquants.

The CES production function was an appropriate improvement if the interest centered on the elasticity of substitution within a production process that used only two inputs, such as capital and labor. However, if the function were extended to the n input case, there remained but one parameter D and, as a result a maintained hypothesis was that the same elasticity of substitution applied to every input pair (see Revankar and Sato for extensions). Agricultural economists are usually interested in disaggregating input categories into more than two inputs. Thus the CES never was extensively used in agricultural economics research. A more flexible functional form was clearly needed for agricultural economics research.

24.10 The Transcendental Production Function and F

Halter, Carter and Hocking (1957) proposed a transcendental production function to depict the three stage production process as represented by the neoclassical theory familiar to any undergraduate agricultural economics student. The transcendental production function is actually a variable elasticity of substitution production function. With proper assumptions with respect to the parameters, the isoquant map for the transcendental production function, and the variant proposed by Debertin, Pagoulatos, and Bradford (1977a, 1977b, p. 8), generate isoquants consisting of concentric rings. This map is quite unlike anything possible with the CES or Cobb-Douglas specifications, which produce isoquants that are everywhere downward sloping.

As was indicated in Chapter 11, the HCH transcendental is

$$24.94 \quad y = x_1^{\alpha_1} x_2^{\alpha_2} e^{\gamma_1 x_1 + \gamma_2 x_2}$$

The Allen elasticity of substitution for the HCH transcendental is

$$24.95 \quad F = [(\sigma_2 + (\sigma_1 x_1))(\sigma_1 + (\sigma_2 x_2))] / [(\sigma_2)(\sigma_1 + (\sigma_2 x_2)^2 + (\sigma_1)(\sigma_2 + (\sigma_1 x_1)^2)]$$

Morishima and Shadow elasticities can be calculated from the Allen measure. This function is readily estimable with data from agricultural production processes (Halter and Bradford). The discussion in Chapter 11 links parameter values to the shape of the function.

Despite some recognition of the HCH functional form in the general economics literature (e.g. Fuss, McFadden and Mundlak, pg. 242), the HCH function is not widely used by economists. Its strength, that it can depict the neoclassical three stage production function, is also its weakness. The fact that, at least for certain parameter values, the function is not monotonically increasing means that the inverse or dual cost curve associated with it is a correspondence, not a function. As a result, parameters of the production process represented by the transcendental cannot be readily derived from the corresponding cost data. Contemporary production theory involves choosing a functional form to represent the production process that is monotonically increasing, and can be readily inverted, such that parameters can be derived from either the cost or the physical input data.

Many agricultural economists continue to emphasize the three stages of the neoclassical production process in undergraduate classes, and continue to be fascinated with stage three, where output declines as incremental units of the variable input are added. In order to take advantage of the duality theorems, contemporary theorists have all but abandoned stage three and therefore the usual assumption made by contemporary theorists is free disposal.

Assuming positive factor prices, no economic conditions could cause the firm to apply units of a variable input beyond the point where output is maximum. Beattie and Taylor (p. 91) indicate negative factor prices could exist, for example, if a farmer were paid to remove a waste product which could be used as a fertilizer. They further contend that a farmer could operate in stage three if a factor price were negative. However, if the factor price were negative, under no circumstances would it be more profitable for the farmer to apply additional units to the crop beyond the point of output maximum, than to dump the waste product consistent with the free disposal assumption.

If fertilizer were free, the farmer would be better off to dump units than to apply it to a crop, if in so doing, yields would be reduced. Again, the free disposal assumption is critical. Contemporary production functions typically increase but at a decreasing rate throughout their range for each variable input. The Cobb-Douglas production might be thought of in this regard as contemporary, rather than neoclassical, but this is also true for the CES and Translog specifications developed much later. The duality concepts are closely linked to the maintained hypothesis of free disposal, and the marginal products that are correspondingly everywhere positive throughout the range of the function.

24.11 Linear in the Parameters Functional Forms and the Translog Production Function

Diewert introduced the concept of linear in the parameters functional forms. While Diewert recognized that advances in computing technology made it possible to estimate functional forms that were non linear in the parameters, little if any new information would be gained about the production process by the use of more complex and computationally burdensome functional forms.

In addition, Diewert recognized the close linkages that exist between various functional forms. One way of looking at various functional forms is in terms of Taylor's series

expansions. For example, the Cobb-Douglas type production function could be written as a first order Taylor's series expansion of $\ln y$ in $\ln x_i$

$$\text{24.96} \quad \ln y = a_0 + \sum_i \beta_i \ln x_i$$

The CES is a first order Taylor's series expansion of y^D in x_i^D (Fuss, McFadden and Mundlak, p. 237). Similarly, the CES could be written in a multiple input setting as

$$\text{24.97} \quad y^D = a_0 + \sum_i \beta_i x_i^D$$

The Translog production function was introduced in 1971 by Christensen, Jorgenson and Lau, and was the logical choice given the difficulties posed by other functional forms. The translog production function is simply a second order Taylor's series expansion of $\ln y$ in $\ln x_i$, whereas the Cobb-Douglas is a first order expansion. The production function as a Taylor's Series expansion can be written as

$$\text{24.98} \quad \ln y = a_0 + \sum_i \beta_i \ln x_i + \sum_{i,j} \beta_{ij} \ln x_i \ln x_j$$

The function had a number of other virtues, in addition to its close linkage to the Cobb-Douglas. It is linear in the parameters, which makes parameter estimation simple. It is normally monotonically increasing with respect to the use of each input under the usual parameter assumptions. However, results depend upon the units in which the x_i are measured. If $0 < x_i < 1$, $\ln x_i < 0$, and under certain positive parameter combinations, the function may not be increasing with respect to the i th input. That the function does not depict the neoclassical three stage production process is viewed as a virtue, not a vice, for fundamental concepts of duality are applicable.

Moreover, there is no maintained hypothesis about the elasticity of substitution between input pairs, and the various elasticity of substitution measures can be derived either directly from the production function, or as is now common, from a dual cost function of the translog form. Thus, it is the production function of choice for agricultural economists who seek to estimate elasticities of substitution between input pairs with little information about the production process other than cost data available to them. If there are both fixed and variable inputs, the translog production function is given as

$$\text{24.99} \quad y = \sum_i \beta_i \ln x_i + \sum_{i,j} \beta_{ij} \ln x_i \ln x_j + \sum_{i,z} \beta_{iz} \ln x_i \ln z_k^\circ + \sum_z \beta_z \ln z_k^\circ$$

where the z_k° represent fixed inputs. The β_{iz} represent the assumed interaction between levels of fixed and variable input use and the assumed constant level of fixed inputs. The term $\sum_z \beta_z \ln z_k^\circ$ is a constant intercept term that performs a role similar to A in a Cobb-Douglas type specification.

Alternately, one might instead rely on duality, and begin with a dual cost function of the translog form. The translog cost function expresses cost as a function of all input prices and the quantity of output that is produced. For a given level of output y^* , the corresponding point on the cost function is assumed to be the minimum cost of producing y^* arising from the expansion path conditions.

The least-cost translog cost function is

$$\begin{aligned} \text{24.100} \quad \ln C^* &= \alpha_0 + \sum_i \alpha_i \ln v_i + \sum_{i,j} \alpha_{ij} \ln v_i \ln v_j \\ &+ \alpha_y \ln y + \sum_{i,z} \alpha_{iz} \ln v_i \ln z_k^\circ + \sum_{y,z} \alpha_{yz} \ln y \ln z_k^\circ \\ &+ \sum_{j,k} \alpha_{jk} \ln z_j^\circ \ln z_k^\circ + \sum_z \alpha_z \ln z_k^\circ + \sum_{y_i} \alpha_{y_i} \ln y \ln v_i \end{aligned}$$

where

(v_1, \dots, v_n) = the vector of input prices

(z_1, \dots, z_n) = the vector representing levels of the fixed inputs

y = output

α = the parameter vector to be estimated

Equation 24.100 is normally estimated from cost share equations which are derived as follows.

The elasticity of total cost with respect to a change in the i th input price is given by

$$\text{24.101} \quad \frac{\partial \ln C^*}{\partial \ln v_i} = \frac{dC^*}{dv_i} \frac{v_i}{C^*} = \gamma_i$$

Hence

$$\text{24.102} \quad \gamma_i = \alpha_i + \sum_{j \neq i} \alpha_{ij} \ln v_j + \sum_{i,z} \alpha_{iz} \ln z_k^\circ + \alpha_{y_i} \ln y.$$

It was not until the translog production and cost functions were introduced in the early 1970s that the importance of Shephard's Lemma for empirical work became apparent. Recognize that γ_i can be written as

$$\text{24.103} \quad \frac{\partial \ln C^*}{\partial \ln v_i} = \frac{\partial \ln C^*}{\partial \ln x_i}.$$

But, since Shephard's lemma states that

$$\text{24.104} \quad \frac{\partial \ln C^*}{\partial \ln x_i} = x_i^*$$

Then

$$\text{24.105} \quad \gamma_i = x_i^* v_i / C^*$$

Notice also, that $x_i^* v_i =$ the total expenditures on input x_i according to the expansion path conditions. Thus, the expression $x_i^* v_i / C^* = \gamma_i = S_i$ where S_i is the cost share associated with the i th input. The series of cost share equations thus becomes

$$\begin{aligned}
 24.106 \quad S_1 &= \alpha_1 + \sum_{j=1}^n \alpha_{1j} \ln v_j + \sum_{k=1}^m \alpha_{1k} \ln z_k + \alpha_{1n} \ln y \\
 &\vdots \\
 S_i &= \alpha_i + \sum_{j=1}^n \alpha_{ij} \ln v_j + \sum_{k=1}^m \alpha_{ik} \ln z_k + \alpha_{in} \ln y \\
 &\vdots \\
 S_n &= \alpha_n + \sum_{j=1}^n \alpha_{nj} \ln v_j + \sum_{k=1}^m \alpha_{nk} \ln z_k + \alpha_{nn} \ln y
 \end{aligned}$$

The cost-share equations are empirically estimated, and include price and output variables and levels of fixed inputs that would normally be readily available from farm records or even census data. If data on the level of fixed inputs are not available, their combined impact is estimated as part of the intercept term.

24.12 Restrictions and Other Estimation Problems

Economic theory imposes a number of restrictions on the estimation process. First, Total Cost = $\sum S_i$. Thus, given total cost and any $n-1$ cost shares, the remaining cost share is known with certainty. Therefore, one equation is redundant, and mechanically, the choice of the equation to be omitted is arbitrary, but the empirical results may not be invariant with respect to the choice of the omitted equation unless an iterative estimation procedure is used (cf. Humphrey and Wolkowitz; Moroney and Toevs; and Berndt and Wood).

As indicated earlier, any total cost function should be homogeneous of degree 1 in input prices. This restriction can be imposed by restricting $\sum \alpha_i = 1$ and $\sum \alpha_{ij} = 0$. Since Young's theorem states that the order of the differentiation makes no difference and the α_{ij} are in reality partial derivatives, a symmetry restriction must also be imposed such that $\alpha_{ij} = \alpha_{ji}$ for all i and j inputs. Finally, the cost share for the i th input is not unrelated to the cost share for the j th input, and a Seemingly Unrelated Regressions approach is the usual choice for estimation of the cost share equations.

24.13 Elasticities of Substitution for U.S. Agriculture

From the parameter estimates of the cost share equations, the corresponding Allen Elasticities of Substitution between input pairs and the related measures can be derived. Brown and Christensen derive the constant output partial static equilibrium cross price elasticity of factor demand as

$$\begin{aligned}
 24.107 \quad E_{ij} &= S_j F_{ij}^A \\
 &= \ln x_i / \ln v_j = (\alpha_{ij} + S_i S_j) / S_i
 \end{aligned}$$

where

$$F_{ij}^A = (\alpha_{ij} + S_i S_j) / (S_i S_j)$$

is the Allen Elasticity of Substitution.

The *AES* estimate is readily derived from the parameter estimates of the cost share equation. The usual approach is to insert the mean of the cost shares for each input category in the data for the sample period in order to obtain the Allen estimates. Once the Allen estimates are obtained, the corresponding Morishima and Shadow Elasticities of Substitution can then be obtained from equations 24.73 and 24.76. Again, the mean of the factor shares for the sample data is introduced into the formulas along with the estimated Allen measure. The Shadow Elasticity of Substitution estimate obtained from this model, that is perhaps the closest to the Hicks' definition, is not quite the long run measure envisioned by McFadden. Inputs in the \mathbf{x} vector other than i and j are treated as variable in the shadow measure. However, inputs in the \mathbf{z} vector are treated as fixed. The true long run measure suggested by McFadden could be obtained if all input categories were treated as part of the \mathbf{x} vector.

24.14 An Empirical Illustration

The empirical illustration of the application of theory presented in this publication is from Aoun, who was concerned with the potential changes in elasticities of substitutions between agricultural inputs over time, particularly energy and farm machinery. Fuss, McFadden, and Mundlak refer to technological change which impacts the partial elasticities of substitution between input pairs as substitution augmenting technological change.

Substitution augmenting technological change that increases the elasticity of substitution between input pairs is desirable in that the producer is given additional flexibility in dealing with changes in the relative prices of the inputs that might occur due to shocks within the factor markets. For example, suppose that the elasticity of substitution between capital and labor within an economy were near zero. The firm would be faced with a situation in which capital and labor would be used in nearly fixed proportions to each other irrespective of relative price levels. Moreover, the firm owner would have little flexibility for dealing with short run variability in input prices over time.

Estimates of elasticities of substitution among input pairs must necessarily rely on data series for a number of years. If there exist shifts in elasticities of substitution over time due to technological change, then the data series for a long period of time can not be relied upon to measure these shifts. If the data series are too short, degrees of freedom problems, multicollinearity between input vectors and instability of regression coefficients upon which the elasticity estimates are derived become issues.

24.15 Theoretical Derivation

Aoun used a translog cost function specified as

$$\begin{aligned} 24.108 \quad \ln C^* = & \alpha_0 + \sum_i \alpha_i \ln v_i + \frac{1}{2} \sum_{yy} \beta_{yy} (\ln v_y)^2 + \frac{1}{2} \sum_i \sum_j \beta_{ij} \ln v_i \ln v_j \\ & + \sum_{yt} \gamma_{yt} \ln v_y \ln t + \frac{1}{2} \sum_{tt} \delta_{tt} t^2 + \sum_{ty} \eta_{ty} t \ln v_y + \sum_i \eta_{it} t \ln v_i \end{aligned}$$

where

C^* = minimum total cost

$i, j = n, l, m, f, e$

y = output

n = land

l = labor

m = machinery

f = fertilizer

e = energy

t = annual time trend variable

v_i, v_j = input prices on $n, l, m, f,$ and e .

The translog cost function is assumed to be continuous, monotonically increasing, concave and homogeneous of degree one with respect to factor prices. Following the analysis by Brown and Christensen, an assumption is made that the translog cost function represents a constant returns to scale technology. This implies the following restrictions

$$\text{24.109} \quad \alpha_y = 1$$

$$\text{24.110} \quad E(\alpha_{yi} = 0 \text{ for } i = 1, 5$$

$$\text{24.111} \quad \alpha_{yy} = 0$$

$$\text{24.112} \quad N_{ty} = 0$$

Partially differentiating 24.108 with respect to the i th input price, assuming that restrictions 24.109 - 24.112 hold

$$\text{24.113} \quad \ln C^* / \ln v_i = \alpha_i + E \alpha_{ij} \ln v_j + (\alpha_{yi} \ln y + N_{it} t$$

$$i = 1, \dots, 5$$

Invoking Shephard's lemma

$$\text{24.114} \quad \ln C^* / \ln v_i = \ln C^* / \ln v_i = (x_i v_i) / C^* = S_i$$

where

S_i = the cost share for the i th input $i = 1, 5$

and

$$S_i = \alpha_i + E \alpha_{ij} \ln v_j + (\alpha_{yi} \ln y + N_{it} t$$

$$i = 1, \dots, 5$$

The restrictions imposed on the estimation were

$$\text{24.115} \quad E \sigma_i = 1$$

$$\text{24.116} \quad E \sigma_{ij} = E \sigma_{ji} = E E \sigma_{ij} = 0$$

$$\text{24.117} \quad E \sigma_{yi} = 0$$

$$\text{24.118} \quad E N_{ii} = 0$$

$$\text{24.119} \quad \sigma_{ij} = \frac{M_h C^*}{(M_h v_i \ln v_j)} = \sigma_{ji} = \frac{M_h C^*}{(M_h v_j \ln v_i)}$$

The Allen measure is derived from the parameter estimates of the cost share equation. The approach used in Aoun is to insert the mean of the cost shares for each input category in the data for the sample period into 24.107 in order to obtain the the Allen estimates. Once the Allen estimates are obtained, the corresponding Morishima and Shadow Elasticities of Substitution can then be obtained. Again, the mean of the factor shares for the sample data is introduced into the formulas along with the estimated Allen measure.

24.16 Empirical Results

Estimates of Elasticities of Substitution for the Allen, Morishima, and Shadow (McFadden) measures were obtained for U.S. agriculture for the three distinct decades 1950-59, 1969-69 and 1970-79, and for the entire period comprising 31 years from 1950 to 1980 (Aoun). Restricted Three Stage Least Squares was the method of estimation. The standard U.S.D.A price indexes for the various input categories was used, except for land, where the index was constructed. A detailed discussion of the sources of data and computational procedures can be found in Aoun. Allen Elasticities are reported for the three distinct decades (Table 24.1) and the Morishima and Shadow elasticities are reported for the period 1970-79 (Tables 24.2 and 24.3). Estimates of the Shadow elasticity of substitution for most input pairs differed significantly from 1, suggesting that the appropriate production function to represent U.S. agriculture is not Cobb-Douglas.

Moreover, the Allen elasticities varied rather substantially from one decade to the next. Of particular interest were the estimates of the elasticities of substitution between machinery (including tractors) and energy for the three distinct decades. The Allen estimates went from -13.240 for 1950-59, to -0.118 for 1960-69 to +13.583 for 1970-79. The remarkable conclusion is that energy and machinery were complements in the 1950s but substitutes during the 1970s according to the Allen measure. The substitution between energy and machinery for the 1970-79 decade was further confirmed by the estimated value of 2.808 for the shadow measure (Table 2), and 1.052 or 5.613 for the nonsymmetric Morishima measure (Table 3). There has been a clear increase in the substitutability between energy and machinery over the three periods for which the estimates are based.

Other changes over the three decades, although perhaps not quite as profound, are also of interest. For example, the elasticity of substitution between labor and energy is clearly trending downward according to the Allen measure, from +5.120 (substitute) for 1950-59 to -10.313 for 1970-79 (complement). Labor and fertilizer, a complement in 1950-59 (+7.950) is clearly a substitute for 1970-79 (+2.125) according to the Allen measure. The signs are in agreement with those for the Morishima and Shadow measures.

Table 24.1. Estimates of the Allen Cross Elasticities of Substitution for the Three Distinct Decades, 1950-59, 1960-69 and 1970-79^a

	F_{nl}	F_{nm}	F_{nf}	F_{ne}	F_{lm}
1950-59	! 1.737** (0.687)	+ 3.789* (1.852)	+8.552**! (1.745)	2.000 (1.457)	! 0.327 (0.910)
1960-69	! 1.440 (2.073)	+ 8.327 (5.558)	+ 2.565 (2.308)	! 0.366 (2.209)	+ 3.865 (4.510)
1970-79	! 0.071 (1.268)	+ 1.484 (1.833)	! 1.083* (0.686)	! .350 (0.999)	+ 10.962** (2.146)
	F_{lf}	F_{le}	F_{mf}	F_{me}	F_{fe}
1950-59	! 7.950** (0.919)	+ 5.120** (0.565)	! 5.950*† (2.823)	13.240** (1.705)	+ 2.158 (1.762)
1960-69	! 1.333 (1.780)	+ 4.586** (1.740)	+ 1.316 (4.207)	! 0.118 (3.669)	! 0.867 (1.700)
1970-79	+ 2.125** (0.745)	! 10.313** (1.210)	! 1.278* (0.811)	+ 13.583** (1.665)	+ 0.455 (0.350)

^aStandard errors in parentheses

n = land $F_{ij} > 0 \Rightarrow$ factor *i* and factor *j* are substitutes
l = labor $F_{ij} < 0 \Rightarrow$ factor *i* and factor *j* are complements
m = machinery * 0.10 significance level by a one-tailed *t*-test
f = fertilizer **0.05 significance level by a one-tailed *t*-test
e = energy

Table 24.2 Morishima Elasticities of Substitution for the 1970-79 Decade^a

Input	Land	Labor	Machinery	Fertilizer	Energy
Land	0.0	1.315 (0.608)	3.949 (0.840)	! 0.356 (0.211)	! 0.152 (0.201)
Labor	! 0.007 (1.009)	0.0	5.285 (0.684)	0.063 (0.076)	! 1.018 (0.120)
Machinery	0.706 (1.241)	2.945 (0.335)	0.0	! 0.378 (0.199)	1.052 (0.168)
Fertilizer	! 0.464 (.672)	1.286 (0.402)	3.567 (0.652)	0.0	! 0.080 (0.107)
Energy	! 0.138 (.902)	! 0.999 (0.385)	5.613 (0.513)	! 0.152 (0.045)	0.0

^aStandard errors in parentheses

Table 24.3 Shadow Elasticities of Substitution for the 1970-79 Decade^a

Input	Land	Labor	Machinery	Fertilizer	Energy
Land	0.0	0.629 (0.654)	3.191 (0.819)	! 0.380 (0.280)	! 0.150 (0.286)
Labor		0.0	4.278 (0.447)	0.574 (0.163)	! 1.012 (0.132)
Machinery			0.0	1.540 (0.355)	2.808 (0.199)
Fertilizer				0.0	! 0.109 (.030)
Energy					0.0

^aStandard errors in parentheses

24.17 Concluding Comments

Contemporary production theory focuses on the duality that exists between the production function and the cost function along the expansion path. Although not developed for that purpose, the Cobb-Douglas production function can be thought of as one of the first forms consistent with the required assumptions for the development of the dual cost function. But it had important disadvantages with respect to the maintained hypotheses with respect to the substitutability of inputs. The CES and Translog specifications represented relaxations of these maintained hypotheses.

The concept of an elasticity of substitution is highly complex. From the basic and familiar two input definition, a number of alternative concepts have been presented. At the same time, this concept is perhaps the most important in all of production economics, and is particularly useful in an agricultural setting. For example, technological change which increases the elasticity of substitution between input pairs would give farmers additional flexibility in dealing with input price variation.

Following the general theoretical approach outlined in this paper, the Aoun study provided some intriguing results with respect to elasticities of substitution between input pairs for U.S. agriculture. The elasticity of substitution between energy and machinery within U.S. agriculture has changed markedly over the three decades from the 1950s to the 1970s. Energy which was a complement for machinery in the 1950s was a substitute by the 1970s. The results provide empirical evidence that the form of technological change within agriculture which increases the elasticity of substitution over time, as suggested by McFadden, has indeed taken place within U.S. agriculture.

This chapter has attempted to show that the premises of contemporary production theory are important to and do have application to problems in agricultural production. What is required is a somewhat different approach than has traditionally been used in research in agricultural production. Instead of the estimation of a Cobb-Douglas type specification on physical input data, a contemporary approach frequently involves the estimation of the factor share equations from the cost data. But this is an advantage for much agricultural economics research in that the cost data is usually more readily available than the physical input data, and is perhaps more reliable as well. The approach should be applicable to studies conducted using data from individual farm records, census data representing small groups of farmers, as well as aggregated studies conducted at a regional or national level.

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