This book is an attack on finitism in all its forms, philosophical and mathematical, from Aristotle and Kant to Brouwer and Wittgenstein. A metaphysics of the actual infinite is offered as the solution to the contemporary crisis in the foundations of mathematics.

Such specifically mathematical topics as convergent and divergent series, the transfinite cardinals and ordinals, Godelian formulas, the continuum and the infinitesimal are investigated in the light of a new approach to Zeno's 'bisection' paradox. Against the background of the actual infinite such specifically metaphysical themes as the following are brought into focus: being, essence, order, the thing in itself, the absolute, materialism and teleology.

The later philosophy of Wittgenstein is subjected to a protracted critique in an effort to justify the metaphysical enterprise.
What I have to do is, as it were, to describe the office of a king; in doing which I must never fall into the error of explaining the kingly dignity by the king’s usefulness, but I must leave neither his dignity nor his usefulness out of account.

WITTGENSTEIN
TO MY MOTHER
for her unfailing encouragement during hard times
Ever since Hume and Kant, and now with Wittgenstein, the credentials of the metaphysician have been subjected to the severest scrutiny, and found wanting. One is almost tempted to liken the metaphysical ascent to the fabulous Indian rope-trick, one end suspended in midair, the other lost in the clouds. Houdini is reported to have said that, though he had often met people who had met people who had seen the Indian rope-trick, despite all his extensive inquiries he had never succeeded in meeting anyone who had seen the Indian rope-trick.

Yet the metaphysical impulse is not readily quenched. If it is checked in one direction, it is liable to break out in another. Abandoned by reason, it may invite the sponsorship of irrationality itself, as we have unhappily seen in our own time. Philosophy in its original sense was understood to be a kind of tropism whereby the finest flower of the intellect was oriented toward the cosmos; it was the opening of a great window onto the universe. It is that very directedness of man's reason ordered toward the cosmological horizon that I should wish to recover in this fairly extensive sheaf of metaphysical reflections. How melancholy today to find the lovers of reason content with what Kant styled critical, and what is currently termed analytical, philosophy, and to find metaphysics—by default—embraced in large measure by the devotees of obscurantism and mystification. If I confess to feeling almost alone in my endeavour to restore the credit of rationalistic metaphysics, I am not unaware of the many latent sympathies that my investigations are likely to awaken, some from sceptical, others from 'critical' or even analytical, slumbers.

But I must not promise too much. Philosophers, past and present alike, have invariably been prone to be long on promises and short on performance. Priding themselves on their 'solutions', they are in fact remembered and cherished for the problems which they raised. Their 'solutions', above all, have proved to be—for us—problems. I know of scarcely one philosopher (Socrates always excepted) who ever raised a problem as a problem. I mean terminally as a problem, not merely by way
of entry into his theme. Thus Zeno himself never viewed his paradoxes as problems; he advanced them only as proofs calculated to establish the impossibility or unintelligibility of motion. There have been dogmatic and there have been sceptical, but there have been no problematic philosophers. More precisely, there have been no problematic philosophers eo nomine, though in fact none has succeeded in being anything else. They have lacked self-knowledge. They have failed to understand the true dignity of their achievements. For the problematic character of philosophy, certainly of all philosophy up to the present, need not be altogether a misfortune. It is the happy suggestion of Leo Strauss that Plato understood the eternal Ideas to be the great range of problems that preside over man's deepest reflections and that it is in being open to those problems, as problems, that he acquires Socratic ignorance, which is the same as Socratic wisdom.

My return to rationalistic metaphysics must then be understood as a return with a difference: it is now expressly problematic. Designed as a protreptic inquiry, it is to be seen as a cosmological lure, and if at the outset I am found to be exploring an ancient paradox, I shall be soon seen to be moving into the very centre of the contemporary crisis in the foundations of mathematics. The centre of that crisis is the concept of infinity. It is here that mathematics and metaphysics intersect with maximum concentration. It is here also, in the concept of infinity, that clarity in the one presupposes clarity in the other. In part, then, our essay may be understood as being addressed to the metaphysical foundations of mathematics; but only in part, for the concept of infinity is found to impinge on almost the whole schedule of ontological questions.

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CHAPTER 1
INFINITY

Part 1: Aristotle and the ‘Bisection’ Paradox

Although Zeno’s celebrated ‘bisection’ paradox or ‘dichotomy’ can only be regarded as an ingenious sophism, it is the kind of sophism which, properly understood, opens up for us a whole range of metaphysical considerations. In the eighth book of the Physics Aristotle presents two formulations of the paradox. The first formulation is altogether familiar. If a man is to walk a distance of one mile, he must first walk half the distance or 1/2 mile, then he must walk half of what remains or 1/4 mile, then again half of what remains or 1/8 mile, &c. ad infinitum. An infinite series of finite distances must be successively traversed if the man is to reach the end of the mile. But an infinite series is by definition a series that cannot be exhausted, for it never comes to an end. Hence the man can never reach the end of the mile, and seeing that the same argument may be applied mutatis mutandis to any finite distance whatever, it is clearly impossible for motion ever to occur.

Less familiar is Aristotle’s second formulation of the paradox. Indeed it appears to be virtually unknown today, and it has certainly never received the kind of concentrated attention which it deserves. In the course of these studies we shall be returning to it again and again, in the most diverse contexts, never perhaps exhausting it completely. This second formulation of the paradox Aristotle reports as follows, in scarcely more than one sentence. ‘If you count the first half of the journey and then the half of what is left and so on, you would have to count an infinite series of numbers before you got to the end of the journey; which is admitted to be impossible.’

1 Physics, 263a 7-11. (Translation by P. Wicksteed and F. Cornford.)

The Oxford translation reads: ‘Some ... would have us grant that in the time during which a motion is in progress it should be possible to reckon a half-motion before the whole for every half-distance that we got, so that we have the result that when the whole distance is traversed we have reckoned an infinite number, which is admittedly impossible.’
the paradox a man is envisaged who not merely divides a finite interval of space into an infinite number of finite sub-intervals (simply by walking) but who also, in the ongoing process of progressive division, successively counts those sub-intervals, one after the other (δὲ ἀριθμῶν καὶ δὲ εἰς τὰ ἡμισελήνης διαμέτρου).

We may even think of the man as recording all of the natural numbers on a sheet of paper (see Fig. 1). While traversing the first half mile, he will divide the sheet of paper in half and record the number 1 in one of the halves; during the next 1/4 mile he will divide the remaining half of the paper again in half and record the number 2 in one of the quarter-sections; during the next 1/8 mile he will divide the empty quarter-section again in half and record the number 3 in one of the eighth-sections, &c. ad infinitum. As the numbers grow larger and larger, they may be said to grow smaller and smaller, so to speak. At the end of the journey, one hour later, all of the natural numbers will be found transcribed on the sheet of paper. Although no mere human being could ever be expected to execute such a prodigious feat (we are too clumsy), may we not design a computer, albeit a metaphysical computer, expressly 'programmed' for that undertaking? But any such utopian experiment is 'admitted to be impossible'—not merely on practical grounds (that is the least of the objections) but on theoretical grounds as well. For the infinite can never be exhausted in toto. Motion, then, must also be allowed to be impossible.

Did Zeno really believe that there is no motion? We can scarcely credit that possibility. Even if we grant that his sceptical conclusions 'appear to follow logically in a dialectical discussion, yet to believe them seems next door to madness when one considers the facts'. How, then, did Zeno understand the import of his paradoxes? As a disciple of Parmenides, he was 'led to transcend sense-perception and to disregard it on the ground that "one ought to follow the argument"'. This, then, was the great legacy of Parmenides to philosophy: 'One ought to follow the argument', the λόγος; one's own personal sentiments are unimportant. The question of belief simply does not arise. At any rate, it does not arise in the way that we are inclined to expect it to arise. For though it is impossible to suppose that Zeno actually believed that there is no motion, it is not difficult to suppose that he really believed his arguments to be irrefutable.

Despite its noble severity, the Parmenidean dictum has not remained uncontested. We may think of Aristotle as having insisted, in opposition, that it is not enough merely to follow the argument: one must also be reasonable. But this appeal to the reasonable cannot absolve us from attending to the argument, and in fact Aristotle undertakes to meet the 'bisection' paradox on its own terms. Earlier in the Physics, in the sixth book, Aristotle offers a rough draft of a solution which will later be qualified, in the eighth book, as merely provisional. In the earlier discussion he is willing to allow that if a man is to walk a distance of one mile he must first walk 1/2 mile, then 1/4 mile, then 1/8 mile, &c. We may style this infinite series as the Z-series. Zeno insists that at each stage of the journey there is always another stage that lies ahead, and hence the man can never exhaust them all. But Zeno chooses to forget (as Aristotle reminds us) that the man will travel the first half mile in (say) one half hour, then the next 1/4 mile in the next 1/4 hour, &c. The infinite series of spatial intervals will be traversed in the course of an infinite series of temporal intervals. There is a one-to-one correspondence between the two series. It will be true to say either that (1) a finite spatial interval (1 mile) has been traversed in a finite temporal interval (1 hour) or that (2) an infinite series of spatial intervals (1/2 mile + 1/4 mile + 1/8 mile...)}
Infinite series of finite intervals of time that must successively elapse if the end of the minute is to be reached. But how can an infinite series come to an end? How can an endless series come to an end? Aristotle's answer to this question is not easy to make out. Let us consult his own words. 'To the question whether it is possible to pass through an infinite number of units [i.e. intervals] either of time or of distance we must reply that in a sense it is and in a sense it is not.' Here, again, we have the famous distinguishing: in one sense, yes, in another sense, no. What exactly are these senses? 'If the units [intervals] are actual, it is not possible; if they are potential, it is possible.' ενελεξαία μὲν γὰρ ὅπως ὁ τὸν ἐνδέχεται, δύναμε ν ἡ ἐνδέχεται. A distinction is thus drawn between actual intervals and potential intervals. 'For in the course of a continuous motion the traveller has traversed an infinite number of units [intervals] in an accidental sense but not in an unqualified sense [per accidem, yes; per se, no.] ἐὰν τὰ συνεχεία καινοίμενα κατὰ συμβέβηκος ἀπειρα διελθηλθείν, ἀπλῶς δ'ονο.'

'For though it is an accidental characteristic of the distance to be an infinite number of half-distances, this is not its real and essential character.' συμβέβηκε γὰρ τῇ γραμμῇ ἀπειρα ημέρα εἶναι, ἥ δ'ὁδοιν ἐντὸν ἐτέρα καὶ τὸ ἑλθα. 'Though what is continuous contains an infinite number of halves, they are not actual but potential halves.' ἐν δὲ τῷ συνεχεί ἐστὶ μὲν ἀπειρα ημέρα, ἀλλ' ὅπως ἐνελεξαία ἡλλα δυνάμεις.'

What precisely is Aristotle saying? I do not believe that the full force of Aristotle's argument has always been adequately grasped by his critics. Let us approach the matter in a somewhat irregular fashion. No one will doubt that we could, in principle, replace our standard decimal notation by the following system. The number 1 will be represented by a line 1/2 inch in length, the number 2 by a line 3/4 inch in length (i.e. 1/2+1/4), the number 3 by a line 7/8 inch in length (1/2+1/4+1/8), etc. We may say that each of the natural numbers has been assigned a unique digit peculiar to it alone. (No digital line can exceed one inch in length.) Imagine now a man equipped with a blank sheet of paper, a pencil, and a straight edge. With one swift motion he draws a straight line down the full length of the page. The line is, say, eight inches long, and the whole operation has been clocked at four seconds. What are we to reply to this scoundrel if he now says, 'Behold! I have written down all of the natural numbers. Did I not draw a line 1/2 inch long, then a line 3/4 inch long, then a line 7/8 inch long, &c., &c.'?
only be nonplussed at his impudence. As if to outrage us further, he will continue, 'Name any number whatever, 8,076? Since the motion of my hand in drawing the line was entirely uniform in velocity from beginning to end, I can even compute for you the exact time at which I drew the line which represents the number 8,076. If you do not trust my accuracy in this matter, we can always employ a mechanical drawing device that will be certifiably uniform in velocity.' Enough! There will be very few, I submit, who will not reject our scoundrel's argument as a miserable sophism. Which is precisely Aristotle's point. In swiftly drawing his line down the length of the page, our scoundrel cannot be said to have actually (ἐνερέχεσθαι) drawn first a line of 1/2 inch, then a line of 3/4 inch, then a line of 7/8 inch, &c. The motion being continuous, the line may be described only per accidens (κατὰ συμβεβηκός) as composed of an infinite number of finite sub-intervals, not per se (ἀπλῶς). The infinite series of finite sub-intervals exists within the line only potentially (δυνάμει), in so far as it is always possible to effect a division at any point in the line; but none of those sub-intervals may be said to exist actually (ἐνερέχεσθαι) unless an actual division is effected.

Even so able a commentator as Ross has failed to appreciate the depth of Aristotle's argument. Very much puzzled by it, he writes, 'I find it difficult to believe that there is any escape along this line. It surely cannot be maintained that a moving particle actualizes a point of space by coming to rest at it. It can come to rest only at a point that is there to be rested at. [Yes, certainly; but does the point exist actually or only potentially? J. A. B.] And when it does not rest but moves continuously, the pre-existence of the points on its course is equally pre-supposed by its passage through them.'

It is imperative to realize that a line is not a collection of points. One has only to note that there exists a strict one-to-one correspondence between the points on a line of one inch and the points on a line of one foot, in order to see that the line of one inch (qua one inch) cannot be essentially characterized as an infinite collection of points. Such a characterization abstracts entirely from the essential metric property of the line. In Aristotle's words, the line of one inch

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1 W. D. Ross, Aristotle's Physics, with introduction and commentary (Oxford, 1936), pp. 74–75.
In the second, 104b 30–33, we read:

Actuality then is the existence of a thing not in the way which we express by 'potentially'. We say that potentially, for instance, a statue of Hermes is in the block of wood and the half-line is in the whole, because it might be separated out.

Aristotle almost seems to be anticipating, and rejecting, a sophism later to be perpetrated in all seriousness.

The sculptor alters the form of a block of marble, not by communicating to it any new qualities, but detaching from it a number of the corpuscles which were included in our conception of the whole. After he has given the last delicate touches that finish the Jupiter, the Venus, or Apollo—the divine form which we admire (as if it had assumed a new existence beneath the artist's hands) is still the same quiescent mass that slumbered for ages in the quarry.

On this view it would be true to say that, despite the fact that no sculptures of Polykleitus have come down to us in the original, at the present time there do exist exact replicas of all his works—hidden within any great block of marble. Aristotle wisely protests that, if these things may be said to exist at all, they may be said, at best, to exist only potentially or per accidens. They certainly do not exist in full actuality. And since 'no sort of shape is present in the solid more than any other . . . if the Hermes is not in the stone, neither is the half of the cube in the cube as something determinate.' So, too, 'the half-line is in the whole' only per accidens.

Are we now entitled to conclude that Aristotle has succeeded in refuting Zeno? I am willing to accept Aristotle's account as proving that no case of continuous passage from A to B is to be understood as an infinite succession of actual sub-intervals. (Any division will certainly be arbitrary.) Alas. Even granting that the 'bisection' paradox has been refuted twice over, first in the formulation in which Zeno himself appears to have couched it and, second, in Aristotle's own, far stronger restatement of the problem, I do not believe that the crux of the difficulty has been adequately identified, much less resolved. There is a third formulation of the paradox that must be considered. This third formulation will be found to escape the brunt of Aristotle's

1 Thomas Brown, Lectures on the Philosophy of the Human Mind, lect. vii, para. 2.

analysis. Put the case of a man (or a god) who walks one mile in one hour non-stop. Let him now walk only 1/2 mile (in 1/2 hour), stopping for a protracted rest of 1/2 hour. Resuming his walk, he now walks 1/4 mile (in 1/4 hour) and again pauses to rest, this time for 1/4 hour. Next, he walks 1/8 mile (in 1/8 hour) and pauses to rest for 1/8 hour, &c. ad infinitum. Following that schedule of intermittent motion, he may be expected to reach the terminus of his journey, having walked (on and off) a distance of one mile, at the end of two hours. Here, then, we do in fact have an infinite succession of actual finite intervals, an infinite series per se, not merely per accidens. This third formulation might have been anticipated from Aristotle's own words: 'If you count the first half of the journey and then the half of what is left and so on, you would have to count an infinite series of numbers before you got to the end of the journey; which is admitted to be impossible.'

For how can an infinite series, i.e. a series without an end, be understood as ever coming to an end?

Part 2: Convergence and the Actual Infinite

Aristotle's failure to solve the 'bisection' paradox requires us to look in other quarters. It is widely, though by no means universally, held today that a definitive solution is at last available through the results of modern mathematics. Russell has been the great popularizer of that persuasion. The names of Weierstrass, Dedekind, and Cantor are repeatedly mentioned; they are said to have broken new ground in our understanding of the infinite. Unfortunately, none of those mathematicians was ever himself so accommodating or so audacious as to advance the claims which others later were to insist upon in their name. According to Cajori, 'we are not aware that any of these three men wrote directly on the paradoxes of Zeno'.

require the most circumspect and delicate handling. These cautionary remarks are not to be taken lightly. They are amply confirmed by the following muddled discussion bearing on our theme.

In his 'Achilles and the Tortoise' Zeno produces an invalid argument depending on ignorance of the theory of infinite convergent numerical series. . . Consider the first half-second as one act of becoming, the next quarter-second as another such act, the next eighth-second as yet another, and so on indefinitely. Zeno then illegitimately assumes this infinite series of acts of becoming can never be exhausted. But there is no need to assume that an infinite series of acts of becoming, with a first act, and each act with an immediate successor is inexhaustible in the process of becoming. Simple arithmetic assures us that the series just indicated will be exhausted in the period of one second. . . Thus this paradox of Zeno is based upon a mathematical fallacy.¹

I am not being captious when I note that, on the one hand, Whitehead is satisfied that 'simple arithmetic assures us that the series . . . will be exhausted in the period of one second', while, on the other hand, he attributes Zeno's fallacy to his 'ignorance of the theory of infinite convergent numerical series'. The two are quite different: 'simple arithmetic' is one thing, the theory of convergent series is something else again. We shall find that this confusion is a great deal more than a mere slip of the pen: it points forward to profound difficulties. 'Simple arithmetic', certainly, is desperately inadequate for the job at hand. Viewing the matter in the most favourable light, let us employ our Zeno procedure so as actually to compute the sum-total of the Z-series. How shall we proceed? In the first

¹ A. N. Whitehead, Process and Reality (Macmillan, 1929), ii, 2, §2. Cf. B. Russell, Our Knowledge of the External World (London, 1914), lecture 6, p. 198. 'The difficulty, like most of the vague difficulties besetting the mathematical infinite, is derived, I think, from the more or less unconscious operation of the idea of counting. If you set to work to count the terms in an infinite collection, you will never have completed your task. Thus, in the case of the runner, if half, three-quarters, seven-eighths, and so on of the course were marked, and the runner was not allowed to pass any of the marks until the umpire said "Now", then Zeno's conclusion would be true in practice, and he would never reach the goal. But it is not essential to the existence of a collection, or even to knowledge and reasoning concerning it, that we should be able to pass its terms in review one by one. This may be seen in the case of finite collections, we can speak of "mankind" or "the human race", though many of the individuals in this collection are not personally known to us.'

1/2 minute we shall compute the partial sum 1/2 + 1/4 = 3/4; in the next 1/4 minute, the partial sum 3/4 + 1/8 = 7/8; in the next 1/8 minute, the partial sum 7/8 + 1/16 = 15/16; &c. At the end of the minute what will we have accomplished? Merely an infinite sequence of partial sums laid out before us. Where will the total sum be found? Nowhere. We may indeed choose to interpret the infinite sequence of partial sums as afford-
is the unique limit of our sequence because, no matter how contracted a neighbourhood we choose to envisage surrounding it, there will always be found terms of the sequence which lie within the neighbourhood. Better: almost all the terms of the sequence will be found lying within any neighbourhood of \( i \) that one might mention, i.e. only a finite number of terms in the sequence can lie outside any neighbourhood of \( i \). Given any small positive rational number \( \epsilon \) that you please, if the neighbourhood surrounding the limit be constituted by an interval \( \epsilon \) in length, then it must always be possible to find more terms of the sequence lying within the neighbourhood than those which lie outside it. We may then say that \( \lim a_n = a \) if and only if

\[
(a - a_n) < \epsilon \text{ for all } n \text{ sufficiently large: } n > N. \]

There must thus always be terms of the sequence \( a_n \) such that the intervals separating those terms from the limit \( a \) shall be less than any positive number \( \epsilon \) that anyone might choose to name. The terms of the sequence \( a_n \) may be said to crowd in upon the limit \( a \) with such thickness that no neighbourhood exists surrounding \( a \) which excludes all or even most of the terms in the sequence.

So much for 'the theory of infinite convergent numerical series'. What is its ontological import? Does it enable us to refute Zeno? No. Zeno never doubted that the successive partial sums of the \( Z \)-series \( 1 + 1/2 + 1/4 + 1/8 + \cdots \) converge toward \( 1 \) as their limit. He merely insisted that, however protracted the sequence of partial sums, the limiting value \( 1 \) will never be actually reached. If anything, Cauchy's account confirms, it certainly does not contest, Zeno's conviction. We have seen that even if we were literally to add together all of the terms in the \( Z \)-series (which is more than any mathematician ever envisaged), even in that case the limit would never be reached. The best we would have is an infinite sequence of partial sums: the total sum would not be forthcoming anywhere in our calculations.

According to Waismann, 'the relation just described' in terms of the epsilon concept 'expresses exactly the same thing' as the expression \( \lim_{n \to \infty} a_n = a \).

However, there is one fact which is very decidedly in its favour: \textit{infinity no longer occurs in it}; on the contrary, it is a system of relations which refer throughout only to finite quantities. From this example there arises an insight of extraordinary import which can be expressed as follows: if the concept 'infinity' occurs in a mathematical statement (in the sense of potential infinite) the same circumstance can also be described by a system of statements which deal only with relations between finite numbers. Consequently, the expression 'infinity' could be entirely banished from the vocabulary of mathematics without thereby sacrificing the minutest part of the content of its laws. Indeed a special attraction may even be found in constructing an infinitesimal calculus in which the concept of infinity is not even applied once, indirectly or directly. 'There is no doubt that such a construction is possible; however, hardly any attempt has been made to do this; rather there are reasons to keep carrying infinity along in the formulas but not in the sense of a primitive concept.

The formula \( \lim_{n \to \infty} a_n = a \) 'proves to be so much more convenient and appropriate than the complicated system of inequalities' resulting from the epsilon notation, viz. \( (a - a_n) < \epsilon (n > N) \) 'that it is advantageous to retain infinity as a façon de parler'.

Waismann is here alluding to the famous letter of Gauss to Schumacher in 1831. Gauss wrote as follows.

As to your proof, I must protest most vehemently against your use of the infinite as something consummated, as this is never permitted in mathematics. The infinite is but a façon de parler; an abridged form for the statement that limits exist which certain ratios may approach as closely as we desire, while other magnitudes may be permitted to grow beyond all bounds... The consequences of Gauss's position—it has come to be known as 'finitism'—can scarcely be exaggerated. It means that a mathematical solution to the 'bisection' paradox is altogether impossible. For we have shown, in our third formulation of the paradox, that Zeno confronts us with a series that is actually infinite. It is in no way reducible to the potential infinite. Infinity figures in the paradox as something very much more substantial than a mere façon de parler. It is not enough that the successive partial sums of the \( Z \)-series converge toward \( 1 \) as their limit. It is absolutely imperative that the total sum of the series equal \( 1 \) in the very same literal sense in which the sum of \( 1/2 + 1/2 = 1 \). The man in our story actually walks the full

distance of one mile. He first travels 1/2 mile, then after resting, he travels 1/4 mile; then after another pause, he travels 1/8 mile, &c. At the end of two hours the sum-total of all the distances he has travelled—literally an infinite series of distances—must unequivocally add up to one full mile. The mathematician who contents himself with a finitist approach to his science is forever debarred from bringing it to bear on the ontological problem that Zeno has raised.

The whole history of mathematics might almost be written around the concept of infinity, the central theme being the various postures adopted toward finitism, both pro and con. Five major phases may be distinguished: (1) the Greeks (2) the seventeenth and eighteenth centuries, with Leibniz and Newton (3) the nineteenth century under the influence of Gauss (4) Cantor and (5) the contemporary crisis in the foundations of mathematics. The first phase, Greek mathematics, is almost exclusively committed to finitism.

The subsequent course of Greek geometry was profoundly affected by the arguments of Zeno on motion. . . . The mathematicians . . . saw that they could only avoid the difficulties connected with them by once and for all banishing the idea of the infinite . . . altogether from their science; thenceforth, therefore, they . . . contented themselves with finite magnitudes that can be made as great or as small as we please.1

Aristotle is thus seen to be the philosopher par excellence of finitism. His failure to solve the ‘bisection’ paradox is nothing less than the failure of finitism itself. Having denied any ontological status to the actual infinite, Aristotle reassures the mathematicians as follows.

Our account does not rob the mathematicians of their science, by disproving the actual existence of the infinite in the direction of increase, in the sense of the untraversable. In point of fact they do not need the infinite and do not use it. They postulate only that the finite straight line may be produced as far as they wish.2

The import of Greek finitism comes through most clearly when we consider its most striking consequence—the Pytha-

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toward one another? What was the ground of their affinity? It may be surmised that the underlying ground was bound up with the following question. Is the universe to be understood as a chaos or as a cosmos, as left to chance or as ruled by reason? It is not accidental that Aristotle's principal argument for the existence of God should rest on the premise that an actual infinite sequence of bodies is unintelligible.

In modern times, by way of contrast, the actual infinite has come to be regarded with suspicion by all or almost all of the hard-headed philosophers, precisely those whom one might suppose to be most akin to the classical materialists. Thus it is Hobbes, an avowed materialist, who denies that we have so much as an idea of the actual infinite, and it is Descartes, the anti-materialist, who insists on its intelligibility. A curious reversal of roles. There has indeed been a strong tendency among modern thinkers of the hard-headed sort to welcome the actual infinite with alacrity whenever the cosmological argument for the existence of God is raised, and to reject it as quite unintelligible in any other connexion, above all in connexion with mathematics. Vanitas vanitatum. With the later pre-Socrates, principally with Zeno and Pythagoras, the actual infinite was thrown very much on the defensive, and with Plato and Aristotle—not least of all in connexion with their attack on the materialists—it was almost pushed to the wall. Nevertheless, Greek materialism in its most sophisticated form, namely the atomism of Democritus, never was willing to relinquish the actual infinite. Thus we find Lucretius arguing at great length and with much vehemence that a finite universe is impossible. What if we were to post ourselves at the supposed limits of the world, he argues, and were then to undertake to hurl a spear beyond? What then? The world as a whole being infinite, he insists that there must be infinitely many atoms distributed throughout.

Seeing that there was so much philosophic sentiment in favour of the actual infinite, why was it that none of the Greek mathematicians, among whom there was doubtless no lack of materialists, ever undertook to subject it to a mathematical treatment? It is not as if there was no philosophic backing for the enterprise. Conversely, why was it that none of the Greek materialists was ever sufficiently interested in mathematics as to project a specifically mathematical physics? The answer to these questions may perhaps be sought along the following lines. If philosophy be the effort to subject the world to a λόγος, then philosophy must divide its attention between world and λόγος. Looking first to the world, as it must, philosophy is predisposed in favour of the actual infinite. Philosophy then looks to the λόγος, and the actual infinite proves to be highly embarrassing. Mathematics being the keeper of the λόγος in its purest form, the watch-dog of rigour, it is innately predisposed against the actual infinite. In so far, then, as Greek philosophy was oriented toward the world, toward natural philosophy, so far was it sympathetic to the actual infinite; and in so far as Greek philosophy was oriented toward the λόγος, toward rigour, dialectic, and the intelligible generally, so far was it hostile to it. Hence it is early Greek philosophy, with its fresh, almost breathless look at the world, that luxuriates in the actual infinite, and it is later Greek philosophy, in its sophisticated preoccupation with logic, that eschews it. This is all very general, of course, and cannot be accepted without much qualification. There are two qualifications, in particular, that must be mentioned. First, we have already seen that the actual infinite persists in classical materialism to the very end, in the teeth of all opposition; and second, it can hardly be said that Aristotle at least (whatever might be said of Plato) is neglectful of natural philosophy, though it must be admitted that the principal charge against Aristotle, ever since Bacon, namely that he corrupted natural philosophy by reading into it the categories of logic, is not without some foundation.

Our account of this tension between world and λόγος vis-à-vis the actual infinite is confirmed, I think, by the subsequent course of modern mathematics and modern philosophy alike.

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1 Cf. Met., 997b 33. 'The earlier thinkers had no tincture of dialectic.' This lack of dialectic need not be understood as any mere crudity. The 'earlier thinkers' may have been rather more like scientists than philosophers, and scientists at all times (even Newton and Einstein) have been regarded by philosophers as philosophically, i.e. dialectically, naive. Although philosophy proper has been identified with dialectic ever since Socrates, the way of dialectic is not the only way: there is also the non-dialectical way of the natural sciences. Aristotle recognizes these two ways in Phys. 204b 3-12. 'We may begin with a dialectical argument and show as follows that there is no such thing [as an infinite body]. . . . If, on the other hand, we investigate the question more in accordance with principles appropriate to physics, we are led as follows to the same result. . . .'
The actual infinite enters mathematics thematically for the first time in the seventeenth century, with Leibniz and Newton, under the auspices of applied, rather than pure, mathematics, that is to say, through the sponsorship of mathematical physics and the return to natural philosophy; and it is later expelled from mathematics, by Gauss and Cauchy, not on any ontological grounds, not through any specific concern with nature and the world, but expressly in the name of rigour. It is true that Cantor undertakes to build a specifically mathematical science of the actual infinite, in large measure independent of natural philosophy and ontology in general, but in our own day finitism is very much in the ascendancy, again in the name of rigour.

In regard to modern philosophy, Hobbes's rejection of the actual infinite is prophetic of what is to follow, for it is not on ontological or cosmological grounds that he bases his rejection, but on specifically epistemological considerations, that is to say in behalf of the λόγος, and with Locke and Kant, the Hobbesian rejection of the actual infinite in epistemological terms comes to dominate modern philosophy. It is against this background that my own programme is to be viewed, for I shall undertake to vindicate the actual infinite in the only way that I believe it can be vindicated, by grounding it in a sufficiently rich ontology. It is precisely the absence of such ontological grounding that has rendered the Cantorian programme, which I shall not only defend but also attempt to enlarge, so highly vulnerable to the attacks of the finitists. In the course of executing our design, I shall be driven to provide an ontological account of both pure and applied mathematics and of natural science as well. If the past history of the problem is any guide, we may expect to be continually bedevilled, as we proceed, by the demands of rigour, on the one hand, and the demands of the world, on the other.

Part 3: Leibniz and Divergent Series

We have seen that it is only in the second phase, in the full flush of the rampant, uninhibited development of modern analysis, that the actual infinite comes to sight as the ontological underpinning proposed for the new mathematics. If Aristotle is the philosopher of finitism, then with some exaggeration Leibniz may be styled as the philosopher of the actual infinite. Unfortunately, there is to be found in the writings of Leibniz no systematic account of his position in any way comparable to the detailed discussions of Aristotle. He has left us only a collection of fragmentary remarks which have been found to be notoriously difficult to piece together into a single coherent system. In his New Essays, II, 17, Leibniz insists that 'an infinity of things exists, i.e. there are always more of them than can be assigned.' In a letter to Foucher in 1693, he writes: 'Je suis tellement pour l'infinit actuel, qu'ai lieu d'admettre que la nature l'abhorre, comme l'on dit, vulgairement, je tiens qu'elle l'affecte partout' (italics his). He continues as follows:

So I believe that every part of matter is, I do not say divisible, but actually divided, and consequently the smallest particle should be considered as a world full of an infinity of creatures.

Leibniz does not mean, of course, that there literally exists a 'smallest particle'. Thus in his letter to Bernoulli (1698) he writes:

Suppose all the sub-divisions of a line, 1/2, 1/4, 1/8, 1/16, 1/32, &c., actually existed. To infer from this series that an infinitesimal term absolutely exists would be an error, for I think nothing more follows from it than that there exists an assignable finite fraction as small as you please. Hence, I conceive points, not as elements of a line, but as limits or termini of a line, bounding further elements.

Leibniz is thus seen to retain the classical position in regard to the mathematical point even while allowing the possibility at least of an actually infinite sequence of line-segments.

In a more extravagant vein Leibniz does not hesitate to say that a parabola is an ellipse—a very odd kind ellipse, to be sure, one focus being at infinity! In exhibiting the parabola as but a special case of the ellipse, Leibniz appeals to the law of continuity which he styles the 'Lydian stone' of metaphysics. It is by means of an actual passage to infinity that figures as specifically diverse as the parabola and the ellipse are found to slide into one another through the law of continuity. In his letter to Varignon (1702) he writes:

1 Cf. however, Monadology, §2, 'There must be simple substances because here are composites'.
INFINITY

In things existing simultaneously there may be continuity even though the imagination perceives only breaks; because many things appear to our eyes to be completely dissimilar and disunited which nevertheless turn out to be perfectly similar and united internally if we could get to know them distinctly. If we consider just the external shape of parabolas, ellipses, and hyperbolas, we should be tempted to believe that there was an immense gap between any two of these kinds of curves. However, we know that they are intimately connected so that it is impossible to insert between two of them some other intermediate kind which may enable us to go from one to the other by more imperceptible nuances.

The import of this discussion is very much qualified if it is taken in conjunction with an earlier discussion, in Leibniz’s letter to Foucher of 1692.

Expressions like ‘Extremes meet’ go a little too far, e.g. when we say that the infinite is a sphere whose center is everywhere and circumference nowhere. Such expressions must not be used for finding the truth.

Hence the continuity. The import of this discussion is very much qualified if it is taken in conjunction with an earlier discussion, in Leibniz’s letter to Foucher of 1692.

Throughout the early development of the calculus a persistent ambiguity may be detected in the use of the infinite, sometimes the one, sometimes the other, approach being indicated, most often, perhaps, no clear distinction being drawn between the two. In the differential calculus the tangent will be seen to be a special, marginal kind of secant (we should prefer to style it as a degenerate case of the secant), and in the integral calculus the circle will be regarded as a regular polygon—with an infinite number of sides each of which is greater than 0 in length but less than any positive rational quantity—an actual infinitesimal. The failure of finitism to meet Zeno’s challenge encourages us to accept Leibniz’s position in the most sympathetic, indeed in the most ontological, light. Let us take a piece of string. In the first 1/2 minute we shall form an equilateral triangle with the string; in the next 1/4 minute we shall employ the string to form a square; in the next 1/8 minute we shall employ the string to form a regular polygon; &c. ad infinitum. At the end of the minute what figure or shape will our piece of string be found to have assumed? Surely it can only be a circle. And yet how intelligible is that process? Each and every one of the polygons in our infinite series contains only a finite number of sides. There is thus a serious conceptual gap separating the circle, as the limiting case, from each and every polygon in the infinite series. How are we to negotiate the conceptual leap required to pass over from the whole infinite series to the circle as the terminus ad quem? How indeed? Si! Fiat! Let us at one great bound make the leap forward to the limit: the devil can take the hindmost. Alas. As soon as we employ the actual infinite in an effort to illuminate divergent numerical series, we find ourselves shaken by further difficulties.

It is perhaps not too much of an exaggeration to say that it was precisely the infinite divergent series, with its peculiar problems, that proved to be the rock upon which the Leibnizian venture finally shattered itself. Hence the return to finitism and the Greeks, with Gauss and Cauchy. We are now to examine link between the first sequence and the second. Leibniz may thus be interpreted as undertaking to combine in one coherent system two very different approaches to the infinite—the ontological approach and the als ob approach. It is precisely this double use of the infinite that I shall be found to advocate in the sequel.
the divergent series in the light of our own ontological preoccupations. What is the sum of the infinite series \( 1 - 1 + 1 - 1 + 1 - \ldots \)? If we take the successive partial sums of the series, we generate the sequence \( 1 - 1 = 0, 0 + 1 = 1, 1 - 1 = 0, 0 + 1 = 1, \&c. \) This sequence of partial sums clearly fails to converge to any limiting value, in terms of the epsilon concept. How then can we compute the sum-total of the series? In 1703 Guido Grandi argued that the sum-total could only be 1/2. He argued as follows. If \( x = -1 \), then the geometric series \( 1 + x + x^2 + x^3 + \ldots \) yields the infinite series \( 1 - 1 + 1 - 1 + 1 - \ldots \). Now the sum of the geometric series \( 1 + x + x^2 + x^3 + x^4 \ldots \) is known to be \( 1 - x \); so that if \( x = -1 \), the sum must be 1/2.

The publication of Grandi prompted a discussion between Leibniz and Wolff, Grandi and Varignon. Wolff asked Leibniz what he thought of the remarkable things which appear in Grandi’s book, and Leibniz offered an opinion in a letter published in 1713 in the Acta Eruditorum. . . . Relying essentially on the authority of Leibniz, Euler later thought that every infinite series must have a definite sum; Goldbach, Daniel Bernoulli and other important mathematicians living in Euler’s time also shared his view. What’s more, Euler explicitly took (in a letter to Goldbach 1745) the value of an infinite series to be equal to the value of the analytic function whose expansion gives rise to the series. The deceptiveness of this conviction is evidenced by the fact that there are distinct functions whose expansion produce the same series. For example,

\[
\frac{1 + x}{1 + x + x^2} = 1 - x^2 + x^3 - x^5 + x^9 - x^{2} + \ldots
\]

is a series which again leads to Grandi’s series for \( x = 1 \); we would therefore be able to ‘prove’ that \( 1 - 1 + 1 - 1 + \ldots = 2/3 \), just as well as 1/2.  

In the circumstances we cannot fail to share the sense of utter dismay and outrage which Abel expresses, in 1828, in a letter to his former teacher Holmboe.

The divergent series are the invention of the devil, and it is a shame to base on them any demonstration whatsoever. By using them, one may draw any conclusion he pleases and that is why these series have produced so many fallacies and so many paradoxes. . . . I have become prodigiously attentive to all this, for with the exception of the geometrical series, there does not exist in all of mathematics a single infinite series the sum of which has been determined rigidly. In other words, the things which are the most important in mathematics are also those which have the least foundation.

What is the ontological import of all this? It may prove to be the proverbial cloud no bigger than a man’s hand. Divergent series present us not merely with a mathematical but also with an ontological crisis. Let us consult an ontological model of Grandi’s series. We have a model ready at hand in our Zeno procedure. In the first 1/2 minute a man takes one complete step forward; in the next 1/4 minute, he takes one back (returning to his original position); in the next 1/8 minute, he again takes one step forward; in the next 1/16 minute he again withdraws one step back, &c. When the gong sounds at the end of the minute, where is the man standing? Another example. In the first 1/2 minute a man places an apple on an otherwise empty table; in the next 1/4 minute, he removes the apple from the table; in the next 1/8 minute he restores the apple to the table, &c. At the end of the minute what is the situation obtaining on the surface of the table? Is it empty? Is the apple to be found there? Or, perhaps, Grandi is right: on top of the table is to be seen but one half of an apple! How absurd. And yet what better solution have we to offer? It is evident that we are reduced to silence. Almost any summation of Grandi’s series would seem to be as reasonable, and as arbitrary, as any other. I say: almost. Let us posit that we have actually performed our ideal experiment with the apple. Surely it would be too bizarre to find, at the end of the minute, 127 apples lying on the table. Or even 1/2 or 2/3 of an apple. We have every right to expect there will be found either one apple or none. But which? No answer is forthcoming.

One is tempted to argue here that the question falls outside the domain of pure mathematics and the a priori. The question can only be answered empirically or, rather, hyper-empirically. Let us then perform the experiment (per impossibile), and let the chips fall as they may. Unfortunately, if the question is to be decided on purely empirical grounds, any answer must be acceptable. Viewing the matter empirically, we might just as well be prepared to find 127 apples on the table as one or none. Are we then to say that, since we know empirically that motion

does occur, the Z-series \(1/2 + 1/4 + 1/8 \ldots\) must add up to 1, not on any \textit{a priori} conceptual grounds, but simply as a matter of brute empirical fact? This is surely too barbaric. We do know that the sum-total of the Z-series cannot be less than 1 (waving actual infinitesimals), and also we know that the sum cannot be greater than 1. Hence, certainly, we may conclude that, if the series may be said to have any total sum at all, it can only be 1. But what of the conditional if-clause? Perhaps it is at this point, and only at this point, that we find ourselves obliged to make a direct appeal to empirical fact. In the case of Grandi’s series, perhaps we may rule out \textit{a priori} all values except 0 and 1 as the sum total of the series, contenting ourselves with a certain limited indeterminacy, allowing either 0 or 1 to be the sum-total, depending on the empirical facts or hyper-facts. This is all rather shabby, I confess, and scarcely calculated to relieve Zeno’s anxieties. What is so very alarming is that both Aristotle and Leibniz have failed to meet the challenge, the one confining himself to the potential, the other embracing the actual, infinite.

I have suggested that the sum-total of Grandi’s series may perhaps be said to be optional, either 0 or 1, depending on the hyper-facts. This suggestion leads to the following curious consequence. Let there be two men each of whom separately, but simultaneously, performs the apple experiment, on separate tables. They begin and end at the same time. It follows then that, at each successive phase of the experiment, \textit{pari passu}, the very same situation will be found obtaining on the one table as on the other: either the apple is seen to be lying on each table or it is seen to be absent from it. A motion picture of the one experiment will be found to be identical, phase by phase, slide by slide, with a motion picture of the other. Yet, at the end of the minute, we are prepared to allow that the one table may be seen to be quite empty while the other may have an apple lying upon it. Is that intelligible?

What happens here is what happens so often in philosophy. Some will answer the question with a violent negative, others with an insouciant affirmative, depending on how they have answered other, apparently quite remote, questions in philosophy. I should suppose that the radical empiricist, taking the question in his stride, will have no qualms in answering it in the affirmative. Most will doubtless share my own sentiments: they will not know what to say. It is difficult to rid oneself of the distressing suspicion that neither 0 or 1 (and hence no value whatever) can be accepted as the sum-total, or even as a sum-total, of the series: if 0, then the series would seem to contain an even number of terms; if 1, an odd number; both of which are surely preposterous. We are now in a position to propose a fourth formulation of the ‘bisection’ paradox. I can imagine Zeno arguing as follows. The infinite convergent series \(1/2 + 1/4 + 1/8, \ldots\) can add up to 1 only on the condition that the infinite divergent series \(1 - 1 + 1 - 1 + 1, \ldots\) adds up to—something, anything! An ontological model is as readily available for the second series as for the first. But it is clear that there is no sum-total for Grandi’s series. Hence there can be none for the Z-series. Motion is thus unintelligible. What shall we say to this? Must we first supply an ontologically intelligible theory of infinite series taken in general, a theory comprehending divergent as well as convergent series, before we are entitled to affirm that the ontological sum-total of the Z-series must be 1? I should be very reluctant to yield to this pressure.

What we have styled the third phase in the history of mathematics (\textit{sub specie infiniti}), the so-called \textit{rigorous} reformulation of the calculus in the nineteenth century, is only to have been expected. The mathematician will always seek to seal off his science from the seditious rumblings of ontology. He will not willingly play the double role of philosopher and mathematician at once. Leibniz is the great exception, launching the calculus expressly as a metaphysical mathematics. We have seen that his employment of the actual infinite cannot be said to have proved altogether felicitous. With the return to finitism, the infinite series is no longer viewed as \textit{actually} infinite. The Z-series now represents no more than a rule for generating an indefinite sequence of partial sums, protracted as far as we wish, but never actually infinite. When we say that the sum-total of the series is equal to 1, we shall mean no more than that the limit of the sequence \(3/4, 7/8, 15/16, \ldots\) is 1. And when we say that the limit of this sequence is 1, we shall mean but one thing—that however contracted a neighbourhood one may choose to envisage surrounding the limiting value 1, we can always generate a \textit{finite} number of terms which lie within the neighbourhood,
a finite number which is greater than any class of terms in the sequence that may be shown to lie outside the neighbourhood. *That and nothing else* is all that we shall mean when we say that the sum-total of the Z-series is equal to 1. If we have gained in rigour, we have lost in ontological import. It is now only by a radical equivocation that we can say that there exists a sum-total both for the finite series \( \frac{1}{2} + \frac{1}{2} \) and for the infinite series \( \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \). The expression 'sum-total' means something very different in the two cases. The failure to note that radical equivocation has misled many into supposing that the modern mathematician has succeeded in refuting Zeno. I have insisted that, if Zeno is to be refuted, it is absolutely imperative that the expression 'sum-total' be employed in one unambiguous univocal sense in the two statements: (1) the sum-total of the finite series \( \frac{1}{2} + \frac{1}{2} = 1 \), and (2) the sum-total of the infinite, indeed \( \text{actually infinite} \), series \( \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1 \). A fifth formulation of the 'bisection' paradox is thus suggested. If the paradox is to be dissolved, then we must preserve a strict univocality of meaning in applying the concept of a sum-total to both the finite and the infinite series. But such a strict univocality is clearly impossible on the basis of finitism. Therefore the force of the paradox remains.

The mathematician succeeds in purchasing rigour in his theory of infinite series only at the price of forfeiting the ontological issue. Therein lies the beauty of the 'bisection' paradox: it points up in a peculiarly trenchant form the conceptual gap between mathematics and reality. Gauss, it will be recalled, did not rule out the actual infinite *tout court*. He insisted merely that it is 'never permitted in mathematics', implying that it may possibly obtain *in rerum natura*. Precisely because Leibniz and Newton were concerned with the nature of things, and not merely with the luxury of a pure mathematics, they were obliged to face up to the hard facts of mechanics. As Cajori remarks, 'Variables arising in mechanics are usually of such a nature that they do reach their limits'.\(^1\) It is not surprising, then, that Newton should write:

Quantities, and the ratios of quantities, which in any finite time converge continually to equality, and before the end of that time


Variables in mechanics not only *converge* toward their limits, in many cases they actually reach them. The 'rigorous' formulation of the calculus in finitist terms is seen to become possible only on the basis of a radical abstraction from those very ontological facts—above all, the facts of motion—which the calculus was originally intended to mathematize. To insist that no rigorous formulation of the calculus is intelligible except on strictly finitist terms, is equivalent to an admission that Zeno was right after all.

With the rise of the non-Euclidean geometries and the shift toward the postulational approach, pure mathematics may be thought to have grown so pure, so purified, so elevated, as to be quite divorced from all ontological reference whatever. Russell's most recent word on the subject is as follows:

I have come to believe, though very reluctantly, that [mathematics] consists of tautologies. I fear that, to a mind of sufficient intellectual power, the whole of mathematics would appear trivial, as trivial as the statement that a four-footed animal is an animal. I think that the timelessness of mathematics has none of the sublimity that it once seemed to me to have, but consists merely in the fact that the pure mathematician is not talking about time.\(^2\)

Zeno, on the other hand, is definitely talking about time. If we are to meet his challenge, we must force the pure mathematician to descend into the flux of time and motion. The current approach as expressed by Russell, altogether snapping the connexion between mathematics and reality, is anticipated in Cantor's dictum: 'The essence of mathematics lies in its freedom'. The classical approach is very much the opposite: the essence of mathematics lies in its necessity.

**Part 4: Cantor and Topological Analysis**

The name of Cantor ushers in the fourth phase in the development of mathematics: the return to the actual infinite.

\(^1\) *Principia*, book I, section I, last scholium.

Launching his theory of the transfinite in 1883 in an essay entitled ‘On Linear Aggregates’, Cantor writes as follows:

It is traditional to regard the infinite as the indefinitely growing or in the closely related form of a convergent sequence, which it acquired during the seventeenth century. As against this I conceive the infinite in the definite form of something consummated, something capable not only of mathematical formulation, but of definition by number. This conception of the infinite is opposed to traditions which have grown dear to me, and it is much against my own will that I have been forced to accept this view. But many years of scientific speculation and trial point to these conclusions as to a logical necessity, and for this reason I am confident that no valid objections could be raised which I would not be in a position to meet.

Cantor's words are very puzzling. If the essence of mathematics lies in its freedom, how is it possible that Cantor was forced against his will as by a logical necessity to effect a radical break with traditions so much cherished by him? Our bafflement can only increase when we read a further discussion by him three years later.

In spite of the essential difference between the conceptions of the potential and the actual infinite, the former signifying a variable finite magnitude increasing beyond all finite limits, while the latter is a fixed, constant quantity lying beyond all finite magnitudes, it happens only too often that the one is mistaken for the other. Owing to a justifiable aversion to such illegitimate actual infinities and the influence of the modern epicuric-materialistic tendency, a certain horror infiniti has grown up in extended scientific circles, which finds its classic expression and support in a letter of Gauss, yet it seems to me that the consequent uncritical rejection of the legitimate actual infinite is no lesser violation of the nature of things, which must be taken as they are.

First, it is a 'logical necessity', now it is an appeal to the 'nature of things', no less, which is to supply the underpinning for the theory of the transfinite. The issue is expressly said to be an ontological one. Finitism is seen as a materialistic doctrine. Was it in an effort to rebut Cantor's charge of materialism that his fierce opponent, the intransigent finitist Kronecker, invoked the Deity in his famous aphorism? 'God created the integers, the rest is the work of man.'

Here we may imagine Cantor mischievously querying Kronecker, 'How many integers did God create? Surely God must have created some definite number of them, namely all that there are. But what is that number?' Cantor denominates it aleph-null. (Did God create aleph-null?) A wildly radical and altogether unprecedented step, that, to regard the class of integers as being of a definite number, a number infinitely larger than each and every integer! The whole question of mathematical existence is thus raised afresh. Can the cardinal number aleph-null really be said to exist? Finitists from Aristotle to Gauss to Brouwer, in our own day, are moved to convict Cantor of illicitly hypostatizing the objects of mathematics. They argue as follows. When Euclid proves that there exist only five regular solids, what does he mean by 'exist'? Surely he means merely that it is possible for only five regular solids to exist in concreto. Mathematical existence is ontological possibility. When we say that there exists an immediate successor for every natural number, what is the ontological import of that assertion? This: given any class of $n$ apples (say), $n$ being any natural number that one might choose to name, it is always possible that another class of $n+1$ apples might exist. What passes for actual existence in mathematical discourse is merely possible existence in ontological discourse. This should not be surprising. The actual objects of mathematical investigation are seen to be pure potentials in ontology. Mathematical discourse promotes and elevates the potential to the status of the actual: hence the high dignity and elevated character of such discourse. We are dazzled when we learn that there exist infinitely many points in any line. Aristotle performs an ontological reduction of that elevated truth, by noting that it means no more than what everyone knows to be an obvious fact, namely that it is always possible (in principle) to divide and sub-divide any line indefinitely. But if it belongs inherently to the very grammar of mathematical discourse that it hypostatize the merely potential, wherein is Cantor to be found egregiously guilty? The finitist will argue that it makes sense to speak of five apples or a million apples but that it makes no sense to speak of aleph-null apples. He will insist that it is not even possible for aleph-null apples to exist. Since any definite number of apples might exist and since it is unintelligible to suppose that aleph-null apples might exist,
it follows at once that aleph-null cannot be a definite number. The definite is always finite.

One may be tempted to reply as follows. Yes, we grant that it is meaningless to speak of aleph-null apples but surely the class of all natural numbers includes certain identifiable elements (e.g. 7 and 186) and excludes other identifiable elements (e.g. apples and negative numbers). May we not then ask, how many elements are contained in the class of natural numbers? The answer can only be aleph-null. And since the answer to the question 'how many?' is always a cardinal number, aleph-null must be a cardinal number. If the argument is conducted along these lines, I must confess that it is the finitist who wins my vote. It is admitted here that, of any actual collection of apples, it is only a natural number that may be predicated; aleph-null is proposed as a predicate applicable, not to any collection of actual things, but rather to mere entia rationis. To say that both 5 and aleph-null are cardinals is equivalent to saying that among the things that exist to which cardinals may be applied are apples and numbers. It is quite objectionable to say that numbers exist. (Certainly there exists a number greater than 5 and less than 7.) What cannot be allowed, what can only prove systematically misleading, is the statement that both apples and numbers exist. Here we are guilty of a category-mistake. All kinds of muddles results from such logical promiscuities. When we say that aleph-null is a cardinal, we mean here no more, ontologically speaking, than that it is possible for any number of apples to exist (‘number’ being here understood as a natural number). That is, we are saying that it is possible that any one of an infinite range of possibilities might actually obtain in the real world: a second-order possibility. When we say that 5 is a cardinal, we are referring to a first-order possibility: it is possible that five apples might exist. To say, then, that both 5 and aleph-null are cardinal numbers, is to be guilty of a confusion of logical, or ontological, types.

I believe that the finitist has had the best of the foregoing argument, but I also believe that it is Cantor who is right and the finitist wrong! I am prepared to agree with the finitist rejection of aleph-null as a cardinal only on the assumption that it is absurd to speak of aleph-null apples. But that assumption is a mistake. The finitist is obliged to say that he knows a priori,

apart from any empirical or scientific evidence, that there exist only a finite number of stars in the heavens. If it is unintelligible and meaningless to speak of an infinite number of stars, then it follows that we know a priori that a spaceship which is launched to explore the heavens must be at some precise finite time in the future (travelling at a uniform velocity) eventually encounter each and every star that now exists. Finitism is thus seen to be a form of apriorism at its worst. It is fatuous to suppose that we know a priori that the stars in the heavens cannot possibly go on and on forever but that at some point in space they must come to an end. The great argument in favour of aleph-null as a cardinal is not that the class of natural numbers is infinite. No. Rather, the decisive argument is that it is quite intelligible and possible that there is no last star in the heavens. The two arguments are very different. We are now entitled, indeed, to say that there exist aleph-null integers, but we do not mean merely that every natural number has a successor. Instead, we mean that it is possible for an infinite number of stars, or apples, to exist. (Does it make any sense to suppose that the number of stars might be \( \aleph_1 \) or \( \aleph_2 \)?)

I want to press this point home in the strongest fashion. I want to say that aleph-null is a cardinal in precisely the same univocal sense in which 5 or a billion are cardinals. This is by no means obvious, even granting the actual infinite as ontologically intelligible. Are we not extending the concept of number when we apply it, not only to finite, but also to infinite collections? How can that be denied? Originally, the concept of cardinal number is applied only to the natural numbers; later the concept is stretched, perhaps justifiably, to cover the actual infinite. The concept of a cardinal surely cannot be said to retain intact its original univocity of meaning. Yet that is precisely, if implausibly, what I wish to maintain. I do not deny that the concept of number is being stretched when it is extended beyond the range of the natural numbers so as to include the rational, negative, real, and imaginary ‘numbers’. Indeed, I should insist that in the pristine sense of ‘number’ zero itself is not a number. I wish to maintain that, whereas aleph-null is a number in the original primitive sense, zero is not. It is thus seen that my standards of univocity are very exacting in
my effort to maintain that aleph-null is a cardinal in the strictest of senses.

Let me first explain my grounds for holding that zero is not a number. A group of men are each given pencil and paper. They are instructed each to write down a certain number of words, any number of words that they please. One of the men writes down nothing at all; but he argues that he has complied with the instructions. He insists that he has written down a certain number of words, namely 0 words. Zero is a number, is it not? In the present context, certainly, zero is a number only by way of a joke. How can a man who has not so much as put pencil to paper be said to have written down any number of words at all? The concept of number in its primitive sense, as it is found in our mother-tongue, as it is employed in common discourse, definitely rules out zero as a number in most, though not all, contexts. I am reminded of the controversy that broke out on this very topic between Husserl and Frege. Frege insisting that zero is a number on the ground that number is the answer to the question ‘how many?’, zero being one admissible answer to that question, Husserl retorting that zero is only a negative answer to the question and hence not a number at all, number being always an affirmative answer to the question. My own sympathies lie with Husserl. If common discourse is to supply our standard, then Frege is surely in error. It must be confessed, however, that common discourse is not altogether unequivocal on this point. Let a census be conducted among all adult males in the community, the object being to record the exact number of children which each has fathered. Will we exclude the names of all childless adult males from the final tabulation? It is not clear what precisely that number is. It may be 92 or 3,023 or 211 or none or 6 or 44,381 or ... , indeed any number. I do not know of anyone who wishes to say that none is a number; though almost everyone, these days, will be prepared to style zero as a number without any hesitation. How very odd. The Greeks certainly had a word for 'zero'—indeed
simply on the basis of a crucial; but the answer to the question is not forthcoming in certain, is introduced into formal mathematics as a number if the disguise were of the very essence. It will be recalled that Russell regards mathematical truths as mere tautologies. Mathematical discourse succeeds in being at best a quasi-number, why does the mathematician enrol it as a number in his science? What is his purpose in stretching our natural concept of number so as to include such an exotic element as none? What are the mathematician's grounds for tautologically accrediting zero as a number? The question is crucial; but the answer to the question is not forthcoming in mathematics proper. The formal, axiomatic, systematic, rigorous, scientific, official exposition of mathematics always omits the all-important question of purpose. Otherwise, the exposition would not be 'scientific' and 'rigorous'. Notice the double strategy employed today to justify the negative and imaginary numbers. Officially, and formally, they are defined into existence by means of a 'construction' out of number-couples. Unofficially, and informally, we are instructed in an 'aside' as to the underlying purpose of the enterprise, namely that closure might be effected.

I am not for one moment doubting that the mathematician has good reasons for counting zero as a number. But where are those good reasons expressly to be found? Being absent from the official exposition of his science, they must be reconstructed from his actual performance and from the incidental patter that accompanies it. In seeking those reasons, in recovering the purpose, the τέλος that animates his science, we are investigating that deeper λόγος which underlies and underpins the 'scientific' λόγος which alone is admitted by the mathematician into the full light. Hence Wittgenstein's profound heuristic: 'Pay attention to the patter by means of which we convince someone of the truth of a mathematical proposition.' In our case the proposition is that zero is a number. 'It tells us something about the function of this conviction. I mean the patter by which intuition is awakened. By which, that is, the machine of a calculating technique is set in motion.'¹ The foundations of mathematics are, in large measure, to be sought in that very patter which the mathematician employs en passant to lure us into what he is liable to regard (mistakenly) as the only serious business before him—the formal axiomatics.

The philosopher will reverse the mathematician's emphasis. For the mathematician, the system is primary, the patter secondary. For the philosopher, the patter is primary, the system secondary—always bearing in mind that 'nothing is more likely than that the verbal expression of the result of a mathematical proof is calculated to delude us with a myth'.² It is the patter which proves to be the connecting link between the informal proto-concept of common discourse, 'at ease' and in multiplicity, and the formal scientific super-concept brought to a

posture of rigid attention. An example of such patter is seen in Frege’s argument designed to prove that zero is a number. Number is the answer to the question 'how many?', is it not? Yes. Is not 'none' one answer to the question 'how many?' Yes. Therefore does it not follow that none is a number? This is, of course, a pseudo-proof. Zero is not proved to be a number at all. Frege's patters has only the appearance of a proof. The exotic word 'zero' is available to conjure with: it enables us to swallow the sophism more readily. If we replace 'zero' by 'none', the sophism is liable to stick in our throats. Mathematical patter is thus largely sophistical. Under the guise of an informal proof, we are being rhetorically redirected by means of a persuasive definition to recast our concept of number into a new form. What is the precise purpose of that rhetoric? What is the goal that is being aimed at? No formal account of mathematics can answer that question. It can be answered only by a teleological account of concept-formation.

Zero is a number in the way that a straight angle is an angle. No one would adduce a straight line as a paradigm case or object lesson or exemplar of what an angle is. An angle is generated by the intersection of two straight lines. Such is the concept in its primary sense. A straight line may be seen as a limiting case toward which a whole sequence of ever-widening angles finally converge; but it is not itself an angle at all. How can one straight line be any kind of angle? Yet the mathematician will feel reluctant to say that the sum of the angles of a triangle is equal to one straight line. Angles and lines are entities of very different orders. How can any number of angles add up to a line? No. It is then not merely useful or convenient, it is also illuminating, to execute an attractive concept-leap whereby the straight line may be styled, through a façon de parler, as an angle for the nonce. The strategy here may smack more of poetry than of science, more of metaphor than of fact, but it is actually both at once compounded together. For it cannot be denied that some kind of fact is being expressed when it is said that the sum of the angles of a triangle is equal to one straight angle.

I am suggesting that the same conceptual dynamics that prompt the mathematician to accredit the straight angle as an angle also moves him to accredit zero or none as a number. In both an evident distinction is operative between the standard, exemplary, paradigm case and the marginal, limiting, boundary case. In both a concept-leap is executed whereby the boundary case, hitherto excluded from the range of instances to which the proto-concept is applicable, is seductively brought under the scope of the super-concept. If this suggestion is correct, it will not be surprising that the Greeks refused to recognize the straight angle. Euclid insists always that the sum of the angles of a triangle is equal to two right angles. These same conceptual dynamics may be found operative throughout modern mathematics at various points. Thus in the differential calculus the tangent may be seen as a kind of secant and, in the integral calculus, a circle may be seen as a kind of polygon. Of course, the tangent is not really a secant, the circle is not really a polygon; but then, in the same way, none is not really a number and a single straight line cannot really determine an angle.

One last example may be noted with particular relish. According to the principle of duality in projective geometry, if the words 'point' and 'line' are interchanged in any theorem, the resulting statement will also be a theorem. The principle of duality can only be established on the basis of a concept-leap. We must extend the concept of what it is for two straight lines to intersect. Parallel lines will now be said to intersect after all—at infinity. This odd locution is strictly on a par with the introduction of zero as a number. Question: What is the exact number of men on the moon? Answer: I believe that there is no number of men on the moon at all. Henceforth 'no number' will count as a number. So, too, we may ask: Where do parallel lines intersect? Answer: parallel lines intersect nowhere. Henceforth 'nowhere' will count as the place where parallel lines intersect.

This last example from projective geometry is especially illuminating. Infinity plays a very different role here from what it does in the theory of convergent series. When we say that the sum-total of the infinite Z-series is equal to 1, we are employing the concept of infinity (even if we are finitists) in a way radically different from the way in which we invoke it in projective geometry. There is scarcely any resemblance between the two. This comes through most clearly if we adopt a non-finitist approach. In the one case we shall insist that the summation of the Z-series must ontologically prove equal to 1 if an actual
infinite sequence of finite sub-intervals is in fact exhausted. In
the other case, it is not in any way supposed that parallel lines
will in fact intersect even if they are extended for an actual
infinite distance. It is certainly true that if we project a three­
dimensional scene on to a flat surface (say by means of a photo­
graph), then railroad tracks which are in fact parallel will seem
to converge in the picture. Indeed, the black lines in the picture
which represent the railroad tracks will truly converge and even
meet at some definite point. But even here, though the lines will
in fact meet, it cannot be said that they will in fact meet at
infinity. They meet at some measurably finite distance. There
are thus two different contexts that are jointly exploited in pro­
jective geometry: the appearance of things and the reality of
things. Under the aspect of appearance, parallel lines do meet
but only at a finite distance. Under the aspect of reality, they
are supposed never to meet even if extended an actual infinite
distance.

I would suggest that it is the 'intersection' (so to speak) of
those two contexts in projective geometry that prompts the
mythical statement that parallel lines meet at infinity. We may
then say that whereas infinity plays an ontological role in the
theory of convergent series, it plays no more than a meta­
phorical role in projective geometry. Mathematics is thus seen
to comprise a 'motley of techniques', and the radical diversity
found among those techniques is liable to be obscured by the
formal, axiomatic approach which, in reducing all statements to
a flat equality as tautologies, leads to the 'mechanization of
mathematics'.

I trust that I may be forgiven this fairly protracted excursion
into the philosophy of mathematics. I have not lost sight of our
principal theme—the metaphysics of infinity. But the investiga­
tion of that theme requires that we consult the mathematics of
infinity which, in its turn, if it is to be mined for its ontological
import, requires that the peculiar grammar of mathematical
discourse be subjected to the 'higher criticism'. Mathematical
discourse is systematically misleading: it may not be taken at
its face value. Words are no longer employed in their literal
senses. They may be said to be employed tropologically. I have

question makes great good sense in their language: 'Is the number of walnuts in this pile greater than the number of peanuts in that pile?' The word 'number' certainly occurs in their language. It is merely that they have no words for the particular numbers 1, 2, 3, 4, 5, &c. Having no system of numeration or counting, they will not be able to answer the question as to the walnuts and the peanuts as we should be likely to do, first by counting the number of walnuts, then by counting the number of peanuts and, finally, by noting which of the two numbers is the greater. Their method will be rather different (though essentially the same). They will attempt to place the walnuts and peanuts in one-to-one correspondence with each other. Should they succeed in that attempt, they will say that the number of walnuts is equal to the number of peanuts. Failing that, they will say (as the case may be) that the number of walnuts is greater or less than the number of peanuts. The number of walnuts will be said to be greater than the number of peanuts if a proper subset of the walnuts can be placed in one-to-one correspondence with all the peanuts. They are capable of going even further. They can say, and verify, that the number of walnuts in this pile added to the number of peanuts in that pile is less than the number of leaves hanging from this tree—all without any system of counting. Although lacking the number-words 1, 2, 3, 4, 5, &c., they have a rich vocabulary which includes the words number, equal, greater, less, addition, subtraction, and we could even have them advance on to multiplication and division as well.

Now I want to imagine a fantastic festival held among the Kumquats. Explorers from the outside world have entered their land. The aborigines are fascinated by the weapons of these explorers and, even more, by the microscopes which they carry, capable of revealing things which are invisible to the naked eye. Enchanted by the microscopes, the Kumquats decide to hold a contest. Each contestant is instructed to write down on a sheet of paper as many X's as he can. The one who writes down the greatest number of X's wins the prize. The X's need not all be visible to the naked eye: microscopes are allowed. Unbeknownst to these people, a god descends among them and enters the contest incognito. The god employs our Zeno procedure so as to write down on his paper an actual infinite sequence of X's, most of which are, of necessity, invisible to the unaided human eye. In submitting his entry, he entrusts the judges with his own microscope which they soon discover to be more powerful than any of the microscopes belonging to the explorers. Indeed, it proves to be a metaphysical microscope the power of which can be increased indefinitely simply by turning a knob. I do not think that it can be doubted that the god's entry satisfies the precise terms of the contest, let them even be construed in the very strictest of senses. Has he not submitted a number of X's? Each entry contains a certain quantity or number of X's. The god's contains the very greatest quantity of all. It is soon verified that the number of X's on his entry is greater than the number of all the other X's, on all the other entries, combined. If a booby-prize were to be awarded to the entry that contained the smallest number of X's, it would not be impossible that some child's entry which was a sheer blank might receive the booby-prize. But this could only be as a joke: a blank sheet cannot be said to contain any number of X's at all. There is no joke in the god's victory: the number of X's that he has submitted is infinite. A number it is, literally. Number means multitude.

Nicomachus in his treatise on number defines number as 'limited multitude'. But Aristotle in Met. X. 1053a 31 drops the qualification, being content to define number as πλήθος μονάδων, a multitude or plurality of individuals. No wonder that the Greeks refused to count even one as a number. Two was the smallest number. This should not seem strange. 'There is a very, very small number of chairs in that room: there are very, very few.' 'How many are there?' 'One.' But one chair is not a very, very small number of chairs: it is not a number of chairs at all, even in our language. If zero remains anomalous even in formal mathematics, much the same is true of one. 'Divide this number of apples (say twelve apples) by three.' That means: 'divide this number into three equal parts.' 'Divide this number by one.' You cannot divide any number into one equal part. The words are senseless. 'Divide this number by one' means 'do not divide this number at all'. So, too, you cannot multiply any number by one. You can only leave it alone. For us, as for the Greeks, one is not a number. It is rather to be understood as the ἀποτέλεσμα of

1 The peculiar status of 0 and 1 is officially recognized in group theory where they are singled out as identity elements.
number, the beginning or source or principle which generates number. For we speak of a number of individuals, not a number of individual(s). When Frege and Russell undertake to 'construct' the natural numbers, it is not accidental that they are compelled to presuppose as an undefined term the concept of an 'individual', i.e. a single individual. Essentially, then, one is presupposed as the \( \Delta \xi \) of their science which itself resists being 'constructed'. 'Here is a tree. Here are two trees. Here are three trees.' Is \( a \) a number? Should the series of number-words read: none, \( a \), two, three, &c? Although one is not a number in the primary sense, I am not denying that in many contexts it may serve to fill the same kind of slot in a sentence which is filled by the proper number-words 'two', 'three', &c. Hence it may come, almost by 'attraction' as the grammarians use that word, to acquire the 'inflection', as it were, of a proper number-word; and seeing that the algebra of equations may almost be described as the science of numerical slots, it is to be expected that one will come to pass in the formal roll-call, no longer as 'individual', i.e. a single individual. Essentially, then, one is not merely a logical or linguistic matter that aleph-null be a number in the literal sense of the word? I think not. I have emphasized that if Zeno's challenge is to be met it is imperative that the word 'sum' retain the very same meaning in the sentence 'the sum-total of the infinite Z-series \( \frac{1}{2} + \frac{1}{2} \) &c., is equal to \( r \)' as it possesses in the sentence 'the sum-total of the finite series \( \frac{1}{2} + \frac{1}{2} \) is equal to \( r \)'. As we pass from finite to infinite series, we must preserve a strict univocity of meaning in all of the concepts that are transferred from the one to the other. Not only 'sum-total' but also 'series', '+', =, and 'number' must all be assigned tropological senses. Thus the sum-total of the Z-series is said to 'equal' \( r \) only in the sense that the limit of the sequence \( \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, &c. \) is equal to \( r \). 'Equality' means something very different here from what it does in the garden-variety statement, 'the sum-total of the finite series \( \frac{1}{2} + \frac{1}{2} = r \). Equality in the one case is defined in terms of convergence, in the other case in terms of strict identity or equivalence. We may thus say that whereas mathematics affords us merely a tropological theory of infinite series, we for our part require an exactly literal theory of infinite series. I would like to think that our effort to exhibit aleph-null as a number in the literal sense might provide some encouragement for the larger, more ambitious programme of transferring intact a whole cluster of concepts—not merely 'number' but also 'series', '+', =, 'sum-total' and 'all'—from the finite to the infinite.

Part 5: The Same Subject Continued

Our tentative conclusion that aleph-null is a number in the literal sense (\( \Delta \pi \lambda \delta \sigma \) or \( \pi \nu \tau \omega \sigma \) or \( \kappa \rho \lambda \omega \), whereas zero and one are numbers only in a derivative or tropological sense (\( \pi \nu \sigma \) or \( \epsilon \rho \mu \varepsilon \nu \sigma \) or \( \tau \rho \iota \sigma \sigma \nu \tau \iota \lambda \) points up the whole problem of
mathematical hermeneutics. Mathematical discourse is like Scripture itself. How are we to distinguish the literal from the tropological? And in the case of the tropological, how are we to extract from it its literal content? We must steer a difficult middle course between the rock of extreme fundamentalism and the whirlpool of an unrestrained 'higher criticism'. I am very much prone to be the fundamentalist in regard to aleph-null, but I am equally insistent on the 'higher criticism' in regard to zero. Our fundamentalism in regard to aleph-null cannot fail to cheer us on in the hope that the whole of Cantor's theory of the transfinite might be accepted in strict fundamentalist terms. Then, indeed, we might expect to ground our metaphysics of the actual infinite in a literal mathematics. Alas, this programme is doomed at the very outset.

Cantor's theory is shot through with the wildest tropes. We have only to consider the sophistical patter which is designed to convince us that the number of even integers is equal to the number of all the integers, odd as well as even. It is argued that, since every integer has a unique double, there exists a one-to-one correspondence between the class of even integers and the class of all integers. Hence the number of elements in the one class must be equal to the number of elements in the other. This pseudo-proof is already anticipated by Galileo. He argues that for every integer there exists a unique square of $n$ ($n^2$); hence the number of elements in the infinite series $1, 2, 3, 4, 5, \&c.$ may be held equal to the number of square integers: $1, 4, 9, 16, 25, \&c.$ Russell does not shrink from affirming that 'it is actually the case that the number of square (finite) numbers is the same as the number of (finite) numbers'.

Significantly, it was this very argument that Leibniz advanced to prove that there cannot be said to be a number of all integers. 'The number of all numbers implies a contradiction, which I show thus: To any number there is a corresponding number equal to its double. Therefore the number of all numbers is not greater than the number of even numbers, i.e. the whole is not greater than its part', which is absurd. Both Leibniz and Cantor, though arguing at cross-purposes, are equally guilty of what Wittgenstein styles a puffed-up proof. 'Our suspicion ought always to be aroused when a proof proves more than its means allow it. Something of this sort might be called a puffed-up proof.' Differing from both Leibniz and Cantor, the ordinary man does not hesitate to say that there are more integers than even numbers. I have asked people devoid of all mathematical sophistication whether the class of integers is equal to, greater than, or less than the class of even numbers. Without exception, they answer that there are more integers than even numbers. Are these people mistaken? All of them are familiar with the altogether banal fact that every integer has a unique double; but they somehow ignore that fact—so momentous both for Leibniz and Cantor—in their answer. By forcibly reminding them of it, it is not difficult to overawe them into surrendering their natural convictions. They can be readily brought to 'see' that, contrary to their first impressions, 'it is actually the case' that there are as many even numbers as there are odd and even numbers.

Why do people find it natural to say that there are more integers than even numbers? Because the class of even numbers is a proper sub-set of the class of integers. Every element in the first class is an element of the second, but there are elements in the second class which are entirely absent from the first. Hence there are more elements—a greater number of elements, even—in the second class than in the first. Is this process of reasoning fallacious? It must be admitted that people will also insist that if $A$ is a proper sub-set of $B$ then it is impossible for the elements of $A$ to be placed in one-to-one correspondence with the elements of $B$. They can thus be driven into a precise contradiction. Was Leibniz right, after all? Can it be true that the 'number of all [natural] numbers implies a contradiction'? If we consult the concept of number as it is employed in our mother-tongue, we must confess that it entails a double standard which would very much seem to rule out aleph-null as a number in the literal sense. On the one hand, the number of integers—let them actually all be exhibited before us by means of our Zeno procedure—must be said to be greater than the number of even numbers (in virtue of the proper sub-set standard), and on the other hand, the number of even numbers must be allowed to

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1 Our Knowledge of the External World, p. 194.

be equal to the number of integers (in virtue of the one-to-one correspondence standard). It is curious that Leibniz failed to see that this 'contradiction' infects not merely the concept of an infinite number but also the concept of any infinite collection whatever. As Dedekind observes, it is the hallmark of any infinite collection that it contains a proper sub-set that can be placed in one-to-one correspondence with the whole set. (If Leibniz's argument were sound, we would have to say that we know a priori that there cannot be an infinite number of stars in the heavens or even, simply, an infinite sequence of stars.) This is all very embarrassing to my own argument.

I have been insisting that aleph-null is a number in the precise, literal sense. This leads to the consequence that the number of even numbers, which can only be aleph-null, must be equal to the number of integers, also aleph-null. Yet our savages will not be readily prepared to accept this consequence. It is true that the god's entry provides the paradigm of aleph-null; but has it been made altogether clear how that paradigm is to be employed? When it is applied vis-à-vis any finite set, there is no problem; but let us suppose that the X's on the god's entry were found to have been written alternately in red and black pencil. Even after being infused with the metaphysical power of executing our Zeno procedure themselves, the Kumquats would not hesitate to say that there were more red and black X's on the god's entry than merely red X's. Are they to be convicted of error? There are at least three distinct criteria operative in our mother-tongue. (1) If A is a proper sub-set of B, then it is said that there is a greater number of elements in B than in A. (2) If A and B can be placed in one-to-one correspondence with each other, then A and B are said to contain the same number of elements. (3) If all the elements of A can be placed in one-to-one correspondence with a proper sub-set of B, then B is said to contain a greater number of elements than A.

This third criterion is especially instructive. It is not merely the case that the class of even numbers can be placed in one-to-one correspondence with a proper sub-set of the class of integers, it is also the case that all of the integers can be placed in one-to-one correspondence with a proper sub-set of the class of even numbers. This third criterion—by itself alone—issues in the demoralizing consequence that there is both a greater number and a smaller number of elements in the class of integers than there is in the class of even numbers.

What, then, are we to say? Shall we adopt the first criterion and yield to the natural temptation to insist that there is a greater number of integers than even numbers? Shall we, rather, adopt the second criterion and insist, with Russell, that 'it is actually the case' that the number of even numbers is equal to the number of integers? Or, finally, shall we adopt the third criterion and, spurning mere consistency, insist that the number of integers is both greater and less than the number of even numbers? Certainly, no one of these answers can be said to instruct us as to what is 'actually the case'. We must reply: distingue—ἐνέργεια πολλῶν—multipliciter accipitur. There is no contradiction, as Leibniz supposed, if judiciously we note that in one sense (according to the first criterion) the number of integers is greater than the number of even numbers; in another sense (according to the second criterion) the number of elements in the one class is equal to the number of elements in the other; and in still another sense (according to the third criterion) the number of the one is both greater and less than the number of the other. (Unhappily, this last cannot fail to leave us very
uneasy.) By thus sorting out these different senses, we avoid the fallacy of univocity of which both Leibniz and Russell are guilty, in failing to note that the concepts $\equiv$, $\succ$ and $\prec$ all cover a family of meanings. Galileo’s position is perhaps the wisest of all. I confess that I have maligned him. His final view is that, since it is possible to regard the even numbers as not only equal in number to the integers but also as either greater or less than the integers, we must conclude that none of these concepts, $\equiv$ or $\succ$ or $\prec$, is applicable to infinite collections. We must content ourselves with saying that the class of integers and the class of even numbers are both infinite, without proceeding to relate them to one another by means of $\equiv$, $\succ$ or $\prec$. Yet this is not quite right either. It is not that $\equiv$, $\succ$ and $\prec$ are not applicable to infinite collections. Rather, none of these concepts is univocally applicable to infinite collections. Applicable they are but only by means of a concept-shift. They are applicable in a certain sort of way, τρόπον των, tropologically.

Must I then relinquish my claim that aleph-null is a number simpliciter? I persist in the conviction that the god has literally written down an infinite number of $X$’s, a number of $X$’s which is demonstrably greater than the number of all the other $X$’s, on all the other entries, combined. In addition, I want to say that there may be an infinite number of stars in the heavens. At the same time I am unwilling to agree with Russell that ‘it is actually the case’ that the number of even integers is equal to the number of integers. My final conclusion on this matter is, on the one hand, that aleph-null is only tropologically and not literally a number and, on the other hand, that the god has literally written down an infinite number of $X$’s. Cantor’s aleph-null is not literally a number because it has built into it one-to-one correspondence as the unique standard of comparison.

Let me round out this discussion with particular reference to the ‘bisection’ paradox. I want to say that the sum-total of the infinite Z-series is literally equal to 1, and I also insist on saying that the number of even numbers is only tropologically equal to the number of integers. The concept of equality thus plays diverse roles in our ontological mathematics of the infinite. Literally predicated in some cases, it is tropologically predicated in others. These tropological locations can be cashed in literal terms. Thus the literal cash-value of the tropological location that the number of even integers is equal to the number of integers, is simply that the one class can be placed in one-to-one correspondence with the other. This last statement requires some comment. The finitist admits that it is theoretically possible to place all of the snakes in Africa in one-to-one correspondence with all of the frogs in America (assuming the two

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1 It is interesting to note that our conclusion stands quite apart from any appeal to the specific character of $\aleph_0$ whereby it is distinguished from $\aleph_1$, $\aleph_2$, &c.
classes to be equal). He does not admit that it is even theoretically possible to place all of the even numbers in one-to-one correspondence with all of the integers. The two cases are very different for him. According to the finitist, it is (at best) only in some tropological sense that this last correspondence can be effected. We, for our part, appealing to our Zeno ally possible to place all of the even numbers in one-to-one correspondence with all of the integers to an ontological import whatever.

It appears to be merely a kind of padding designed to give shape to the whole system. Very different is the trope in transfinite arithmetic whereby the infinite sequence of even number is said to be equal in number to the infinite sequence of integers. This principle certainly possesses an ontological cash-value when it is reduced to its literal content. Finally, of greatest ontological weight is the' theorem in analysis that the sum-total of the infinite Z-series is equal to \( \frac{1}{2} \). I insist that this last is literally true as it stands.

To insist is one thing, to render intelligible is another. It may be objected that I have done nothing but insist. The failure of finitism to meet Zeno's challenge—witness the neighbourhood concept in the theory of limits—does not of itself guarantee the intelligibility of the actual infinite. Furthermore, the intelligibility of the actual infinite—let it be granted—does not of itself guarantee a solution to the 'bisection' paradox. If I perform the partial summation \( \frac{1}{2} + \frac{1}{4} = \frac{3}{4} \), and then proceed to \( \frac{3}{4} + \frac{1}{8} = \frac{7}{8} \), and on to \( \frac{7}{8} + \frac{1}{16} = \frac{15}{16} \), &c. &c., I can never succeed in pushing through to the total sum. Even granting an actual infinite sequence of partial sums, the sum-total proves to elude us even after the whole sequence of partial sums has been exhausted. Hence it is impossible for the sum-total of the Z-series to equal \( \frac{1}{2} \) in the very same, univocal sense in which the sum-total of the series \( \frac{1}{2} + \frac{1}{2} = 1 \). But we cannot be satisfied with anything less than strict univocity here. Our failure to secure univocity even on the basis of the actual infinite, constitutes for us a sixth version of the 'bisection' paradox.

The failure of finitism to meet Zeno's challenge was only to have been expected. Quite unexpected and keenly disheartening is the prospect of the actual infinite equally failing to meet the issue. Such a double failure would seem to leave us without any resource whatever. I should be most reluctant to concede the failure of the actual infinite to carry us through to a solution. Although Cantor does not appear to have addressed himself directly to our problem, others have not hesitated to apply his theory of the actual infinite to the matter at hand, and they have claimed a resounding success for their efforts. I am prepared to be warmly sympathetic to these efforts. Having broken sharply with the finitists, I may be said to have almost nowhere else to turn. At once the most revealing and most candid account along these lines, is to be found in Cajori's useful 'History of Zeno's Arguments on Motion', sub-titled 'Phases in the Development of the Theory of Limits'. An enthusiastic Cantorian, Cajori concludes his history with the following remarks:

As now we pause and look backward, we see that a full and logically correct explanation of Zeno's arguments on motion has been given by the philosophers of mathematics. Looking about us, we see that the question is still regarded as being in an unsettled condition. Philosophers whose intellectual interests are remote from mathematics are taking little interest in the linear continuum as created by the school of Georg Cantor. Nor do they offer a satisfactory substitute. The main difficulty is not primarily one of logic; it is one of postulates or assumptions. What assumptions are reasonable and useful? On this point there is disagreement. (Italics mine.)

Our hopes are dashed even before they can be raised. Deeply pessimistic, we consult the details of that 'full and logically correct explanation' that is held out to us with one hand even as it is withdrawn with the other, for 'the main difficulty is not primarily one of logic'.

Cajori discusses Zeno's paradox of Achilles and the tortoise as follows:

\[ \text{1 Amer. Math. Monthly, 1915, pp. 296–7.} \]
Suppose the tortoise has an initial start of 10 ft. and that it travels 1 ft. per second, while Achilles travels 10 ft. per second. Forming the series A, whose terms represent each the distance Achilles travels to come up to the place where the tortoise was at the beginning of the time interval under consideration, and letting the series T represent these time-intervals, we have

\[
A \quad 10 \quad 1/10 \quad 1/100 \quad 1/1000 \quad \ldots \\
T \quad 1 \quad 1/10 \quad 1/100 \quad 1/1000 \quad 1/10000 \quad \ldots
\]

Both geometric infinite series are convergent; the sum of the terms of each series approaches a finite number as a limit. Now comes the ever-present, delicate question whether the sum actually reaches its limit. It is to be observed that this question arises in each series, that one series does not help the other. If the sum of A reaches its limit, so does the sum of T, but the possibility of the sum of A reaching its limit is a consideration independent of T. In this sense the consideration of time does not enter the critical part in the explanation of the 'Achilles'. Whether the sum of A reaches its limit or not is a matter of pure assumption on our part. If the limiting value 1/10 ft. is assumed to be included in the aggregate of numbers which the distance-variable may take, then of course the variable reaches its limit; if 1/10 ft. is not assumed as a value which the variable may take, then of course the limit is not reached. It is here that we must receive a suggestion from our sensuous observations; we know from our knowledge of motion as supplied to us by our senses that Achilles travelling with a uniform finite velocity in the same direction will within a finite time reach the distance 1 1/9 ft. from his starting point. This information, supplied to us by our senses, enables us to choose, of the two possible alternative assumptions offered by theory (as mentioned above), the one which makes the variable sum A conform with the known sensuous phenomena. On this assumption the region of the convergent process has a limit which is contained in the aggregate of values the variable can take, and... the limit is reached by the variable. Viewed from the standpoint of the theory of infinite aggregates and of the Georg Cantor conception of the continuum, the 'Achilles' is almost a self-evident proposition. (Italics mine.)

The whole discussion is a manifest fraud. The mathematics is certainly bogus, mere trumpery. What is the hard cash-value of Cajori's patter? Simply that we know as a matter of plain empirical fact that Achilles will succeed in overtaking the tortoise; hence we are entitled to affirm that the series not merely converges toward the limit but also actually reaches it. This is an argumentum ad rem, not an argumentum ad rationem. Cajori is thus guilty of an ignotatio elenchi. The hard facts in the matter are as available to Zeno as they are to us. The problem is to render those facts intelligible, not simply to reiterate them.

Out of all this wreckage there is one small point that merits our closest attention. Cajori writes: 'If the sum of A reaches its limit, so does the sum of T, but the possibility of the sum of A reaching its limit is a consideration independent of T. In this sense the consideration of time does not enter the critical part in the explanation of the "Achilles".' This is a serious mistake. Time is of the essence here. There are two cases that must be radically distinguished. The first may be styled Achilles in the small, the second Achilles in the large. Let us suppose that Achilles were to travel 10 ft. in the first second, 1 ft. in the next second, 1/10 ft. in the third second, &c. At that rate Achilles will never succeed in travelling the full distance of 11 1/9 ft. He will converge closer and closer to the limit without ever actually reaching it. In the familiar case of Achilles in the small, we have the two series

\[
A \quad 10+1 \quad +1/10 \quad +1/100 \quad +1/1000+ \ldots \\
T \quad 1 \quad +1/10 \quad +1/100 \quad +1/1000 \quad +1/10000 \quad + \ldots
\]

In the case of Achilles in the large, we have the two series

\[
A \quad 10+1 \quad +1/10 \quad +1/100 \quad +1/1000+ \ldots \\
T \quad 1 \quad +1 \quad +1 \quad +1 \quad + \ldots
\]

In both cases the A-series is exactly the same. What distinguishes the two cases is precisely the T-series. This fact is big with consequences. I, too, have been assuming that for Zeno's challenge to be met it must belong to the very essence of a convergent series that it actually reach its limit. But the A-series definitely does not reach its limit in the case of Achilles in the large. Hence it cannot belong to the very essence of a convergent series as such, quite apart from any constraining conditions, that it actually reach its limit. Nor can it be said that the very essence of a convergent series entails that the series may be allowed only to converge toward its limit without ever actually reaching it. In the case of Achilles in the small, the limit is actually reached.

Let the question be raised: does an infinite convergent series actually reach its limit or not? We can only answer: it all

1 Ibid. p. 218.
depends on the ontological conditions; in some cases, yes, in other cases, no. If pure mathematics be understood to be a science that abstracts from these ontological conditions, then neither our question nor our answer can fall under the scope of pure mathematics: they must now be assigned to applied mathematics. I have urged that the sum-total of the $Z$-series must be literally equal to 1. Now I must qualify my position. Only under certain ontological conditions (corresponding to the case of Achilles in the small) can that fundamentalist and literalist position be held to obtain. There are other ontological conditions (corresponding to the case of Achilles in the large) in which the sum-total of the series must be admitted to be only tropologically equal to 1. How are we then to construe the locution in pure mathematics that the sum-total of the series is equal to 1. Is it to be understood literally or tropologically? Here we must introduce the concept of a neutral mode of discourse. Seeing that the pure locution can be cashed, in one context, in literal terms and, in a different context, only in tropological terms, the pure locution itself must be said to be neutral in respect to both interpretations. In itself it is then to be understood neither literally nor tropologically but, instead, neutrally, so as to be capable of either interpretation, depending on context. This neutral mode of discourse is in itself unintelligible. It is intelligible only as a dummy or schema which is essentially oriented toward its diverse ontological uses. The pure mathematics of convergent series is thus seen to be essentially grounded in the applied mathematics. The connexion between pure mathematics and ontology is thereby restored.

In considering the $A$-series in the context of Achilles in the large, we find that the finitist approach is beautifully applicable. The series converges toward its limit; it never reaches it. The neighbourhood concept is precisely what is demanded here. Indeed, the case of Achilles in the large affords us a perfect ontological model of the merely potential infinite. We must all be finitists in this context. Very different is the case of Achilles in the small. Here the $A$-series not merely converges toward its limit; it actually reaches it. Within this context the actual infinite is imperative. What are we now to say of a pure mathematics which abstracts from both contexts? What kind of infinite is operative here? The potential infinite or the actual infinite? Clearly we are not allowed to play favourites as between the two. And yet there is no third kind of infinite which stands to the other two as genus to species. There is no third, neutral infinite which presides over the other two. I am led to the conclusion that a pure mathematics of convergent series, so pure as to abstract from the very distinction between the potential and the actual infinite, is an impossibility. We must resign ourselves to a double theory of convergent series, one which studies convergent series which are only potentially infinite, the other which studies convergent series which are actually infinite.

Does an infinite convergent series actually reach its limit? An infinite convergent series which is actually infinite actually reaches its limit; but an infinite convergent series which is merely potentially infinite, does not actually reach its limit. The very concept of an infinite convergent series is thus seen to be ambiguous. ‘Infinite’ may mean actually infinite or it may mean potentially infinite. Distingly. If the ‘sublimity’ of pure mathematics ‘consists merely’ (as Russell holds) ‘in the fact that the pure mathematician is not talking about time’, then the infinite of pure mathematics can be neither a potential infinite nor an actual infinite, seeing that both are here bound up with time. This pure infinite must then be neutrally patient of either ontological interpretation, which is surely impossible.

**Part 6: Brouwer and the Paradoxes**

If the names of Aristotle, Leibniz, Gauss and Cantor may be allowed to mark four of the five phases in our abridged history of the infinite in mathematics, then it is the name of Brouwer that looms largest as we turn to the contemporary scene. Not that Brouwer’s intransigent finitism can be said to be the only, or even the dominant, school of thought in current mathematics. His is doubtless an extreme position, but even his great opponent Hilbert has ‘conceded that the propositions of classical mathematics which involve the completed infinite go beyond intuitive evidence’, and though Hilbert has ‘refused to follow Brouwer in giving up classical mathematics on this account’, it may be argued that his refusal is rather more a matter of form than of substance.
Hilbert distinguishes between two types of statements in classical mathematics: real statements and ideal statements.

The real statements are those which are being used as having an intuitive meaning; the ideal statements are those which are not being so used. The statements which correspond to the treatment of the infinite as actual are ideal. Classical mathematics adjoins the ideal statements to the real. . . . [They serve] to complete its structure and simplify the theory of the system . . . . The delicate point [for Hilbert] is to explain how the non-intuitionistic classical mathematics is significant, after having initially agreed with the intuitionists [the school of Brouwer] that its theorems lack a real meaning in terms of which they are true.\(^1\)

Hilbert is thus seen to be in essential agreement with Brouwer on the ontological issue. Ontologically speaking, Hilbert is no less a finitist than Brouwer himself—this despite his praise of Cantor's theory as 'the most admirable flowering of the mathematical spirit' and his insistence that 'we must let no one drive us from the paradise that Cantor has created for us'.

Four schools of thought are to be found operative today in current researches into the foundations of mathematics. These are (1) the logicism of Russell, (2) the radical finitism of Brouwer, (3) the formalism of Hilbert, and (4) the concept-analysis of Wittgenstein. Even more than Hilbert, Wittgenstein may be shown to have quite fallen under the spell of Brouwer's radical finitism after attending Brouwer's lectures in Vienna in 1928. Of the four schools, Russell's alone dares to make any kind of commitment in support of the actual infinite, and that commitment is so injudicious as hardly to inspire confidence. The object of the Principia Mathematica is to exhibit classical mathematics as a purely a priori science based exclusively on the formal principles of deductive logic. But no comprehensive account of classical mathematics is possible (Cantor's theory of the transfinite being included in classical mathematics) without some decision being reached as to the mathematical, if not to the ontological, status of the actual infinite.\(^2\) It would be vain to suppose that one could reach any decision on this matter purely by consulting the formal principles of deductive logic. Are we to confine ourselves to the potential infinite or are we allowed to exploit the actual infinite as well? This question is almost certainly trans-logical in character and demands a trans-logical answer (logic being understood here in the narrowest sense).

We are not surprised to find that in the Principia Mathematica Russell and Whitehead, in their effort to preserve Cantor's theory of the transfinite, are obliged to contaminate the logical purity of their programme by expressly introducing one great extra-logical and trans-logical principle. This is the awkward 'axiom of infinity', which postulates the actual existence as a plain matter of fact of an infinite number of concrete individuals in the world. The 'axiom of infinity' is not understood as an axiom in the classical sense, i.e. a self-evident truth. The authors of the Principia freely acknowledge that they are altogether ignorant as to the exact number of individual things which exist in the world. It may be finite or it may be infinite, as far as they know. The 'axiom of infinity' is thus seen to be an arbitrary ontological postulate which is introduced by sheer fiat to preserve the scope of classical mathematics. The whole machinery creaks and groans, but we have not yet mentioned the worst. Following Frege, Russell and Whitehead define the number 2 as the class of all couples, the number 3 as the class of all triples, the number 4 as the class of all quadruples, &c. Now what if only one individual thing were to exist in the world? The number 2 would then be a class without any members: it would be a null-class. Likewise, with the number 3, and the number 4, &c. Under these penurious ontological conditions, the following theorem would be true: 2 = 3. For the class of all couples would then be identical with the class of all triples—vacuously identical, to be sure. The axiom of infinity is invoked not merely to prop up Cantor's theory of the transfinite. It is required to support the very simplest school arithmetic. In order to guarantee that \(n+1 \neq n\) for every natural number \(n\), it is found necessary to ensure that no natural number \(n\) should be left void as a class without members. An actual infinite number of individual things must then be presumed to exist.'

Frege's solution was different, though no less misguided. Eschewing the axiom of infinity, he guaranteed that no natural

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\(^2\) Quite apart from Cantor, the actual infinite is found in Dedekind's definition of an irrational number as an infinite class of rationals—Brouwer will have none of this.
number could collapse into a null-class by the following stratagem. Even if there were no individual things existing at all (we might style this possibility as the null-world), then at least the null-class would 'exist', namely the class of all concrete individual things that actually exist, in this case a class without any members. Moreover, there would also 'exist' the class which contains the null-class as its only member. This gives us our first non-empty class. Also there would exist the class which contains the following two members: (1) the null-class and (2) the class that contains the null-class as its only member, &c. \textit{ad nauseam}. In this way an infinite sequence of classes will be found to 'exist' (by a kind of creation \textit{ex nihilo}) in our null-world. Finally, there will be seen to 'exist' the class of all classes—this supreme class will contain an actual infinite number of members. Here is logic with a vengeance. No wonder that Hermann Weyl was moved to say of the logistic programme of Frege and Russell, 'Mathematics is no longer founded on logic but on a sort of logician's paradise'. We can only sympathize with Brouwer for brandishing Occam's razor and blasting the whole house of cards to the ground.

I would suggest that at least some of the excesses of Frege and Russell might be avoided if we recall our earlier remarks on mathematical existence. When we say that \( a \neq a + 1 \), what do we mean? Surely we are asserting merely a hypothetical, not a categorical, proposition. We are saying that if it should be the case that two objects \( A \) and \( B \) exist and also that three objects \( C \), \( D \) and \( E \) exist, then it follows that the two objects cannot be identical with the three objects. We are affirming nothing as to the actual world. Mathematical existence is ontological possibility. So, too, I would urge that the actual infinite may be employed in mathematics without any ontological commitment being made as to the actual number of existing things. We have merely to allow the ontological possibility of the actual infinite.\(^1\) How can that possibility be denied? Who will dare affirm that, apart from all empirical evidence, he knows \textit{a priori} that only a finite number of stars exist in the heavens? It is quite possible that the number of stars is infinite.

\(^1\) But if the concept of possibility is required here, does it not then follow that the 'logicizing' of mathematics can only be implemented through a \textit{modal} logic?

As we turn to Brouwer's radical finitism, it is not be to supposed that we have here any mere renewal of Aristotle and Gauss. In 1908 Brouwer published a paper entitled 'The Untrustworthiness of the Principles of Logic' in which he impugns not only the logic of Aristotle but the logic of Frege and Russell as well. Modern and ancient logic fall equally under his indictment. Of the three hallowed laws of logic—the law of identity, the law of contradiction, and the law of the excluded middle—it is only the first and second that he is willing to accept without qualification. On the strength of the law of the excluded middle nothing would seem more self-evident than that the sequence of digits 77777 either occurs somewhere in the decimal expansion of \( \pi \) or it does not. \textit{Tertium non datur}. This is precisely what Brouwer—\textit{horrible dictum}—denies. Not that he is able to propose any third possibility; he simply banishes the whole business into the outer darkness. The consequences are positively cyclopean. Thus he refuses to admit that any two real numbers \( r \) and \( r_1 \) must either be equal or unequal to each other. We should be inclined to dismiss Brouwer's scepticism as a mere aberration, were it not for the fact that great minds have suffered themselves to be seduced by it. Wittgenstein writes as follows:

Isn't it like this? The concepts of infinite decimals in mathematical propositions are not concepts of series but of the unlimited technique of expansion of series. We learn an endless technique: that is to say, something is done for us first, and then we do it; we are told rules and we do exercises in following them; perhaps some expression like 'and so on \( \text{ad inf.} \)' is also used, but what is in question here is not some gigantic extension. \textit{These are the facts}. And now what does it mean to say: '\( \phi \) either occurs in the expansion or it does not occur'? ... To ask this is to ask for a rule regarding the occurrence of \( \phi \). ... Here it happens that our thinking plays us a queer trick. We want, that is, to quote the law of excluded middle and to say: ... 'In the decimal expansion of \( \pi \) either the group “77777” occurs or it does not—there is no third possibility.' That is to say: 'God sees—but we don't know.' But what does that mean?—We use a picture; the picture of a visible series which one person sees the whole of and another not. The law of excluded middle says here: It must either look like this or like that. So it really—and this is a truism—says nothing at all but gives us a picture. And the problem must now be whether reality accords with the picture or not. And this picture
seems to determine what we have to do, what to look for, and how—but it does not do so, just because we do not know how it is to be applied. Here saying 'There is no third possibility' or 'But there can't be a third possibility!' expresses our inability to turn our eyes away from this picture. . . . A picture is conjured up which seems to fix the sense unambiguously. The actual use, compared with that suggested by the picture, seems somewhat muddied . . . the form of expression we use seems to have been designed for a god, who knows what we cannot know; he sees the whole of those infinite series. . . . For us, of course, these forms of expression are like ornamental trappings, which we put on perhaps, but with which we cannot do much, since we lack the effective power which would give this venture meaning and purpose. In the actual use of expressions we make detours, we go by side-roads. We see the straight highway before us, but of course we cannot use it because it is permanently closed.¹

Neither Brouwer nor Wittgenstein denies that it might be possible to prove that the sequence 7777 does not occur anywhere in the decimal expansion of π or, alternatively, that one might succeed in proving—say by actually performing the expansion to a million places—that the sequence does in fact occur in some specifiable region in the series. What they insist upon is that we have no a priori guarantee that one or the other of those proofs must—even in principle—be available. It might be the case that, no matter how far out we might undertake to generate the decimal expansion of π, the sequence 7777 would never be forthcoming. Moreover, we cannot assume that a mathematical proof of that fact must necessarily lie waiting for us in some Platonic heaven. This is the hard core of Brouwer's position. There is no a priori guarantee that every mathematical problem must be capable of a solution. We cannot but assent (however grudgingly) to this much of the argument. The great question is, what leads Brouwer and Wittgenstein so much farther, so as to deny that the sequence 7777 either occurs in the decimal expansion of π or it does not. Wittgenstein's motivation is not difficult to reconstruct. Even in his later thought, he was never able to shake off the postulate of logical positivism that the meaning of a proposition is its method of verification. What was explicit at the time of his early Tractatus, went under-

and behaviourism are quite similar trends. Both say, but surely, all we have here is... Both deny the existence of something, both with a view to escaping from a confusion.¹

We have not yet mentioned the most powerful source of Brouwer’s radical finitism. This is the alarming cluster of paradoxes that rocked the foundations of mathematics and logic early in this century. The most famous of these, the Russell paradox, may be briefly recalled. Appealing to the law of the excluded middle, we may say that every class must either be a member of itself or not a member of itself. Ordinary classes, such as the class of tables, are not members of themselves: the class of tables is not a table. Very different is the class of all classes; it cannot be an ordinary class, for the class of all classes is itself a class and hence a member of itself. Let us now consider the Russell class, the class of all ordinary classes. It must either be an ordinary class or not an ordinary class, it is either not a member of itself or it is a member of itself. Which is it? It is evident that the Russell class cannot be an ordinary class. For if it were an ordinary class, then two things would follow: (1) the Russell class, being by definition the class of all ordinary classes, would contain itself a member, and (2) the Russell class, being by hypothesis an ordinary class, would not contain itself as a member, seeing that an ordinary class by definition is a class which is not a member of itself. A formal contradiction is seen to result from the assumption that the Russell class is an ordinary class. By the law of the excluded middle, it follows that the Russell class is not an ordinary class. Q.E.D. This is virtually a paradigm case of a proof by the method of reductio ad absurdum, a type of proof which is widely employed in mathematics. Unhappily, Russell has shown that it is false to say that the Russell class is not an ordinary class. For if the Russell class is not an ordinary class, then two things must follow: (1) the Russell class must be a member of itself, seeing that any class which is not an ordinary class is by definition a class which is a member of itself, and (2) the Russell class cannot be a member of itself, seeing that the Russell class by definition only includes ordinary classes. By the law of the excluded middle, it follows that the Russell class must be an ordinary class. Q.E.D.


I have designedly couched Russell’s paradox in a form congenial to Brouwer: the law of the excluded middle has been suggested to be the villain in the piece. It must be admitted that if Russell’s paradox proves nothing else, it does succeed in proving that the rule of tertium non datur is not universally applicable; certain restrictions must be placed on its use. What are those restrictions? Brouwer insists that the paradoxes arise from the use of the tertium non datur rule in connexion with the actual infinite. This is by no means self-evident. Does the actual infinite play any role in Russell’s paradox? Any theory of classes which permits a class of all classes would seem, almost inevitably, to sanction an infinite number of classes. There will be one class that contains all numbers divisible by 2, another that contains all numbers divisible by 3, &c., ad infinitum. Despite this fact, it can be shown that the infinite plays no essential role in Russell’s paradox. The paradox can be transplanted intact out of the theory of classes into a very different context where the infinite is entirely absent. Let us consider Grelling’s version of the paradox. Of all the words in the English language (the number is finite), there are a few that can be predicated of themselves. Thus the word ‘noun’ is a noun. Words of this sort will be called autological. Most words cannot be predicated of themselves. Thus the word ‘house’ is not a house. Words of this kind will be called heterological. Now let us inspect the word ‘heterological’. Is it autological or is it heterological? It will be readily seen that if it is autological, then it is heterological, and if it is heterological, then it is autological. The essence of Russell’s paradox is thus shown to be altogether independent of the infinite.

Weyl is surely mistaken when he argues (speaking of the potential infinite and the actual infinite):

That we blindly converted one into the other is the true source of our difficulties, including the antinomies—a source of more fundamental nature than Russell’s vicious circle principle indicated. Brouwer opened our eyes and made us see how far classical mathematics, nourished by a belief in the ‘absolute’ that transcends all human possibilities of realization, goes beyond such statements as can claim real meaning and truth founded on evidence.¹

¹ Quoted by S. C. Kleene, op. cit. p. 49.
Weyl is correct in insisting that Russell failed to locate the true source of the paradoxes when he traced them to the principle of self-reference. On Russell's view the paradigm of all the paradoxes is to be found in the Cretan liar or Epimenides paradox: 'The present sentence is false'. Is that sentence true or false? It may be noted that if one dismisses it as meaningless, then it proves to be false. For if it is meaningless, then it is false. In any event, there are many cases in which self-reference is quite innocuous. The following sentence is entirely self-referential but its truth is unimpeachable: 'The present sentence consists of seven words'. Less fictitious is the following: 'All English sentences begin with a capital letter.' This is a true statement which is to be found in any school grammar. Russell's theory of logical types, by ruling out all self-referential statements, does succeed in banishing the paradoxes but only at the price of banishing the innocent along with the guilty. Self-reference may be a necessary, it is certainly not a sufficient, condition for the logical paradoxes. If neither Brouwer nor Russell has succeeded in disclosing the true source of the paradoxes, what indeed is the true source? I do not know. No one seems to know. For our purposes, it has been sufficient to show that the infinite must be exonerated of all responsibility.

Our own approach to Brouwer's problem should be obvious enough. Does the sequence 7777 occur in the decimal expansion of \( \pi \)? We have only to employ our Zeno procedure and write out the complete expansion. Let us then scan the whole series with a metaphysical microscope. One minute will suffice to yield the answer to Brouwer's question. Is the number of 7's in the expansion finite or infinite? Count them. Wittgenstein writes: 'In the actual use of expressions we make detours, we go by side-roads. We see the straight highway before us, but of course we cannot use it because it is permanently closed.' Permanently closed? I suggest that we open it up and travel its full length. It is very true that 'the form of expression we use seems to have been designed for a god', and I freely admit that I am appealing to an "absolute" that transcends all human possibilities of realization'. On the other side, it cannot be too often repeated that if the actual infinite is to be ruled out as unintelligible, then we must conclude that we know a priori that there can only be a finite number of stars in the heavens. The denial of the actual infinite leads to consequences no less 'metaphysical' than its affirmation.

To show that the actual infinite plays no role in the logical paradoxes, is one thing; to show that the actual infinite is altogether free of paradox, is something else again. There are not only logical paradoxes; there are metaphysical paradoxes as well. Not one but six metaphysical paradoxes have been generated out of Zeno's "dichotomy". How successful have we been in our efforts to strike them down one after the other? Let us review them synoptically and tie together some of the loose ends that have been left dangling. The first of the six is attributed by Aristotle to Zeno himself. How is motion possible? How can an infinite sequence of finite spatial intervals be traversed in a finite time? Aristotle answers that in one sense only (\( \text{katà to } \pi\sigma\sigma\nu \)) is the time finite. In another sense (\( \text{katà dìalpèsw} \)) the time itself is infinite, and it is in this sense alone that the distance may be said to be infinite; \( \text{katà to } \pi\sigma\sigma\nu \), the distance is finite. By sorting out these different senses, time and space are found to be strictly isomorphic: the paradox is dissolved. An infinite sequence of finite spatial intervals can be traversed in an infinite sequence of finite temporal intervals.

The second paradox is somewhat more difficult. Laying aside the question of space entirely, how can a minute of time ever elapse, the minute being successively composed of an infinite sequence of finite intervals? How can an infinite series come to an end? Aristotle answers that in any case of continuous uninterrupted motion the time is divisible only per accidens (\( \text{katà soubèthkèn} \)) into an infinite sequence of finite intervals. It is only potentially (\( \text{dunamei} \)) infinite. Actually (\( \text{euergetai} \)) there is no such succession of sub-intervals that elapse. Zeno's division is arbitrary—possible, certainly, but not actual. It may be noted that in his answer to the first paradox Aristotle is content to distinguish—as if co-ordinately—the two senses in which a finite interval of space or time may be said to be, on the one hand, finite and, on the other hand, infinite. In his answer to the second paradox those two senses are assigned very different ontological weights. The sense in which the finite interval is finite (\( \text{katà to } \pi\sigma\sigma\nu \)) is also the sense in which the interval actually exists (\( \text{euergetai} \)). The sense in which the finite interval is infinite (\( \text{katà dìalpèsw} \)) is merely the sense in which the
interval possesses the potential of being indefinitely divided (bouleus).

The third paradox leads us beyond Aristotle's own inquiries. Here we posit the case of a man or a god who, in traversing a finite interval of space in a finite duration of time, actually divides both the one and the other into an infinite succession of finite sub-intervals. How can a series which is actually infinite ever come to an end? It will be recalled that in the press of our investigation we neglected this paradox, leaving it altogether unanswered. I shall attempt to answer it now. Despite the fact that in admitting the actual infinite here we are breaking radically with Aristotle's finitism, I believe that a solution to the paradox is quite feasible along Aristotelian lines. How can a series without an end ever come to an end? It is obvious that the paradox cannot be dissolved unless an equivocation is exposed in the two uses of the word 'end'. We must show that the fallacy of equivocation is being committed and that the same word is being employed in two very different senses: διέχως λύγεται, dupliciter accipitur. When we say that the series is without an end, what do we mean? We mean that each term in the series has an immediate successor that lies within the series. When we say that the series comes to an end, what do we mean? We mean that certain terms are reached, lying outside the series, which fall beyond each and every term occurring in the series: these post-serial terms are successors (though not immediate successors) of the terms in the series. Distil/guo.

To say that a series without an end can come to an end, is to be guilty of a formal contradiction. Let us rather say that a series of which each term in the series has an immediate successor in the series can, under certain conditions, elapse in toto so that terms may be reached which, lying beyond each and every term in the series, are successors of all the terms in the series. Under certain conditions. If the Z-series is enacted in the large, then it is not possible for those post-serial terms ever to be reached. Time is of the essence. If we are asked precisely how those post-serial terms can in fact be reached, we have no other recourse but to exhibit the Z-series in the small as an object-lesson. I am not so bold as to offer this 'solution' as a definitive answer to the 'bisection' paradox. Questions doubtless remain. I merely contend that it dispels the paradox in its third formulation. It exposes the equivocation that is operative in our bewilderment as to how a series without an end can ever come to an end.

Our answer to the fourth formulation of the paradox is perhaps not as elegant as one might wish, but I think that it is adequate, though only barely adequate. It is demanded of us that we assign a sum-total to Grandi's series enacted in the small. Ontologically, Grandi's series is quite on a par with the Z-series. If the latter is intelligible, then the former must also be intelligible. And yet it is evident that no determinate sum-total can be assigned to Grandi's series. How then can one be assigned to the Z-series? Our answer proceeds as follows. If both series be enacted in the large, then in the literal sense neither series can be said to have a sum. However, the Z-series does possess a sum in a tropological sense, in virtue of its convergence. Grandi's series lacks even a tropological sum, which should not be surprising, seeing that it is an alternating divergent series. It would be unreasonable to expect the distinction between convergence and divergence to have no ontological import. As in the case of the two series enacted in the large, so also in the case of the two series enacted in the small, the distinction between convergence and divergence is ontologically significant. In this case the Z-series must possess a sum in a literal sense, whereas Grandi's series can only possess a sum in a tropological sense—there is certainly no summation or accumulation of the successive terms of the series here. What there must be, of course, is an ontological issue or terminal result to Grandi's series. Merely because we are obliged to confess that the terminal result is logically indeterminate, does not entail the consequence that it is ontologically unintelligible. In any case, why must we allow our distress over the indeterminacy inherent in alternating divergent series to infect and subvert our confidence in convergent series?

The fifth and sixth formulations of the paradox are both addressed to a single problem. It is clear that the sum-total assigned to the infinite Z-series in the small must be a sum-total in the very same literal and univocal sense in which a sum-total

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1 In the case of Grandi's series place an apple on a table today, remove it tomorrow, restore it the next day, &c.
is assigned to the finite series \( \frac{1}{2} + \frac{1}{2} \). But how is that possible? According to the fifth formulation of the paradox, it is insisted that only a tropological sum-total is feasible here, on the ground that the only sense in which we can speak of the sum-total of the Z-series as being equal to 1 is the sense in which the limit toward which the series of partial sums converges is 1. Our answer is that this account is applicable only to the Z-series enacted in the large where a finitist approach is perfectly in order. There is a core of truth in the fifth formulation, namely that a finitist approach is incapable of assigning a literal sum-total to a convergent series.

The sixth and last formulation of the paradox has embarrassed and alarmed us more than any of the others. Here the actual infinite is admitted, but it is argued that it proves no more successful than the potential infinite in meeting Zeno's challenge. If the Z-series is to be assigned a literal sum-total, then we must be able—in principle—to add together all of the terms in the series \( \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \ldots \) so as to produce a literal sum-total of 1. It is not denied that, in some sense, we can actually succeed in adding together all of the terms in the series by means of our Zeno procedure. Unfortunately, we seem to be confined to an infinite sequence of partial sums. How can we push through to the total sum? It is not enough to infer the existence of a total sum from the actual infinite sequence of partial sums. The inference, as an inference, is persuasive enough, but it may be objected that we have merely extended the concept of a sum-total to cover an outlying, critical, refractory case. I believe that this paradox can be effectively dissolved in the most decisive manner, but not without a radical shift in notation. Let us replace our decimal system by a notation that is entirely non-digital in character. I am referring now to a linear system. Let the number 1 be represented by a line 1 inch in length, the number 2 by a line 2 inches in length, the number 86 by a line 86 inches in length, the number \( \frac{7}{2}/3 \) by a line \( \frac{7}{2}/3 \) inches in length, the number \( \sqrt{2} \) by a line \( \sqrt{2} \) inches in length. Three types of number—the natural, the rational and the real—are thus assigned a homogenous notational representation all on a par with one another. Let us now compute the sum of the finite series \( 2 + 86 + \frac{7}{2}/3 \). We have only to draw a line 2 inches in length, then extend it 86 inches further and, finally, extend it another \( \frac{7}{2}/3 \) inches. The final result will be a line segment representing the sum-total of the series. If we wish to translate that result into our decimal notation we have only to measure the line segment by means of a standard yardstick.

Viewed now in terms of our linear notation, the Z-series in the small is no longer a peculiar case. Let us first draw a line 1/2 inch in length, let us then extend it 1/4 inch, and then another 1/8 inch, &c. ad infinitum. At the end of one minute, employing an analogue computer, we shall have succeeded in writing down an entirely literal sum-total of the whole series. The final line segment will be found to measure 1 inch in length. At last we have succeeded in our project of transferring intact a whole cluster of concepts—series, summation, equality—from the finite to the infinite without relying on any kind of trope whatever. We are now entitled to affirm that 'simple arithmetic' establishes the sum-total of the Z-series in the small as being quite literally equal to 1. Notationally speaking, finite and infinite are found to be on a par within a single homogeneous system. It should not be surprising that our linear notation is seen to be superior to all others. Here alone do we have an exact ontological image of the facts in the case. The line that we draw corresponds pari passu to the distance actually being travelled in the 'dichotomy'.

One last gnawing doubt remains. It is easy enough to prove that the final line segment in our linear computation of the sum-total of the Z-series in the small must be assigned a value greater than any rational value less than 1, also that it cannot be assigned any rational value greater than 1. Are we now quite certain that it must necessarily be assigned a value exactly equal to 1? Yes, certainly, on the assumption that there exist no actual infinitesimal quantities. Actual infinitesimals may not enjoy the best repute in contemporary mathematics, often being regarded almost as pariahs in the commonplace of number; but no one has presumed to prove that they cannot exist. Moreover, there are always mounting pressures in the most diverse areas in mathematics that urge us on to enfranchise them. (One recalls the notorious 'horn' angles of classical antiquity.) How can we prove that the linear sum-total of the Z-series is exactly equal to 1? Must we not admit a small margin of indeterminacy here? Either the sum-total is exactly equal to 1 or it is equal to 1 minus...
an actual infinitesimal quantity. It may be suggested that the only way we can settle the question as to the exact value of the linear sum-total of the Z-series is to measure the final line segment with absolute accuracy.

*Absolute accuracy* is that an intelligible goal to aim at? Yes, but only on the condition that we waive the actual infinitesimal. We can first measure the line with a crude instrument, to the accuracy of \(1/10\) of an inch, as a first approximation. Then we can employ a finer instrument and measure the line to the accuracy of \(1/100\) of an inch, as a second approximation, \&c. *ad infinitum*. At the end of one minute we can report whether or not the line is exactly equal to one inch (plus or minus an actual infinitesimal quantity). Our Zeno procedure enables us to execute a kind of metaphysical measurement which issues in almost absolute accuracy. The only difficulty is that this metaphysical accuracy falls short of *perfect accuracy*, seeing that the actual infinitesimal is too small to be detected even by an infinite succession of corrigible instruments each of one grade greater accuracy than the preceding one. The best we can do is to measure a line to the nearest rational or even irrational value: each succeeding instrument will pin down the value to an additional decimal place. There may be some who will feel that this inherent incapacity of our metaphysical measure to detect the presence of an actual infinitesimal, entitles us to rule it out of consideration, at least in the present context. If the presence of an actual infinitesimal is even metaphysically unverifiable, how can it be said to exist? One may wish to appeal to the principle that to be is to be knowable: to be meaningful is to be verifiable, albeit metaphysically verifiable.

The issue here is complicated by the fact that, even if not verifiable, the presence of an actual infinitesimal is, at least in some cases, falsifiable. You can certainly *refute* my claim that the edge of this sheet of paper is of infinitesimal thinness. Furthermore, assuming that the edge is indeed of an actual infinitesimal thinness, the use of our metaphysical measure will succeed in disclosing that the thinness of the edge is certifiably *less* than any rational or irrational value. Infinitesimals, then, are detectable after all.

In any event, it is certainly evident that the linear sum-total of the Z-series is a line of some definite length—there is nothing indeterminate about the line itself. The god in our story who travels first a distance of \(1/2\) mile, then after resting travels an additional \(1/4\) mile, \&c., does succeed in travelling a definite finite distance in the span of two hours. He may not indeed have travelled one full mile. At the very least, he will be found, at the end of the two hours, to be removed from his goal by a distance so infinitesimally minute as to be altogether undetectable except (it may be) under metaphysical scrutiny. He has now only to move forward that infinitesimal distance, in a duration of time equally infinitesimal, and he will have reached the absolute end of the mile.

How I wish that I might announce that at long last the ‘bisection’ paradox has been finally dissolved and dispelled. Six formulations of it have been explored in some detail, and I believe that we have succeeded, more or less, in dispatching them all. But how can we guarantee that a seventh, more deadly still, might not arise to overwhelm us? Zeno is like the Hydra: strike off one head and another, more dangerous, takes its place. It may be that we have failed to strike to the root of the evil. I propose to return to the charge in a later chapter. We are very far, as yet, from having pulled out *all* the metaphysical stops.
CHAPTER II

HYPER-MATHEMATICS

Part 1: Infinite Proofs

In a novel by Arthur Schnitzler one of the characters undertakes to prove that there really is no death.

Leinbach had discovered a proof that there really is no death. It is beyond question, he had declared, that not only the drowning, but all the dying, live over again their whole past lives in the last moment, with a rapidity inconceivable to us others. This remembered life must also have a last moment, and this last moment its own last moment, and so on; hence, dying was itself Eternity: in accordance with the theory of limits one approached death, but never got there. . . . A questionable figure, this Doctor Leinbach. . . .

Granting the whimsical premise, Leinbach's 'proof' is altogether sound if one assumes with the finitist that a convergent series can only approach its limit indefinitely, being forever debarred from reaching it. In which case one will indeed enter into every neighbourhood of death but death itself, though eminently approachable, must remain the unattainable limit of the convergence.

Apart from Schnitzler's poetic tour de force, there are three distinct, and independent, anticipations of our Zeno procedure with which I am acquainted. The earliest of the three we have found recorded in Aristotle. The other two are recent: C. S. Peirce in 1910, and Hermann Weyl in 1927. Peirce is engaged in clarifying our conviction that 'a die thrown from a dice box will with a probability of one-third, that is, once in three times in the long run, turn up a number (either tray or size) that is divisible by three'. What is to be understood here by 'the long run?' How long is it? That is Peirce's question. He answers that the long run cannot be seen to be less than 'an endless series of trials . . . [for] it is necessary that the die should undergo an endless series of throws from the dicebox' if the probability quotient one-third is to be verified. But how are we to conceive of a 'die . . . thrown an endless succession of times . . . with a finite pause after each throw?' Surely that is quite impossible (verification being the object). Yes, but the impossibility is merely a physical, and not a logical impossibility, as was well illustrated in that famous sporting event in which Achilles succeeded in overtaking the champion tortoise, in spite of his giving the latter the start of a whole stadium. Achilles proved, contrary to Zeno, that it is quite possible 'to traverse the sum of an infinite multitude of finite distances, each in a finite time'. Hence it follows that 'no contradiction is involved in the idea of an endless series of finite times or spaces having but a finite sum, provided there is no fixed quantity which every member of an endless part of that series must each and every one exceed'.

In the following discussion Peirce almost seems to be engaged in programming our metaphysical computer to deal with his problem.

I will now describe the behavior of the die during the endless series of throws, in respect to turning up numbers divisible by three. It would be perfectly possible to construct a machine that would automatically throw the die and pick it up, and continue doing so as long as it was supplied with energy. It would further be still easier to design the plan of an arrangement whereby a hand should after each throw move over an arc graduated so as to indicate the value of the quotient of the number of throws of three or six that had been known since the beginning of the experiment, divided by the total number of throws since the beginning. It is true that the mechanical difficulties would become quite insuperable before the die had been thrown many times; but fortunately a general description of the way the hand would move will answer our purpose much better than would the actual machine, were it ever so perfect.

After the first throw, the hand will go either to \( \frac{0}{3} = 0/1 \) or \( \frac{1}{3} = 1/1 \); and there it may stay for several throws. But when it once moves, it will move after every throw, without exception, since the denominator of the fraction at whose value it points will always increase by 1, and consequently the value of the fraction will be diminished if the numerator remains unchanged, as it will be increased in case the numerator is increased by 1, these two being the only possible cases. The behaviour of the hand may be described as an excessively irregular oscillation, back and forth, from one side of \( 1/3 \) to the other . . .


The question that now arises is this. Are we entitled to conclude that, at the end of one minute (assuming the infinite series of throws to be exhausted in that time) the pointer-reading on the arc or dial must be \( \frac{1}{3} \)? We are here positing that the die is entirely unbiased and free-falling. What if the terminal pointer-reading should be some fraction less than or greater than \( \frac{1}{3} \)? Are we then justified in insisting that the die must have been biased? Does an affirmative answer to this last question issue necessarily from our concept of chance? Peirce's discussion is curiously vacillating. First he answers yes, then he answers no, then he answers yes again, then he answers no again, and in the end everything seems left quite up in the air.

Though 'the probability of the die turning up a three or a six is not sure to produce any determination of the run of the numbers in any finite series of throws, . . . when the series is endless [then] we can be sure that it will have a particular character'. Apodictically sure? No. Even when there is an endless series of throws, there is no syllogistic certainty, no 'mathematical' certainty . . . that the die will not turn up a six obstinately at every single throw. It might be that if in the course of the endless series, some friends should borrow the die to make a pair for a game of backgammon, there might be nothing unusual in the behavior of the lent die, and yet when it was returned and our experimental series was resumed where it had been interrupted, the die might return to turning up nothing but six every time.

On this view there is no essential difference between a finite and an infinite series: in neither case does the probability quotient of \( \frac{1}{3} \) guarantee anything whatever as to the actual behaviour of the die. It can no longer be said that the probability quotient \( \frac{1}{3} \) means that 'once in three times in the long run' the die will turn up three or six. Even an infinite series is evidently not long enough for 'the long run'.

It is not surprising that Peirce cannot rest in that position: it has proved to be self-stultifying. He moves on. Granting that the unbiased die might register a six at every throw throughout the infinite series, this is only true 'in the sense that it would not violate the principle of contradiction if it did. It sanely would not, however, unless a miracle were performed; and moreover if such miracle were worked, I should say (since it is my use of the term
Brouwer) exhibited a hyper-mathematical 'solution' to the question, does the sequence 7777 occur anywhere in the decimal expansion of π? And there is a passage in Hermann Weyl—albeit buried in a marginal discussion—in which the full sweep of our programme is boldly anticipated. Weyl writes as follows: If the segment of length 1 really consists of infinitely many subsegments of lengths 1/2, 1/4, 1/8, ... as of 'chopped off' wholes, then it is incompatible with the character of the infinite as the 'incompletable' that Achilles should have been able to traverse them all. If one admits this possibility, then there is no reason why a machine should not be capable of completing an infinite sequence of distinct acts of decision within a finite amount of time; say, by supplying the first result after 1/2 minute, the second after another 1/4 minute, the third 1/8 minute later than the second, etc. In this way it would be possible, provided the receptive power of the brain would function similarly, to achieve a transversal of all natural numbers and thereby a sure yes-or-no decision regarding any existential question about natural numbers!

Oddly enough, Weyl's anticipation of our Zeno procedure is relegated to small print in the original German edition, and it is bracketed in the English translation. The import of these brackets is explained by him as follows: 'Sections of historical and supplementary interest not necessary to the main course of development of the book, set off in the German edition by small print, are indicated in this volume by opening and closing brackets, such as these.' As for the 'main course of development of the book', Weyl does not hesitate to say 'mathematics with Brouwer gains its highest intuitive clarity'.

Weyl is thus seen both to anticipate and to reject our Zeno procedure, and one would suppose that he would present specific grounds for his rejection. No such grounds are offered. He contents himself with insisting that we must 'renounce the mystical error of expecting the transcendent ever to fall within the lighted circle of our intuition... A field of possibilities open into infinity has been mistaken for a closed realm of things existing in themselves. As Brouwer pointed out, this is a fallacy, the Fall and Original Sin of set-theory, even if no paradoxes result from it.'

Apart from this general dread of metaphysics, I am able to explain Weyl's specific rejection of our Zeno procedure only as follows. Weyl seems to repeat Aristotle's mistake. Like Aristotle, he appears to assume that the intelligibility of the Zeno procedure rests on the premise that continuous motion is to be understood as an actual succession of chopped-off wholes. Like Aristotle, he seems to believe that the denial of that premise entails the rejection of the Zeno procedure. But we have already seen that the Aristotelian position as regards continuous motion—namely that it is potentially divisible without being actually divided—in no way militates against the intelligibility of the Zeno procedure as an actual succession of chopped-off intervals.

Acquiescing in the Zeno procedure, a whole range of problems in mathematics, currently unsolved, yield readily to a metaphysical solution. Is every even number the sum of two prime numbers? This question was first raised in 1742 by Goldbach in a letter to Euler; it has remained unanswered to the present day. No general proof of the theorem has been discovered nor has anyone succeeded in producing a counterexample which would refute it. Every even number that has been tested has been found to satisfy the theorem. (Thus 20 = 13 + 7.) Enlisting our Zeno procedure, it will be readily evident that we are now in a position (so to speak) to provide a decisive answer to Goldbach's question, be it in the affirmative or in the negative. Given any assigned even number, a finite series of very obvious operations will serve to determine whether or not it is the sum of two primes. By suitably programming our metaphysical computer, we can inspect successively each even number in turn, in each case deciding whether there exist two primes of which it is the sum. Within the space of one minute we should be able to report that Goldbach's theorem is true, if indeed it is true; and if it is false, we shall have succeeded in isolating an even number which is demonstrably not the sum of any two primes.

A second prime number theorem that has resisted all efforts at proving (or disproving) is the following: there exists no greatest prime twin. The numbers m and n (n>m) are said to be prime twins if (1) m and n are both prime and (2) n = m + 2. Prime twins are thus seen to be 'adjacent' primes (e.g. 17 and 19). In any comprehensive theory of proof it will be noted that there is absent from Goldbach's theorem a significant feature

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2 Ibid. p. vi.
3 Ibid. p. 54.
found to be present in the prime twin problem. If Goldbach’s theorem is false, then quite by accident someone might come upon a counter-example which by the crudest of methods could be shown to explode the theorem in ad hoc fashion. That is altogether impossible in the case of the prime twins. Even if there exists a greatest prime twin, a general proof will be required to establish that no greater one exists in the infinite range of numbers lying beyond it. Despite this ‘logical’ difference in the two problems, the second is scarcely more difficult than the first for our metaphysical computer. Having written down all of the natural numbers, we shall apply to the series the sieve of Eratosthenes, first striking out all multiples of 2 (except 2 itself), then all multiples of 3 (except 3 itself), then all multiples of 5 (except 5 itself), &c. Only prime numbers will remain, all others having been sieved out of the series. The primes being laid out successively before us, we shall mark each pair of prime twins as we encounter it in our progress through the series via our Zeno procedure. Is there a greatest prime twin? We can answer yes or no only after we have progressed through the entire series of primes.

There is a third kind of problem, significantly different from both the others, which lends itself with equal ease to our procedures of super-proof. Here it is a question of pinning down a uniquely defined number which we already know to exist, though no finite method is currently available for tracking down and identifying the number. A good example of this kind of problem is found in the Vinogradoff number. Vinogradoff has shown that there exists a number V such that every successor of V is the sum of at most four primes. This result will be seen to be an immediate corollary of Vinogradoff’s proof that only a finite number of integers can exist which are not the sum of at most four primes: the hypothesis that there is an infinite class of such integers leads to a contradiction. It is one thing, however, to establish the existence of V, it is quite another thing to exhibit it to view. Vinogradoff’s proof provides us with no method for locating it in the infinite range of integers. Here, again, our metaphysical computer is equipped to dispatch the problem at almost one blow, though it must be emphasized that nothing short of a serial inspection of all the natural numbers will suffice. The Vinogradoff number deserves special attention in the light of Brouwer’s cyclopean finitism. Brouwer refuses to admit that Vinogradoff has succeeded in establishing the actual existence of V. He argues that Vinogradoff’s proof is ‘non-constructive’ in character, and hence invalid. Merely to show by a reductio ad absurdum that the denial of V leads to a contradiction, does not suffice to convince Brouwer that the number V must necessarily exist. He demands a definite recipe which will enable us actually to ‘construct’ V, i.e. exhibit it bodily to view. All the more welcome, then, is our Zeno procedure in that it is calculated to satisfy Brouwer’s demand.

We cannot delay examining a critical question which the reader has already raised for himself. Is our Zeno procedure powerful enough to solve all mathematical problems? Are we in a position to programme our metaphysical computer so that it can dispatch each and every problem in pure mathematics as it arises? (Somewhat cavalierly, we may rule out Peirce’s problem as not belonging, in the strict sense, to pure mathematics.) The lure of a universal algorithm has teased the minds of mathematicians ever since the time of Vieta who did not hesitate to affirm that, on the basis of the new algebra, all problems can be solved. It will be evident that there exist problems in pure mathematics for which our simple, direct method of serial inspection, whereby we examine all the integers one by one, is certainly inadequate: witness Fermat’s last theorem. In the margin of his copy of Diophantus he noted that there are no positive integers a, b, c, and n (n>2) such that \(a^n + b^n = c^n\). Fermat added, ‘I have found for this a truly wonderful proof, but the margin is too small to hold it’. Subsequent research has established the truth of Fermat’s last theorem for all values of \(n<619\); but no general proof of the theorem is available. Moreover, we cannot be certain that Fermat did not make a slip in his ‘wonderful proof’: it is possible that a counter-example exists.

If our metaphysical computer is to be programmed to deal with this problem, we must certainly employ a procedure rather more sophisticated than any mere serial inspection of the integers taken one by one. Here it is a question of examining every ordered quadruple of integers to see if there might not exist in the infinite class of such quadruples some one or more which satisfies the equation \(a^n + b^n = c^n(n>2)\). There are several ways
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of proceeding at this point, and though the one I shall suggest may not be the simplest, it will enable us to introduce a technique that will prove of great use in the sequel. Let us assign to every quadruple a unique natural number which will serve as its index. Thus we wish to assign to the quadruple \( a = 12, b = 8, c = 11, n = 8 \) (which corresponds to the Fermat ‘equation’ \( 12^8 + 8^8 \neq 11^8 \)) a unique number such that, given the number, we can unpack from it the quadruple from which it was derived. We can then list all the natural numbers and check off the index numbers as we encounter them in the series. Finally, we shall unpack the index numbers one by one and thereby test every quadruple in turn.

The index number of any quadruple is derived in two steps. First, we translate the quadruple out of our decimal notation into its equivalent in a binary system. Here we have only two numerals, 0 and 1, so that 0 = 0, 1 = 1, 2 = 10, 3 = 11, 4 = 100, 5 = 101, 6 = 110, 7 = 111, 8 = 1000, 9 = 1001, 10 = 1010, 11 = 1011, 12 = 1111, 13 = 10,000, &c. The quadruple 12-8-11-8 finds its equivalent in the binary notation as 1111-1011-1000. The second step consists in stringing together the numbers of this new quadruple while plugging up the gaps with some numeral other than 0 or 1, say 7. Thus 1111-1011-1000 now yields the number 1,111,710,007, 101,171,000. This new number is the unique index number, in our decimal notation, assigned to the quadruple 12-8-11-8. Each index number will be seen to consist of three 7’s interspersed among a series of 0’s and 1’s. It can be decoded simply by sorting out the series of 0’s and 1’s into four distinct groups (the gaps being indicated by the 7’s) and then translating each group back into our decimal notation. We are now in a position, after we have written down all of the natural numbers, to mark each index number as we encounter it. The first will be 17, 171, 711. Decoded, this yields 1-1-1-3 which in turn yields the Fermat ‘equation’ \( 1^3 + 1^3 = 1^3 \).

Having exhibited an effective procedure—in some cases, quite simple, in others, somewhat more elaborate—for solving four refractory problems in pure mathematics, the Goldbach, prime twin, Vinogradoff, and Fermat problems, we may now return to the question we have left pending. Can we build into our metaphysical computer an omnibus programme designed to dispatch every mathematical problem that can possibly arise? Connected with this is a second question. What of theorems in geometry? Here we encounter problems that are not exclusively numerical in character. Do they fall outside the scope of our Zeno procedure? A good example is the four colour problem in topology, proposed by Möbius in 1840. It is believed to be impossible for a map to be drawn on a plane surface which requires more than four colours, each region being assigned a colour different from that assigned to any neighbouring region. The four colour theorem has been proved for all maps containing less than thirty-eight regions, and it has been demonstrated that five colours will always suffice. But what of the four colour theorem itself, in its general form? Has our Zeno procedure any relevance in a matter of this kind? I propose to explore a new route that bids fair to disclose a solution not merely to the four colour problem but to any mathematical problem whatever that may be suggested (Peirce’s included). More than that, this new programme may even be thought to provide a universal algorithm capable of solving every philosophical problem as well. Can it be that there are no bounds at all to our metaphysical pretensions? No limits to our megalomania?

In exploring this new route, we shall have the opportunity to face up to an objection that has been rankling the fastidious reader. What could be more philistine, more churlish, really, than our super-proofs? Let it be granted that we have established the truth of Goldbach’s theorem, having tested every even integer in turn. Will the mathematician rejoice? Certainly not. We have shown merely that Goldbach’s theorem is true, we have not shown why it is true. We have been viewing the domain of numbers as if it were a branch of natural history, and our procedures of investigation have been crassly empirical (if hyperbolically empirical). Our proofs have been proofs by complete enumeration, reminiscent of the crudest kind of scientific induction. Examining each crow in turn, we have concluded that all crows are black. It is not merely that our super-proofs are altogether unfeasible from any practical point of view. That is perhaps the least of the objections. Far more serious, it is found that on purely theoretical considerations they are inferior in dignity to the finite proofs of classical mathematics.
One has only to compare Euclid’s beautiful proof of the fact that there is no greatest prime number with our own super-proof of that same fact. Euclid’s reveals the ground of the fact, it lays bare the cause, it explains precisely what it is in the very essence of any prime number that requires that there always exist another prime lying beyond it. All that is absent from our super-proofs.

It must be confessed, however, on the opposite side of the ledger, that the super-proof enjoys an advantage which is not to be despised: it can be guaranteed in advance. How very different it is with the classical proof which depends on the whim of Fortune, it lies in the lap of the gods. There is no effective procedure in classical mathematics for guaranteeing an answer to Goldbach’s question. Diligence and industry are not enough. Apart from the happy conjunction of genius and chance, there is no prospect of a solution. There are thus advantages on both sides. If only they could be united—the power of the one with the dignity of the other. It is here that our new programme offers much to recommend it: it affords us the proof of that same fact. Euclid’s reveals the sense.

One may now to assign a unique number to every letter of the alphabet, no three can be included in this infinite library, at least one volume exclusively devoted to a crisp, clear finite proof (or disproof) of the prime twin theorem? Shall we not come upon succinct, trenchant solutions of the classical type to the Goldbach, Vinogradoff, and Fermat problems? Even to the four colour problem, though it fails to fall under the scope of our super-proof? And will not Peirce’s problem find its definitive solution in a beautiful monograph on the topic? Name any philosophic issue: free will, God, infinity. Will not each be finally settled, beyond revision, beyond controversy, in some masterly treatise found in our infinite library? (I am assuming, perhaps rashly, that we shall have the wit to recognize a solution when we encounter it.)
Alas! Those are by no means merely rhetorical questions that I have raised. An affirmative answer cannot be said to be self-evident. Assuming Goldbach’s theorem to be true, is it altogether absurd to suppose that there might not exist in our metaphysical library or indeed anywhere at all a finite proof of the theorem? Let every even number be the sum of two primes—does it follow that there must be a finite proof of that fact? What if Goldbach’s theorem were true but only as a cosmic accident? If such were the case, if there were in fact nothing whatever in the very essence of an even number as such (qua even) that entailed the existence of two primes whose sum it must be, then indeed the dignity of our super-proof of Goldbach’s theorem would be enormously enhanced. This suggestion is the less implausible in the face of Gödel’s new almost legendary researches. Gödel has shown that in any logico-axiomatic system strong enough to express the arithmetic of natural numbers there will always exist a true theorem that predicates a certain property of every natural number but which cannot be proved to be true on the basis of the axioms in the system. More precisely, the Gödelian theorem can very definitely be logically deduced from the axioms but only by means of an infinite proof. There are thus seen to be infinite proofs of theorems for which no finite proofs exist in the system. Gödel’s findings may be said virtually to cry out for the introduction of our Zeno procedure.

Very briefly, Gödel’s strategy is as follows. Given any logico-axiomatic system designed to express the arithmetic of natural numbers, we are to assign a unique index number to each proof (i.e. each finite proof) in the system: these we may style code numbers. Also, we are to assign a unique index number to each proof (i.e. each finite proof) in the system: these we may style proof numbers. Gödel now proves that within the system itself there exist the following formulas: (1) ‘the formula “\(x+2 = 4\)” has the code number 448’, and (2) ‘the proof number of the formula with code number 448 is 932’. Both 1 and 2 are perfectly innocent formulas, and they may be readily proved to be true or false as the case may be. One has merely to unpack the putative code and proof numbers 448 and 932. By an ingenious device, Gödel now demonstrates the existence in the system of an embarrassing formula with a certain code number, say 8,043, which reads: ‘There does not exist a proof number for the formula with code number 8,043.’ This is the famous Gödelian sentence.1

Assigned the code number 8,043, it asserts that there exists no proof number (and hence no finite proof in the system) of the formula with code number 8,043. If the G-sentence is true, then it cannot be logically deduced in any finite number of steps from the axioms of the system. If it can be so deduced, then it must be false. Moreover, a direct contradiction will be seen to result at once. Hence, assuming the system to be consistent, the G-sentence must be both true and unprovable, but unprovable only in the sense that no finite proof of the sentence exists in the system. Provable in the system it certainly is, as soon as we allow infinite proofs.

How will the infinite proof look? Something like this. Let \(P\) represent the following predicate: ‘not the proof number of the formula with code number 8,043.’ Then the infinite proof will read: ‘Since the number 1 is \(P\) (reasons given), and the number 2 is \(P\) (reasons given), and the number 3 is \(P\) (reasons given), and \ldots, and \ldots &c., therefore it follows (since 1, 2, 3, 4, 5, \ldots) that there does not exist any number \(z\) which is the proof number of replacement formula \(x\) of replacement formula \(y\).’

Within the system there will now be found the following formula: \(\sim(\exists z) Pz\). This formula states that there does not exist any number \(y\) which is the proof number of replacement formula \(z\). (It may be noted that \(z\) occurs as an unbound variable here.) The present formula may be suitably described as a proto-Gödelian sentence. It has a code number of its own, say 200. There will then exist in the system the key formula \(\sim(\exists y) P200y\). What does this formula state? It states that there does not exist any proof number of replacement formula 200. But what is replacement formula 200? It is the formula \(\sim(\exists y) P200y\). Hence the key formula \(\sim(\exists y) P200y\) states that there does not exist any proof number of the formula \(\sim(\exists y) P200y\). It will be evident that replacement formula 200 must both be true and unprovable in the system.
... &c., comprise all the natural numbers) that there exists no proof number of the formula with code number 8,043.' Consider now the formula which reads: 'Every natural number is P.' Predicating a property of every natural number, this true formula cannot be proved true by any finite proof in the system.

Given Gödel's results, is it not quite possible that Goldbach's theorem might be altogether unverifiable except on the basis of an infinite proof? And what of the four colour theorem? Let us suppose that, in reading through all the books in our metaphysical library, we find that no volume exists which supplies a finite proof of the theorem. We shall also suppose that there exists no volume in the library that details a procedure for constructing a counter-example to the theorem, i.e. a map containing at least thirty-eight regions which requires that five colours be employed if each region is to be assigned a colour different from that assigned to any neighbouring region. Are we not now entitled to conclude that the very absence of any volume-by-volume inspection of every book in our metaphysical library does not actually contain all possible English books. It does contain all finite English books, but not all books simpliciter. We are capable of writing certain important infinite books, witness the super-proof of Goldbach's theorem, to which no catalogue numbers have been assigned. In fact, it can be proved that it is theoretically impossible for a unique natural number to be assigned to each infinite book. This is not to say that each infinite book may not be assigned a unique index number of any kind whatever. We shall find that the real numbers will prove serviceable for that purpose. For the present, we have only to emphasize that our metaphysical library, as currently constituted, precludes all volumes of infinite length.

How serious is that limitation? Is our original programme in any way impaired? We have been seeking an effective omnibus procedure for solving all mathematical and even all philosophical problems. I do not think that we need be discouraged by the mooted possibility that perhaps for certain mathematical problems there simply do not exist anywhere, even in the mind of God, any finite proofs at all. A premium may reasonably be placed upon a finite proof only on the condition that it 'exists'. If we can actually prove, in a particular case, the non-existence of a finite proof, then the construction of an infinite proof will in that case be doubly significant, significant first for establishing the simple truth of the theorem and, second, for showing that it is true only per accidens, as it were, thereby verifying it as a cosmic accident in the realm of pure ideas. We may introduce the concept of a 'complex infinite proof' to signify proofs with that double significance. Such a complex infinite proof would be supplied in the case of Goldbach's theorem if we were to establish, first, the non-existence of any finite proof of the theorem in our metaphysical library (this very fact being established by means of an infinite bibliographical proof) and, second, the simple truth of the theorem. In regard to this second step, we have the option of adopting either a super-proof or an infinite bibliographical proof (this last showing that no counter-example exists). It may be noted that that option does not exist in the case either of the prime twin or of the four colour theorem. The four
colour theorem (assuming that no finite bibliographical proof exists) lends itself only to the infinite bibliographical proof: the super-proof is not relevant. The prime twin theorem lends itself only to the super-proof: the irrelevance here of the infinite bibliographical proof springs from what we have earlier styled as the 'logical' difference between the prime twin and the Goldbach theorems. The mere absence of any finite bibliographical disproof of the prime twin theorem does not entail, as it very definitely does entail in the case of Goldbach's theorem, the truth of the theorem.

It does not appear that the exclusion of all infinite books from our metaphysical library has in any way jeopardized the success of our programme. Not that we have been able to rest content with finite bibliographical proofs. Confronted with the possibility that finite proofs may be altogether unavailable in certain cases, we have had to fall back upon the infinite bibliographical proof as a pis aller (though we have shown how it may be converted into a kind of triumph). Our metaphysical library does not, of course, include any volumes which contain infinite bibliographical proofs. Each infinite proof that we construct will constitute a 'book' that lies outside of the library, though it will be related to the library in a peculiarly intimate way. Being directly parasitic upon it, it may be filed in a metaphysical annex to the library.

What now of philosophical questions? What if no decisive finite solution to the 'bisection' paradox should be available? Is there any prospect of constructing an infinite bibliographical solution to it? I think that it may be argued that no serious philosophic question can ever be decisively answered with anything less than an infinite argument. Let me explain. We all know that no philosophic thesis, however cogent, has ever been so copiously argued—be it in print or in speech—that all possible objections have been anticipated and refuted. The argument as expressly stated is always a mere abridgment of the full story. The reader is never relieved of the obligation to think for himself: he must exercise his wits toward elaborating

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1 Indeed, if we can prove that the four colour theorem is true only as a cosmic accident, then it will be verified as a synthetic a priori truth, i.e. as an apodictic universal proposition the denial of which is not self-contradictory.

the argument with such versatility that he finds himself capable of meeting the most various and wide-ranging objections that may be raised against it. Indeed, the test whereby we measure the depth of comprehension in one who affects to understand a philosop hic position is to expose him to a highly diversified battery of objections, many from unexpected quarters. The degree to which one understands a philosopher (be it Plato or Aristotle or Hume or Kant) is precisely the degree to which one is capable of meeting those objections. The mastery of a philosophic argument consists, then, in the acquisition of a settled habitus that empowers one to defend the position—not in a spirit of contention but with all the aplomb of sweet reason—against the widest possible range of attacks. The ideal limit, and surely the only indefeasible warrant that can conclusively confirm any philosophic theory, must be the inherent capacity, built into the theory, of meeting all objections. I do not wish to suggest that the full number of possible objections—"I mean serious objections—need be actually infinite. Let us grant that they are finite. But the great question is this. How can one ever know that he is capable of meeting every serious objection that might be proposed? This is no mere frivolous anxiety of a callow sceptic. Experience amply confirms the fact that theories which may be altogether fortified against all frontal attacks are only too liable to collapse in the heat of an indirect assault. How, then, can we plug up all the holes? Is there any way in which we might elaborate our thesis so copiously, so comprehensively that every possible objection is expressly anticipated and expressly refuted? Yes. We have only to consult all the books in our metaphysical library. One concentrated minute of intense research should suffice to enable us to canvass every possible objection to our thesis. The complete and exhaustive statement of our position—the master statement—will prove infinite in length. For it is not sufficient that all objections be answered; one must also show that none has been omitted. Hence all the books in our library must be passed in review one by one, and they must all be included in toto in our master statement. Ideally, every philosophic argument is thus seen to require an exposition of infinite length. It will be recalled that in discussing the highest kind of knowledge, on the fourth level of the Divided Line, Plato insists that 'unless one can run the
gauntlet of all objections ... he apprehends only a shadow, if anything at all, which is given by opinion and not by science’ (Rep. VII, § 534). Knowledge absolute would seem to be possible only through the exercise of our Zeno procedure. Unhappily, we must allow for the distressing possibility that there may not exist a single serious philosophic thesis which has in fact the power to ‘run the gauntlet of all objections’. Perhaps for every thesis there exists a counter-thesis of equal weight.

It may be protested that all this epistemological bru-ha-ha is quite unnecessary. Surely anything worth saying in philosophy can be said in less than a billion words (I am purposely leaving a wide berth). Now it can be easily proved that the sum-total of all possible books or discourses each consisting of a billion English words or less, must altogether be finite in number. We have only to admit that no serious objection to a philosophic thesis requires more than a billion words for its adequate statement, and it is evident that we have no need of our Zeno procedure in order to compile an exhaustive dossier of all possible objections to all possible philosophic theses. Furthermore, if we also admit that the answer to any objection will likewise never require more than a billion words, if follows that the sum-total of all philosophic statements and counter-statements can be effectively compiled in a finite duration of time quite apart from any procedure of metaphysical research. The whole philosophic enterprise is thus brought to a final end, there being nothing of any importance left to say. This sanguine prospect rests on some fairly dubious assumptions. To mention only one, it is here tacitly assumed that philosophy and science are radically disjoint. But much of philosophy is directly or indirectly parasitic on the results of science and mathematics. The simple statement of a philosopher that ‘final causes are no longer employed in modern natural science’ presupposes that all of modern natural science has been investigated in detail, if not by him personally, then by the community of scholars. The full elaboration of the philosopher’s statement will thus require more than a billion words if it is to be backed up by an exhaustive study of every single item in modern science. Moreover, the philosopher’s statement will not mean precisely the same thing a thousand years from now as it does today: modern science is an aggressively expanding system. It is highly doubtful that philosophy can ever be brought to a state of definitive perfection at any time prior to the perfection of natural science itself. But perhaps we can hasten the perfection of science by means of our Zeno procedure. Our metaphysical library certainly includes every possible scientific treatise, not omitting the detailed explanation of every possible scientific experiment.

Now it is not enough here to read through and meditate upon each of these treatises; one must actually execute all of the experiments. Apart from the experiments, the treatises are of no value. Why not discharge that programme in one concentrated minute of intense experimental research? Let us bring all of science, all of philosophy, and all of mathematics to one great triumphant conclusion. Already we have catalogued in our metaphysical library every poem, every novel, indeed every work of imaginative literature, that could possibly be composed in the English language.

Part 2: The Non-Denumerable

Is there any limit to our megalomania? There are doubtless many questions that may be raised here, many protests that may be directed against our metaphysical aggrandizement. I propose to explore only one of several lines of inquiry. It is surely impossible to ignore the immense consequences of Cantor’s profound diagonal theorem. For it is that theorem which constitutes the most immediate and most massive barrier against the onrush of our ambitions. Narrowly mathematical in its origin, it may be shown to have serious repercussions that extend far beyond mathematics proper. On the basis of Cantor’s theorem it follows that, though it is metaphysically possible for us to write down all of the natural and even all of the rational numbers by means of our Zeno procedure, it is altogether impossible—metaphysically impossible—for us to write down all of the real numbers. It is certainly easy enough for us to record the complete decimal expansion of √2. Let us assign 1/2 minute to the task. In the next 1/4 minute we shall record the complete decimal expansion of √3; in the next 1/8 minute, the complete decimal expansion of √5; &c. ad infinitum. At the
end of the minute we shall have succeeded in writing down an infinite number of complete decimal expansions. But not all of the real numbers or, what amounts to the same thing in the present context, not all of the irrational numbers will have been recorded. Thus \( \pi \) will be omitted from the list. This omission is neither surprising nor important. If I write down the complete Z-series of rational numbers, 'most' of the rational numbers will nonetheless remain unrecorded. Yet it is quite possible to record all the rational numbers. We have only to assign to each rational number a unique natural number as its index. Using 4/5 as an example, we first translate the fraction into our binary notation. 4/5 is now represented as \( \frac{1}{100} \). Next, we string together numerator and denominator (in that order), employing a 7 as a plug. The unique index number of 4/5 is thus seen to be 1,1,00. Now we have only to write down all of the natural numbers and, in checking off the index numbers as we encounter them, we shall translate them, one by one, into our standard notation. Thus we shall succeed in recording all of the rational numbers, not in order of magnitude certainly (for that is impossible) but nonetheless in a perspicuous serial order. Cantor has shown that no such procedure is possible in the case of the irrational numbers. The rational (like the even) numbers are said to be denumerably infinite. Being listed in a series, they can be placed in one-to-one correspondence with the whole series of natural numbers. They can thus be denumerated or 'counted' in an extended sense of the term. That the real numbers are non-denumerably infinite, that they cannot be placed in one-to-one correspondence with the natural numbers, is established by Cantor on the strength of the following lemma. For every denumerably infinite set of real numbers there exists a constructible real number that lies outside the set. Hence the class of all real numbers cannot be denumerable. Are we now entitled to infer, from the non-denumerability of the real numbers, that it is metaphysically impossible for us to write them all down? I wish to insist upon this point, though I do not suppose that the inference is at once self-evident simply on the strength of the diagonal theorem. Other theorems in transfinite arithmetic must be brought into the picture, and there are doubtless certain ontological presuppositions that lie in the background of the inference. I am certainly not capable of fetching all those presuppositions out into the foreground. I can only trust that a binding proof of the inference is available in our metaphysical library. One objection may be anticipated in advance. Let us write down an infinite number of complete decimal expansions in \( \frac{1}{2} \) minute, then in the next \( \frac{1}{4} \) minute another infinite series of complete decimal expansions, in the next \( \frac{1}{8} \) minute still another, &c. At the end of the minute we shall have not one but an infinite number of square arrays. Is there any possibility of recording all the real numbers be derived parasitically from the infinite diagonal of digits. Let the first digit of our new number be 5, seeing that the first digit of the first number in the array is 4; let the second digit of our new number be 7, seeing that the second digit of the second number is 6; let the third digit be 8, seeing that the third digit of the third number is 7, &c. In this fashion we shall construct a real number which must differ from each of the real numbers in the array in at least one decimal place. This new number, \( 578501 \ldots \), lies outside the square array. It is thus proved that for every denumerable set of real numbers there exists a constructible real number that lies outside the set. Hence the class of all real numbers cannot be denumerable.
by this method? None whatever. The diagram in Fig. 2 represents all of the square arrays that we have succeeded in
writing down in the minute. Each asterisk represents the complete decimal expansion of one real number. Each vertical
line of asterisks represents one square array. The path of the arrow in Fig. 3 indicates a rule for placing all of these real
numbers into one-to-one correspondence with the natural numbers. On the basis of that one-to-one correspondence we
can pack all of these real numbers into a single square array, and then we can construct a new real number by means of the
diagonal procedure. Considerations of this kind lead me to conclude that in the real numbers we at last encounter a
domain of entities that transcends even the Zeno procedure in all its scope and power. Cantor's diagonal theorem is thus seen
to be as much a metaphysical as a mathematical principle. This is not the first time that we have been obliged to recognize
that for us a proof of the G-sentence is available within the system. Although no finite proof of the G-sentence is possible in the system, an infinite proof of it is readily constructible. The existence of infinite proofs suggests at once the possibility of constructing infinite theorems as well, and it will not be difficult to exhibit an infinite theorem for which no proof whatever, not even an
infinite proof, is at all possible within the system. We are thus to construct what may be styled a hyper-Gödelian sentence.
It will be seen that this new construction, unlike Gödel's own, requires no ingenuity whatever. First, we must assign to every
infinite proof a unique proof number and to every infinite theorem a unique code number. We have already shown that
to each sequence of words there corresponds a unique sequence of natural numbers, and though we have hitherto considered
only finite sequences of words in this connexion, it is evident that to every infinite sequence of words there will correspond
an infinite sequence of natural numbers. Simply by prefixing a decimal point to any infinite sequence of numbers (allowing for
the 7's designed to plug up the gaps), we succeed in constructing a unique real number which will serve as the code
number or the proof number of the infinite sequence of words, be it an infinite theorem or an infinite proof. Whereas the
finite sentences in the system are assigned natural numbers, the infinite sentences are assigned real numbers. The hyper-
Gödelian sentence will now be constructed as follows: 'There does not exist a number, be it real or natural, which is the proof
number of the formula with code number 

\[ 407279757207573574771675721752157167277 \ldots \]

What precisely is this formula with code number '407279757207 &c.? If we unpack this code number we generate the following
formula: 'There does not exist a number, be it real or natural, which is the proof number of the formula with code number 

\[ 40727975720757357477167572157167277 \ldots \]

It is clear that if the hyper-G-sentence is true, then there cannot exist any proof of the sentence in the system, be it finite or infinite. On the other hand, if a 'proof' of the sentence does exist within the system, then not only must the sentence be false but the system itself must be infected with self-contradiction. Assuming the system to be consistent, it follows that the hyper-G-sentence is both true and incapable of being proved true in the system. Strictly speaking, it must be emphasized that the hyper-G-sentence does not assert its own unprovability (that is merely our interpretation). What the sentence does assert is simply that there does not exist a number with a certain defined property P, and here it may be asked why we cannot apply the same kind of procedure which we found to be so effective in the case of the

\[ \ldots \]

\[ \ldots \]
The original G-sentence asserts that there does not exist any natural number with a certain defined property P, and we verified the truth of that sentence simply by testing each of the natural numbers in turn. If we are now to verify the truth of the hyper-G-sentence, we must examine each of the real numbers in turn. More precisely, it will be necessary to examine not all of the real numbers but simply all real numbers of a certain sort, namely those which have a 7 for every second or third digit (these 7's being the plugs) and which never have two or more 7's occurring in succession. This sub-set of real numbers, those which are possible proof numbers, can be easily shown to be itself non-denumerable. The radical difference between the G-sentence and the hyper-G-sentence (the one being provable, the other not) is reflected in the fact that whereas all of the natural proof numbers are denumerable, the class of real proof numbers is non-denumerable. It is this non-denumerability of the real proof numbers that renders it metaphysically impossible for us to examine each of them in turn. Any proof of the hyper-G-sentence (if so wild a thought may be allowed to cross our minds) would have to be not only infinite but, above all, non-denumerably infinite in scope. It is not to be supposed that such a transcendent proof must be ruled out simply on the ground that a contradiction would result. There is no contradiction here. The hyper-G-sentence does not rule out any proof whatever of the formula with code number 40727975207573574... It merely rules out any denumerably infinite proof of the formula. Proof numbers have been assigned only to proofs of denumerable extent. Even if the possibility of non-denumerable proofs were to be admitted (say in the mind of God), it is clear that such proofs could not be assigned real proof numbers.

Having constructed the hyper-G-sentence, it would seem that (waiving the divine proof of non-denumerable extent) there could remain to us but one method for establishing the truth of the sentence. Indeed, we have already provided a conditional proof of the sentence. We have shown that the sentence must be true if the system is consistent. This leads us to the general problem of consistency. If we can succeed in establishing (outside the system) the consistency of the axioms of the system, then we shall have at hand a meta-proof of the hyper-G-sentence. This meta-proof will not be logically deducible from the axioms of the system, it will not be a proof in the system proper (hence there can be no embarrassing contradiction). It being evident that the hyper-G-sentence must be true if the system is free of contradiction, a meta-proof of the sentence results at once if we can certify that the axioms are mutually consistent. Is such a consistency-proof possible? Is there any way of guaranteeing in advance that a given set of axioms or premises does not contain a hidden inconsistency? Limiting ourselves to the finite formulas and finite proofs of the system, it will be seen that, being denumerably infinite in number, we can examine all of them in turn. Each finite formula and each finite proof being assigned a unique natural number, we can canvass all of the finite proofs in the system so as to determine whether there exists both a 'proof' of a formula \( p \) and a 'proof' of the formula \( \neg p \). Let us list all of the finite formulas in one column and all of the finite proofs in a parallel column. Beginning with the first formula in the list, we shall run down the column of proofs in an effort to find either a proof or a disproof of the formula. In the unhappy event that we track down both a 'proof' and a 'disproof' of the formula, the system will be exposed as inconsistent. In the absence of such a contradiction in the case of the first formula, we shall undertake to test the second formula in the list, then the third, &c. In this fashion we can establish the consistency of the system by means of an infinite argument. At the same time we have at our disposal a 'decision' procedure for deciding whether a given finite formula is neither provable nor disprovable in the finite sub-system. Such formulas—the original G-sentence is one—are known as 'undecidable' formulas. It will be easy for us, in the course of establishing the consistency of the system, to make a collection of all the (finite) 'undecidable' formulas. Shall we then be able to construct an infinite proof (or disproof) for each of the formulas (I am referring, of course, only to closed well-formed formulas) which has been found to be 'undecidable' in the finite sub-system?

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1 More precisely, we have been assuming that each infinite formula and each infinite proof is of order-type \( \omega \). Our discussion presupposes that restriction. Complications set in as soon as we lift that restriction and allow infinite proofs of any transfinite order-type, whatever.
This question obliges us to consider the relation between the finite sub-system and the infinite sub-system, and in investigating that relation, we find that the problem of consistency is rather more complex than it has appeared. In establishing the internal consistency of the finite sub-system, have we established the consistency of the system as such—absolutely? Were we able to answer this question in the affirmative, we would be entitled to conclude that we had succeeded in constructing a proof of the hyper-G-sentence. What we are asking is this. Granting that no finite theorem contradicts any other finite theorem in the system, might there not be a contradiction between, on the one hand, a finite theorem and an infinite theorem and, on the other hand, one infinite theorem and another infinite theorem? Consider the following case. Let there be in the system a finite proof of the finite theorem T which asserts the existence of a certain natural number (say the Vinogradoff number) with a certain property. Let this proof be non-constructive in character. Moreover, let there be absent from the system any finite proof which will enable us to pin down the number. Finally, we shall posit the existence of an infinite proof which, by scanning each of the natural numbers in turn, certifies that there is no natural number with the specified property. It is clear that the theorem T is quite worthless. A system infected with this kind of contradiction is called omega-inconsistent. Merely to establish the internal consistency of the finite sub-system, is thus seen to be manifestly insufficient. It is imperative that we secure omega-consistency as well. But how is that to be achieved? Certainly, we cannot canvas all of the infinite theorems in the system: they are non-denumerable. Now it may be felt that we can verify omega-consistency simply by devising an infinite proof for every non-constructive proof in the finite sub-system. Quite apart from other difficulties, there is the following important case to be considered. Let there be a non-constructive finite proof of a theorem which asserts the existence of a certain subset of the real numbers which possess a certain property (I am thinking particularly of a denial of the continuum hypothesis). We cannot be expected to canvass every subset of the real numbers, in the infinite sub-system, so as to underwrite the finite ‘proof.’ Yet it is quite possible that the finite ‘proof’ may be worthless: no such subset of real numbers may in fact exist.

Considerations of this kind lend force to Brouwer’s suspicion of non-constructive proofs. In the absence of a guarantee of omega-consistency, it is difficult to feel altogether confident about them. The threat of omega-inconsistency is especially significant in that it points up the close connexion between the finite and the infinite sub-systems: they are not to be thought of as simply adjoining one another in an external way.

In constructing the hyper-G-sentence, we have shown that in any consistent logico-axiomatic system strong enough to express the arithmetic of natural numbers there will always exist true formulas which are not only unprovable in the system but perhaps unprovable absolutely. This prospect of absolute unprovability follows upon the difficulty of establishing the absolute consistency of the system, both in its finite and its infinite sub-systems. Although the consistency of the finite sub-system has seemed easy enough to certify, even that limited consistency might be thought to be jejune when a contradiction might break out in the system viewed as a whole. But it is not only the case that there are infinite formulas which are unprovable, there may be also finite formulas which are unprovable, and I mean unprovable in the infinite as well as in the finite sub-system. Thus I can see no reason for assuming that a proof, be it finite or (denumerably) infinite, of the continuum hypothesis (or its denial) need be forthcoming anywhere in (or even out) of the system. The continuum hypothesis asserts the non-existence of any subset of the real numbers which is both (1) non-denumerable and (2) incapable of being placed in one-to-one correspondence with all of the real numbers. The continuum hypothesis may indeed be true, and yet no proof of it may be available, even through our Zeno procedure. Effective in dealing with the denumerably infinite, the Zeno procedure is inadequate for coping with the non-denumerable. Although it is quite possible that the continuum hypothesis might be both true and unprovable, it is equally possible that a proof of it, even a finite proof, might be discovered by us in our search through the system.  

1 Correction: it has just been established by Paul Cohen that the continuum hypothesis cannot be proved true on the basis of the standard axioms of set theory. In 1939 Gödel showed that the continuum hypothesis cannot be proved false on the basis of those same axioms. But one or the
a finite formula which we can guarantee to be unprovable? Here I would suggest the following device. The code number of the hyper-G-sentence we found to be easy to construct, and it should not be difficult to state the rule—a finite rule—whereby we generate the successive digits of that code number. Let that rule be called $R(h-g)$. Then having stated that rule, will there not be a finite formula in the system which reads, ‘There does not exist a number, be it real or natural, which is the proof number of the formula with the code number which is generated in accordance with the rule $R(h-g)$’? This abbreviated version of the hyper-G-sentence, being itself finite, will not have a real number as its own code number: its code number will be a natural number. I do not think that this abbreviated hyper-G-sentence can be provable in the system, even by an infinite proof. For if it were provable, it would seem that the hyper-G-sentence proper could then be logically deduced from it, and a contradiction would result. An even simpler method for constructing a finite hyper-G-sentence is as follows. We have only to recast the original G"odelian sentence so that it now reads: ‘There does not exist a number, be it real or natural, which is the proof number of the finite formula with code number 9,000’. The code number of that very sentence is found to be 9,000.

Although all of the G-sentences may seem too factitious to be interesting in their own right, there is at least one which must be taken seriously. This is the finite formula in the system which may be construed by us as asserting—indirectly—that the system is absolutely consistent. The consistency-formula (as we may call it) asserts that there do not exist any two numbers $y$ and $y_1$ (real or natural) such that $y$ is the proof number of some formula $p$ and $y_1$ the proof number of the negation of $p$. If the consistency-formula could be proved within the system, then all of the G-sentences would also be provable on the strength of it. We have seen that the G-sentences must be true if the system is consistent. That very hypothetical can be expressed (and even proved) within the system. The consistency-formula must then be unprovable within the system for it constitutes the antecedent of the hypothetical. Proving the antecedent means proving the consequent, the result being a contradiction.

Having failed in our projected programme of solving all mathematical problems, our metaphysical library must now be viewed in a new light. We saw that knowledge absolute requires that one have the capacity to ‘run the gauntlet of all objections’. What if there be certain objections that cannot be stated in less than an infinite number of words? We have already admitted the existence of treatises which are of an infinite length, why not also infinite objections to those treatises? If this suggestion has any sense in it, then we shall be obliged to canvass not only all of the finite but also all of the infinite books that might possibly be written in the English language. Otherwise, we cannot be certain that we are in a position to meet every objection that might be raised against us. Each infinite book being assigned a unique real number as its index number, the collection of all infinite books is seen to be non-denumerable. It will thus be impossible for us to write down all of these books, much less to read them. However many infinite books we may succeed in composing, there will always remain another infinite book that has yet to be written. Knowledge universal and absolute would seem ever to be receding from us into the transcendent horizon of the non-denumerably infinite.

other must be a true formula, either the continuum hypothesis or its denial (pace Brouwer). Unless some way can be found to prove the formula by revising or supplementing the axioms (say by the use of infinite techniques), the G"odel-Cohen theorem entails that a true formula exists which is both absolutely unprovable and synthetic a priori.
CHAPTER III

COSMOLOGY I

Part I: Empiricism and the World

The world is everything that is the case, it is the sum-total of what exists. What statement could be more self-evident, more luminously true, or more platitudinous? It is the supreme banality, so invincibly true as to be quite trifling. Precisely what we mean by the word 'world' is the sum-total of what exists, everything that is the case. True by definition, the statement is a mere tautology that cannot but invite a vast yawn. How deceptively! Lurking within that cosmological platitude there lies a whole nest of perplexities.

If the world is everything that is the case, if the world is the sum-total of what exists, then the world must be either finite or infinite, it is either limited or unlimited. Here, then, is a second platitude that follows upon the first as an immediate corollary: the world is either finite or infinite. Like the first, this second platitude would seem to be an evident tautology. By the rule of tertium non datur the world must be either limited in its extent or it must be not limited. Surely there can be no problem here. How can a tautology present a problem? We can only be puzzled by Hobbes’s remarks on this topic. Cum quæratur an mundus finitus an infinitus, nihil in animo est sub voce mundus. It would seem, then, that it is not the word 'infinite' which is senseless but rather the word 'world'. Is Hobbes denying that the world must be either finite or infinite? Is he proposing some third alternative? No. Can it be that he believes that there simply is no world at all? This last suggestion is doubtless wildly extravagant, but when we turn to Kant, we find precisely this conclusion being elicited. Kant expressly argues that, since it is unintelligible to suppose that the world is either finite or infinite, it must be the case that no objective world exists! Kant is the first philosopher in whom the world comes to sight as a problem, simply as world. Other philosophers had certainly put in question the external world but only qua external, not qua world. Only in Kant is the problem thematically central. In Hobbes it is certainly marginal, and yet his account, for all its incoherence, repays examination.

In a later chapter, entitled ‘Of the World and Of the Stars’, Hobbes writes as follows.

Now every object is either a part of the whole world or an aggregate of parts. The greatest of all bodies, or sensible objects, is the world itself; which we behold when we look round about us from this point of the same which we call the earth. Concerning the world, as it is one aggregate of many parts, the things that fall under inquiry are but few; and those we can determine, none. Of the whole world we may inquire what is its magnitude, what its duration, and how many there be, but nothing else. . . . The questions concerning the magnitude of the world are whether it be finite or infinite, full or not full; concerning its duration, whether it had a beginning or be eternal, and concerning the number, whether there be one or many; though as concerning the number, if it were of infinite magnitude, there could be no controversy at all. Also if it had a beginning, then by what cause and of what matter it was made; and again, from whence that cause and that matter had their being, will be new questions; till at last we come to one or many eternal cause or causes. And the determination of all these things belongeth to him that professeth
the universal doctrine of philosophy, in case as much could be known as can be sought. But the knowledge of what is infinite can never be attained by a finite inquirer.¹

In this account the great cosmological questions are at least acknowledged to be intelligible: they may be said to 'fall under inquiry', and if 'as much could be known as can be sought', 'the determination of all these things' would be secured by the 'universal doctrine of philosophy'. Unhappily, in regard to the universe as a whole, there is a radical disproportion between 'the things that fall under inquiry' and 'those we can determine', the former being 'but few', the latter being 'none'. But if the cosmological questions are allowed to 'fall under inquiry', it can no longer be true that, in raising these questions, nihil in animo est sub voce mundus. We are not speaking mere gibberish. Our mind is not reduced to a blank. There must be some meaning attached to the vocable mundus. Moreover, it is only the 'finite inquirer' who is debarred from 'the knowledge of what is infinite', not the inquiring mind as such. In the earlier chapter Hobbes seemed to be saying that we could not so much as ask, much less answer, the cosmological questions. In the present discussion he seems to allow that the questions may intelligibly be asked.

What precisely renders the cosmological questions so desperate? For Hobbes, as for Kant, the source of all the difficulties lies in the problematical character of the infinite; and sharing Hobbes's conviction, Locke protests that we have only a 'positive idea towards infinity' and that 'what lies beyond', the actual infinite as such, 'lies in obscurity and has the indeterminate confusion of a negative idea'. At the very opposite extreme, Descartes argues in his third Meditation that not only do we possess a 'clear and distinct idea' of the actual infinite but that our idea of the infinite must be logically and even epistemologically prior to our idea of the finite. As the imperfect is intelligible only by way of negation of the perfect, so, too, Descartes holds that the finite cannot be understood as finite except by way of negation of the infinite. Epistemologically speaking, the perfect and the infinite are on a par in contrast to their negations, the imperfect and the finite. Descartes was doubtless aware that, nominally, it is of course the infinite which serves as the negative of the finite, not the other way around, which he insists is the proper order. Our language is thus seen to be systematically misleading in regard to the finite-infinite dichotomy, though altogether satisfactory in regard to perfect and imperfect. It is not surprising that Descartes should insist that our concept of the actual infinite must be an 'innate idea' which cannot be derived from any empirical evidence. We are not able to recognize anything in our experience as finite unless we can apprehend it as being limited, as being bounded. But we cannot think of anything as bounded unless we have a prior notion of what it is for a thing to sweep on altogether free of limits. A clear and distinct idea of the infinite is thus held to be an a priori condition presupposed by all experience of the finite as finite.

I suspect that Descartes' argument could only have proved very embarrassing to Locke in his effort to ground all of our ideas in experience. Locke writes as follows:

Finite, then, and infinite, being by the mind looked on as modifications of expansion and duration, the next thing to be considered is—How the mind comes by them. As for the idea of finite, there is no great difficulty.... The difficulty is, how we come by those boundless ideas of eternity and immensity, since the objects we converse with come so much short of any approach or proportion to that largeness.

What is Locke's answer? How do we come by our idea of the boundless? As if to vindicate Descartes, Locke finds himself obliged to confess that on the basis of experience we cannot arrive at that idea at all. The best we can achieve is 'the indeterminate confusion of a negative idea'. Despite this 'indeterminate confusion' in our idea of the infinite, Locke is no more prepared than Hobbes to insist that the partial, if not total, unintelligibility with which our idea of the infinite is infected, entails the necessary consequence that the world must be conceived to be finite. Agreeing with Hobbes, Locke cannot rule out the possibility that the world is actually infinite. How then can he deny that he possesses an idea of that very possibility which he is able to entertain?

As between Descartes and Locke, I feel that it is Descartes who has had the better of the argument; but I am also persuaded that

¹ Elements of Philosophy, ch. 26, § 1.
the truth lies somewhere between the two. Descartes is right: our idea of the finite presupposes an idea of the infinite. But it must also be added that any idea of the infinite equally presupposes an idea of the finite. Finite and infinite are seen to be correlative notions, neither being intelligible apart from the other. Consider the following case. We are confronting a wall that extends far beyond our field of vision both to the right and to the left. We raise the question, does the wall come to an end or does it continue on and on without limit? No empiricist will wish to say that he knows the answer to this question a priori. The logical empiricists of our own day, who flatter themselves on having advanced far beyond Locke, will refuse to be embarrassed by the hypothesis that the wall might in fact be infinite. They will accept the hypothesis quite readily as being perfectly intelligible, on the ground that, if it is not empirically verifiable, it is certainly empirically ‘falsifiable’, and that is quite sufficient. They demand of any hypothesis no more than that it be such that evidence might in principle be adduced either to support it or to refute it.

Very different in their eyes is the cosmological question, is the world finite or infinite? This question will be ruled out as ‘metaphysical’ in an opprobrious sense. It is written off as altogether meaningless, on the ground that no conceivable evidence might be adduced either in support or in confutation of either alternative. In the case of the wall, it make sense to suppose that it might be infinite, for if it were not infinite, evidence establishing its finitude might be readily forthcoming. In the case of the world, however, they insist that it makes no sense to suppose that it, too, might be infinite, for no empirical evidence could possibly be adduced, not merely to confirm, but even to refute the hypothesis. It is thus held to be senseless either to suppose that the world is infinite or to suppose that it is finite. I submit that this advanced empiricism is no less incoherent than Locke’s. The empirical thesis that the wall is infinite logically entails the cosmological thesis that the world is infinite. If it is meaningful and intelligible to suppose that the wall is infinite, then it must also be meaningful and intelligible to suppose that the world in infinite. Precisely what we mean by an infinite world is a world in which an infinite wall might exist. If an infinite wall exists in fact, then the world must be big enough to accommodate it. I am not suggesting that there is no ‘logical’ difference between the empirical thesis that the wall is infinite and the metaphysical thesis that the world is infinite. Let us grant that the metaphysical thesis is both unverifiable and unfalsifiable, in contradistinction to the empirical thesis which admits of falsification even as it defies verification. I insist only that if the empirical thesis is meaningful, the metaphysical thesis must also be meaningful: the one logically entails the other. In the face of these logical facts, I cannot see how one can persist in pinning down the meaningful to what is merely open to empirical evidence.

This conclusion should come as no surprise. Much attention has been addressed to the thesis that all crows are black. Empiricists have insisted that it is in principle unverifiable. Granting even that one has succeeded in examining every crow, how could one ever know that every last crow has in fact been canvassed? Unwilling to write the thesis off as meaningless, the empiricists enlarge their criterion of intelligibility so as to accommodate the empirically falsifiable. The thesis that all crows everywhere are black, is thus said to be meaningful, on the ground that if it were false one might succeed in establishing its falsity by exhibiting a white crow. (A false proposition cannot be meaningless.) How did this whole problem arise in the first place? It arose because we live in an open universe. It is easy enough (in principle) to verify that all crows in Mexico are black. It is only when we insist that all crows as such—all crows in the universe—are black that we confront a great cosmological problem. The empirical order is seen to open up on to the cosmological order, the one falls within the other. It is vain to attempt to snap the connexion between the two. The empirical impossibility (if it is an impossibility) of verifying that all crows everywhere are black, is strictly derivative from the cosmological problematic.

The logical route that leads us to the recognition of a cosmological order that transcends the empirical, is found in the tension between two principles that are equally cherished by contemporary analytic philosophy. On the one hand, we have an empirical theory of meaning and, on the other hand, we have the logical requirement of a contrast theory of meaning. These two principles are essential to any logical empiricism.
According to the first principle, a concept is meaningful only if it is capable of some possible empirical application. According to the second principle, a concept is meaningful only if its logical opposite is also meaningful. These two principles prove to be incompatible in the crucial case of the finite-infinite dichotomy. Our concept of the finite is capable of a wide range of empirical employment—the finite wall is but one case in point. Very different is its logical opposite, our concept of the infinite. It is this concept which inherently transcends the empirical order (in the small as in the large) and opens up to us the logical prospect not only of an infinite wall but of an infinite universe as well. Thence arises the cosmological problematic, for if an infinite universe must be acknowledged as an intelligible possibility, then its logical opposite, a finite universe, must be equally intelligible; and yet neither of the two would seem to allow of any empirical employment for the concepts finite and infinite.

We may say that the correlative terms finite and infinite are peculiar in that one of the terms denotes an empirical, whereas the other denotes a metaphysical, concept. If we contrast the hard empirical thesis that this wall before me is finite and the clearly metaphysical thesis that the universe is infinite, it follows that the denial or falsity of the empirical thesis logically entails the truth of the metaphysical thesis. The empirical is seen to be logically bound up with the metaphysical. Does the world have a beginning in time? Here is another cosmological question which, since Kant at least, has been ruled out by empiricists as meaningless. Not content to adopt a modest scepticism which confines itself to insisting that no evidence could possibly be forthcoming to settle the question, one way or the other, our logical empiricists have undertaken to anathematize the very question itself as spurious. Once again a metaphysical thesis, the thesis that the world is without a beginning in time, may be shown to be logically entailed by an empirical thesis or, at least, by the denial of an empirical thesis. The metaphysical thesis may be both unverifiable and unfalsifiable, but it is certainly entailed by the denial of the simple empirical thesis that this oak tree had a beginning in time. It is easy enough to verify that this oak tree had a beginning in time, and if its denial is admitted to be intelligible, namely that this oak tree has always been in existence, then one has already envisaged the possibility that the world itself is without a beginning in time. No one (least of all an empiricist) will wish to say that he knows a priori that the sun and the moon and the stars must necessarily have come into existence at some time in the past. But this is equivalent to the admission that it is altogether intelligible to suppose that they may have always been in existence, which in its turn is equivalent to the further admission that the world itself may perhaps be without a beginning in time. That man can only be engendered by man, is a perfectly humdrum empirical thesis that commonsense accepts; but it logically entails that the world can have no beginning. For if I was engendered by my parents, they in turn must have been engendered by theirs, &c. ad infinitum. Finite and infinite being correlative concepts, the empirical is seen to be unintelligible apart from the cosmological.

Although Kant is the first philosopher expressly and thematically to rule out the cosmological horizon as inherently meaningless, he is in some measure anticipated by Hobbes and Locke in their insistence that we can have no idea of the actual infinite. Leibniz protests, quite rightly, that this is a mistake—we must not confound having an idea of something with having an image of it. Certainly, we cannot imagine anything to be actually infinite, in the sense of picturing it in toto in our mind's eye; but we are quite capable of having an idea of the actual infinite, in the sense of supplying a precise definition of it.1

When we entertain the possibility that there may be an infinite number of stars in the heavens, we may mean by 'infinite' any one of the following: (1) the sequence of stars does not come to an end—there is no last star, (2) for every star there exists another star that lies beyond it, (3) for any natural number \( n \), there exist \( n \) stars, (4) there exists a relation of one-to-one correspondence between all stars and a proper sub-set of all stars. These four definitions are not all on a par: the first alone is couched in negative terms; the second, third, and fourth are couched in positive terms. When Locke insists that we have only a negative idea of the infinite, he is doubtless thinking only of the first definition, and when Descartes insists that we have a

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1 New Essays, II, ch. 29, § 15.
positive idea of the infinite, he may be thinking of any one, or all, of the others. Nominally, the word 'infinite' is the negate of the word 'finite', but logically it is not difficult to reverse the relationship, so as to view the infinite in a positive light. When we say that there may be only a finite number of stars in the heavens, we may mean by 'finite' any one of the following: (1) the sequence of stars does come to an end—there is a last star, (2) there exists a star such that no star lies beyond it, (3) there exists a natural number \( n \) such that it is not the case that there are \( n \) stars, (4) there does not exist a relation of one-to-one correspondence between all stars and any proper sub-set of all stars. Here it is the finite which proves to be the negative idea, in three out of four of the definitions. Although we are tempted to regard the actual infinite as being, at best, a highly problematical concept, it would seem to be as readily susceptible of a \( \lambda yos \), a rational account, as its correlative, the finite: the one is as intelligible as the other. Equally intelligible, they are not, however, equally empirical. Indeed, it is only the concept of the finite which would seem to lend itself to any empirical application.

The preoccupation with empiricism leads almost inevitably to the Kantian denial of the cosmological horizon. If a concept is intelligible only if it is capable of being cashed in experience, then it becomes difficult to see how the very concept of the world as such can be retained. Is the world—the world as a whole—a possible object of experience? This question is equivalent (for our purposes) to the following: is it possible to verify whether the number of stars which exist are finite or infinite? To answer this question in the negative, as Kant does, is not in itself surprising. Kant is guilty of an enormity only at the point when he draws the stupefying conclusion that the world simply does not exist! The argument may be blocked out as follows. (1) The world as a whole is no possible object of experience; (2) Only what is a possible object of experience may be rationally supposed to exist; \( \therefore (3) \) The world as a whole may not be rationally supposed to exist. We may suggest that very much as Hume subjects the concept of causality to the test of experience and finds it deficient, so, too, Kant subjects the concept of \( \text{world} \) to the same test and finds it equally or, rather, far more grievously defective—at least there is empirical backing for our concept of causality. Kant is quick to insist that, in his denial of objective reality to the world, he is not lapsing into any mere subjective idealism: he does not wish to say that 'everything is in our mind'. His is rather a 'critical' idealism. Kant does not doubt that we have empirical evidence that entitles us to assert that tables and chairs and stars really exist. He insists only that it is unintelligible—in the last analysis—to suppose that these things exist in themselves, independent of mind. They are all essentially mind-dependent. However bizarre the Kantian position may be, I would suggest that it continues to infect contemporary analytic philosophy, albeit in a clandestine form. No one, certainly, has argued against idealism with greater force than Wittgenstein, and yet it may be seen to persevere in his thought on a subterranean level. Consider the following discussion.

Suppose children are taught that the earth is an infinite flat surface; or that God created an infinite number of stars; or that a star keeps on moving uniformly in a straight line, without ever stopping.

Queer: when one takes something of this sort as a matter of course, as it were in one's stride, it loses its whole paradoxical aspect. It is as if I were to be told: Don't worry, this series, or movement, goes on without ever stopping. We arc as it were excused the labor of thinking of an end.

'We won't bother about an end.'

'It might also be said: 'for us the series is infinite.'

'We won't worry about an end to this series; for us it is always beyond our ken.'

Wittgenstein is very decidedly not prepared to take in stride the possibility that there may exist an infinite number of stars. Nor is he willing to infer from the 'paradoxical aspect' of the infinite (whatever that 'paradoxical aspect' might be: he does not explain) that there can only exist a finite number of stars. As with Kant, so with Wittgenstein, the cosmological order is ruled out as unintelligible. For Wittgenstein, no clear sense can be assigned either to the hypothesis that there are an infinite number of stars or to the counter-hypothesis that there are only a finite number of stars. To be sure, a \( \text{use} \) may be assigned to the locution 'there are an infinite number of stars', but that use will

\[1\text{Foundations of Mathematics, IV, § 14.}\]
be purely procedural—certainly not ontological—in import. Thus the locution may be employed to mean, 'We won't worry about an end to this series; for us it is always beyond our ken'. Wittgenstein's position is seen to be a clandestine form of idealism.

If there is an objective world that exists in its own right whether or not we apply our minds to it, if this objective world is the natural world of commonsense that we all accept, then it is not only true that objectively real stars exist in the heavens but it must also be the case that the sequence of stars is either finite or infinite—there is either a last star or every star has a successor. So, too, if there is an objective wall that confronts us, that wall must be either finite or infinite. To rule out the cosmological horizon is tantamount to ruling out the natural world as an objective thing in itself. Kant insists that we replace our cosmological mode of discourse by an experiential or operational mode of discourse. Experience exhibits to us a brick wall. Further experience may disclose to us an end to the wall or it may unfold for us—indefinitely—a protracted continuance of the wall. All that is allowable. At any given time the range of our experience is finite, and it may be enlarged indefinitely. We must not, however, speak of the wall as an objective thing in itself. To speak of the wall as an objective thing in itself entails the consequence that either the wall is actually finite or it is actually infinite—either a finite or an infinite number of actual bricks compose the wall. Kant insists that we cease to speak of τὰ δύναμεν we must now speak rather of τὰ φαινόμενα. It is not merely the cosmological order that is ruled out, the ontological order is equally dismissed. The bricks in the wall are no longer viewed as objective entities existing in their own right. As objective entities, the bricks must be either finite or infinite in number. In ruling out the actual infinite, our logical positivists, our operationalists, even Wittgenstein himself, are all clandestine Kantians in disguise.

I am urging that we accept 'as a matter of course' the possibility that there are an infinite number of stars and, a fortiori, that the world is infinite. Why not? To deny this possibility is equivalent to lapsing into idealism unless one is prepared to argue that it is an a priori truth that only a finite number of stars can possibly exist. The cosmological order is seen to infect the empirical. Our attitude to this brick here in this wall is radically transformed if we rule out the cosmological horizon: no longer are we able to regard the brick as a thing in itself, existing in its own right independent of any experience that we may have of it. Limiting oneself to the empirical issues in a reduction of the empirical to the phenomenal. Our attitude to reality in the large is not unrelated to our attitude to reality in the small. On the one hand, there is the ontological status of the world as a whole to be considered; on the other hand, there is the ontological status of this brick to be considered. The status of the one determines the status of the other. It is a great mistake to think that we can remain hard realists in regard to the brick and at the same time rule out the cosmological horizon. Hard realism in regard to the brick presupposes a recognition of a world that transcends the empirical.

Part 2: Hilbert and Kantian Idealism

In urging the intelligibility of the actual infinite, I have no intention of denying that there is a certain 'paradoxical aspect' or at least quasi-paradoxical aspect connected with it. There is a beautiful story of Hilbert's that points up the oddity (to put it mildly) of the whole conception. In his lectures on infinity, Hilbert, himself a finitist, as we have seen, recounts the following episode that occurred at the Grand Hotel. The Grand Hotel contains an infinite number of rooms. One night a traveller applied at the hotel for a room, only to be told by the desk clerk that, as every room was occupied by a lodger, no accommodations were available. Fortunately, the manager overheard the conversation and, intervening, suggested that perhaps something might be done. Lifting the office telephone (which connected him at once with all the rooms in the hotel), the manager made the following announcement: 'It is regretted that, due to unforeseen circumstances, each lodger will be required to vacate his present room and remove to the next (which connected him at once with all the rooms in the hotel), the manager made the following announcement: 'It is regretted that, due to unforeseen circumstances, each lodger will be required to vacate his present room and remove to the next room further up the hall. The man in room 1 will move to room 2, the man in room 2 will move to room 3, and so forth.

In general, if your room number is \( n \), you will move to room number \( n+1 \). Turning now to the traveller, the manager said, ‘I think that we shall be able to accommodate you in room number 1.’ The mind cannot but boggle at this tour de force. Hilbert may be forgiven for having ‘conceded that the propositions of classical mathematics that involve the completed infinite go beyond intuitive evidence.’ Surely the Grand Hotel is only a joke. How can we be expected to take it seriously? And yet, if we are to admit the ontological possibility that there may be an infinite number of stars, we must equally admit the possibility of a hotel with an infinite number of rooms. We shall then be obliged to accept Hilbert’s story in full earnestness.

Viewed in a purely abstract fashion, the Grand Hotel would seem to be no more than an anecdotal embellishment of Cantor’s theory of the transfinite. Actually, it is a great deal more. For the first time we are exposed to the ontological marvels of the actual infinite. When one is engaged in pure mathematics in the modern vein, nothing is further from one’s mind than ontology. One learns how to subtract 8 from 5, one learns how to extract the square root of \(-4\), one learns that \( \aleph_0 + 1 = \aleph_0 \). These marvels, these mysteries which dazzle us at first—when we are still naive enough to suppose that mathematics is an ontological science—prove to be mere banalities upon closer inspection. Growing sophisticated, we come to think of mathematics as a system of conventions: certain arbitrary definitions being laid down, certain conclusions can be logically deduced from them—that is all. Now it is quite true that after one has been initiated into the inner sanctum of the imaginary numbers, one discovers that the shrine is empty: there is nothing at all marvellous about \( \sqrt{-4} \). One is tempted, then, to assimilate all of mathematics to the model of the imaginaries. Henceforth, case-hardened and blasé, one is fortified in advance against any kind of wonder in mathematics. Even Cantor comes to be readily digested in turn. It all seems to be of one piece. I believe that this approach is a mistake. Mathematics comprises a ‘motley of techniques’, and the beauty of the Grand Hotel lies in the fact that an ontological marvel is here being presented which—quite genuinely—merits all of our wonder. Unfortunately, in Cantor’s own formulation of the actual infinite the true wonder of it all is very much disguised by his tropological use of language. We are expected to see that the number of lodgers accommodated in the hotel is precisely the same both before and after our traveller has been assigned a room, this in the teeth of our natural inclination to say that the number or multitude of lodgers has come to be greater—by one—than what it was before. At the same time we cannot deny that the number of rooms remains exactly the same. Every room that was previously occupied is still occupied, and certainly there was no vacant room in the hotel which is later pressed into service.

In our earlier discussion of Cantor we noted that there are operative in our mother-tongue diverse criteria which may be employed to establish that two collections \( A \) and \( B \) contain (or do not contain) an equal number of elements. In that discussion we failed to mention the very strongest criterion of all, namely, that two collections \( A \) and \( B \) will be said to contain the same number of elements if every element of \( A \) is an element of \( B \) and every element of \( B \) is an element of \( A \). It is this criterion which obliges us to admit that the number of rooms in the Grand Hotel remains exactly the same before and after. But if this is admitted, must we not also confess that Cantor is right after all, that the number of lodgers must remain exactly the same? No. We may choose to persist in our conviction that the number of lodgers has grown greater by one. Unhappily, this recalcitrance in the face of Cantor’s pressure entails a severe penalty: we are then obliged to admit that the same number of rooms—indeed, the same rooms—may now accommodate one number or multitude of lodgers and now a different, and even greater, number or multitude of lodgers, though in both cases, before and after, there is a strict one-to-one correspondence between the collection of rooms and the collection of lodgers. If one is tempted to protest here that this account is incoherent and senseless, he must remember that, however incoherent and senseless it may be, it is no more incoherent, it is no more senseless than Cantor’s own account. I am not denying that, taking the long view, Cantor’s approach will be seen to be the lesser evil of the two. I insist only that it is not literally, but (at best) only tropologically, the case that \( \aleph_0 + 1 = \aleph_0 \). It is interesting to note that the operator \( + \) as it occurs in the formula retains the exact same meaning that it
bears in the formula $5 + 1 = 6$. The difference lies in the operator $\times$. It is here that there is a concept-shift.

It is not to be supposed that I am merely rehearsing our earlier discussion of Cantor in which, having laid bare the tropological character of his discourse, I left the impression that once the hard core of his position was stripped of all excrescences the residue would be found to be altogether banal. That could only be a mistake. It is precisely at the point at which we succeed in *discounting* what is tropical in his account, precisely at that point are we in a position at last to marvel at the true prodigy. Therein lies the great virtue of the Grand Hotel. Stripped of all poetry, the sheer prose of the episode cannot but dazzle the imagination. The real force of the formula 'aleph-null $+ 1 = $ aleph-null' is to be found enacted in the Grand Hotel, it is not to be found in the trope whereby the concept $\times$ is assigned a super-sense, a sense parasitic, certainly, on its pristine meaning but at the same time significantly divergent from it. Very different are the imaginary numbers. We are amazed to learn that there exists a number which when multiplied by itself proves to be equal to $-4$. Extraordinary! Later we discover that what is meant here by 'number' and what is meant here by 'multiplication' is something radically distinct from what we originally meant by 'number' and 'multiplication'. In this case, once the conceptual trope has been discounted, nothing remains, certainly nothing of any ontological import remains.

Two types of number may thus be distinguished: ontological numbers and operational numbers. The irrationals and the transfinite cardinals are examples of the first. The negative and imaginary numbers are examples of the second. Viewed formally, the theory of the transfinite and the theory of the imaginaries would seem to be on a par. Definitions being stipulated, conclusions follow in both cases. Viewed ontologically, the two theories are strikingly different. In being introduced to the imaginary numbers, we are not learning anything substantive of which we were previously ignorant. This is not to deny that they are very useful for purposes both of theory and of practice. They are of value as a technique. Quite different is the Grand Hotel—here our eyes are being opened to an ontological possibility to which we were previously blind.

That possibility may be further elaborated upon in the following sequel to Hilbert's story. Each of the lodgers in the Grand Hotel has but one small glass of water, he has no other drink and in fact let us posit that there is no other drink available in the whole universe. Is the human race to be despaired of in this crisis? Is everyone to die of thirst? Not if we assume that a man is able to travel at any finite velocity that he pleases. Each man can be supplied with one million glasses of water! If we appeal to what I have called the 'very strongest criterion' of equality, we must admit that the number of glasses after the redistribution is precisely equal to the number before; and yet it would not be incorrect to say in another sense (what sense?) that the number has greatly increased, indeed that it has increased a million-fold. Actually, we can even undertake to assign to each lodger no less than an infinite number of glasses of water. For any infinite collection can be decomposed into an infinite number of disjoint sub-sets each of which contains an infinite number of elements.

Have we here a *reductio ad absurdum* of the very idea of the actual infinite? To be sure, the concept of the actual infinite is free of any internal self-contradiction in the strict sense—otherwise we could affirm *a priori* that there are only a finite number of stars—but seeing that the actual infinite entails the farce of the Grand Hotel, we may be almost irresistibly driven to adopt some kind of Kantian idealism as a pis aller. The problem is, how to rule out the actual infinite without insisting that only a finite number of stars can possibly exist; and it is here that idealism offers a ready solution.

The objects of experience, then, are never given in themselves, but only in experience, and have no existence outside it. That there may be inhabitants in the moon, although no one has ever perceived them, must certainly be admitted. This, however, only means that in the possible advance of experience we may encounter them. For everything is real which stands in connection with a perception in accordance with the laws of empirical advance. They are therefore real if they stand in an empirical connection with my actual consciousness, although they are not for that reason real in themselves, that is, outside this advance of experience. . . . If we regard the two propositions, that the world is infinite in magnitude and that it is finite in magnitude, as contradictory opposites, we are assuming that the
world, the complete series of appearances, is a thing in itself that remains even if I suspend the infinite or the finite regress in the series of its appearances. If, however, I reject this assumption, or rather this accompanying transcendental illusion, and deny that the world is a thing in itself, the contradictory opposition of the two assertions is converted into a merely dialectical opposition. Since the world does not exist in itself, independent of the regressive series of my representations, it exists in itself neither as an infinite whole nor as a finite whole. It exists only in the empirical regress of the series of appearances, and is not to be met with as something in itself.\(^1\)

Speaking quite personally, I may say that I am so much out of sympathy with idealism that, given the choice between idealism and the Grand Hotel, I should not hesitate to plump for the Grand Hotel. We are confronted with a choice between two oddities, and of the two I feel that the Grand Hotel is the less objectionable. Others will doubtless feel differently. Certainly, if one has already been attracted to idealism, say in connexion with the problem of perception, then our present discussion will be likely to confirm him in his sentiments. I doubt if anyone would be prompted to opt for idealism on the strength of the Grand Hotel alone. Our natural acceptance of the world as a thing in itself is so very powerful that we should be willing to relinquish our ‘transcendental illusion’ only at pistol point. Is that pistol perhaps to be found in our idea of the actual infinite? There is a curious ‘contradiction’ embedded in our natural understanding of the world. From Aristotle to Locke to Kant to Wittgenstein the actual infinite has been ruled out as unintelligible not only by many deep philosophers but also, and above all, by the plain man when it is brought to his attention. An actual infinite number of stars, is that really conceivable? The plain man invariably protests that the very thought of that enormity lies beyond his powers. On the other hand, if asked whether the stars in the heavens at some point come to an end or whether they proceed on and on without limit, the plain man will certainly reply that he cannot be expected to know the answer to that question. This very rejoinder—‘I do not know’—presupposes here the intelligibility of the actual infinite. Commonsense is thus seen to admit

of the actual infinite as a hidden presupposition of its acceptance of the natural world, but when that hidden presupposition is brought forth into the light, commonsense shrinks from it like the plague, and when the full consequences are spelled out, as in the story of the Grand Hotel, the plain man must confess himself baffled. He does not wish to allow the intelligibility of the Grand Hotel nor does he wish to insist that he knows a priori that there are only a finite number of stars, and yet he must opt either for the one or the other if he is to persist in his conviction that the world is a thing in itself. At this point he becomes ripe for idealism.

Although it falls outside the compass of these studies to adjudicate the dispute between idealism and realism, I believe that I am able to show that idealism, even granting it to be true, is incapable of banishing the actual infinite. The actual infinite can be reinstated within idealism itself. What seems to be a real option, the choice between idealism and the Grand Hotel, proves to be no option at all. Toward establishing this thesis it will be found to be convenient to explore a further strand in Kant’s thought—the celebrated antinomies.

Kant’s insistence that the world is neither finite nor infinite (and hence not a thing in itself) springs from at least two distinct lines of inquiry.\(^1\) First, there is the ontological principle of empiricism—to be is to be a possible object of experience. There being held to be no possible experience which is capable of exhibiting the world either as finite or as infinite, it is then concluded that the world cannot be either finite or infinite. Kant’s answer to the ontological question (What is it to be?) dictates his answer to the cosmological question (Is the world finite or infinite?) Logically independent of this first line of approach is a second line of argument, derived from the First Antimony.

From this antimony we can . . . obtain . . . [an] indirect proof of the transcendental ideality of appearances—a proof which ought to convince any one who may not be satisfied by the direct proof given in the Transcendental Aesthetic. This proof would consist in the following dilemma. If the world is a whole existing in itself, it is

either finite or infinite. But both alternatives are false (as shown in the proofs of the antithesis and thesis respectively). It is therefore false that the world (the sum of all appearances) is a whole existing in itself. From this it then follows that appearances in general are nothing outside our representations—which is just what is meant by their transcendental ideality.\footnote{Op. cit. p. 246.}

It is, above all, through his doctrine of the ‘antinomy of pure reason’ that Kant proposes to blot out the cosmological horizon. The ‘system of cosmological ideas’, in Kant’s own words, is inherently shot through with ‘pseudo-rational assertions’.

Whether we maintain that ‘the world has a beginning in time and is also limited as regards space’ or that ‘the world has no beginning in time, and no limits in space, being infinite as regards both space and time’; whether we maintain that ‘every composite substance in the world is made up of simple parts, and nothing anywhere exists save the simple or what is composed of the simple’ or that ‘no composite thing in the world is made up of simple parts, and there nowhere exists in the world anything simple’; whether we maintain that ‘causality in accordance with laws of nature is not the only causality from which the appearances of the world can one and all be derived, [but that] to explain these appearances it is necessary to assume that there is also another causality, that of freedom’ or that ‘there is no freedom, since everything in the world takes place solely in accordance with laws of nature’; whether, finally, we maintain that ‘there belongs to the world, either as its part or as its cause, a being that is absolutely necessary [God]’ or that ‘an absolutely necessary being nowhere exists in the world, nor does it exist outside the world as its cause’—whether we maintain one or the other, thesis or antithesis, in all these cases we are confronted with ‘pseudo-rational assertions’ that arise out of ‘the proud pretensions of reason when it strives to extend its domain beyond all limits of experience’. The ‘system of cosmological ideas’ degenerates into a self-stultifying ‘conflict of the transcendental ideas’. Metaphysics has been traditionally divided into three branches, ontology, cosmology, and theology. Cosmology is central. We have already seen how ontology and cosmology are bound up together, and it is evident from the four antinomies that the system of cosmological ideas opens up onto the theological question.

In our preoccupation with the first antinomy, we must examine at close quarters Kant’s ‘proof’ that the world cannot possibly be infinite. I believe that that ‘proof’ is fallacious, and I propose to detect in it at least five distinct errors. Although the exposure of any one of these errors is sufficient to explode the argument, it will be found that each in its own right—some perhaps more than others—affords an illumination that enables us successfully to enlarge the scope of our inquiry. Kant’s ‘proof’ that the world cannot be infinite is as follows.

Let us . . . assume . . . that the world is an infinite given whole of co-existing things. Now the magnitude of a quantum which is not given in intuition as within certain limits can be thought only through the synthesis of its parts, and the totality of such a quantum only through a synthesis that is brought to completion through repeated addition of unit to unit. In order, therefore, to think, as a whole, the world which fills all spaces, the successive synthesis of the parts of an infinite world must be viewed as completed, that is an infinite time must be viewed as having elapsed in the enumeration of all co-existing things. This, however, is impossible. An infinite aggregate of actual things cannot therefore be viewed as a given whole, nor consequently as simultaneously given.

It will be noted that one crucial premise in the argument is that it is ‘impossible’ that ‘an infinite time . . . be viewed as having elapsed’. This is a mistake. Kant’s ‘proof’ of this premise will be seen to be vitiated by the fallacy of equivocation.

If we assume that the world has no beginning in time, then up to every given moment an eternity has elapsed, and there has passed away in the world an infinite series of successive states of things. Now the infinity of a series consists in the fact that it can never be completed through successive synthesis. It thus follows that it is impossible for an infinite world-series to have passed away. . . .\footnote{Op. cit. pp. 217–18.}
in the present; which contradicts the very meaning of an infinite series.

The word ‘end’ occurs twice in the argument, and unfortunately it is employed in two different senses. It is important to distinguish three types of infinite series. First, there is the series represented by the positive integers. Here we have a series which possesses a terminus a quo or beginning, namely the number 0, but which lacks a terminus ad quem or end. Second, there is the series represented by the negative integers. Here we have a series which possesses a terminus ad quem or end, again the number 0, but which lacks a terminus a quo or beginning. Finally, there is the series represented by all of the integers, positive and negative. Here we have a series that lacks both a terminus a quo and a terminus ad quem. These three types of infinite series have their temporal embodiments. The first is found in the infinite stretch of time that lies ahead in the future; the second in the infinite stretch of time that lies behind in the past; the third in the whole infinite stretch of time comprising past, present and future. It will now be seen that it is quite possible that ‘an infinite time ... be viewed as having elapsed’ in the past. An infinite series must indeed be endless, but this means simply that it must lack at least one terminus, if not a terminus ad quem, then certainly a terminus a quo. The infinite past constitutes a beginningless series. Kant must show that a beginningless series cannot come to an end; but this cannot be shown, for once one grants the existence of a beginningless series, in actu exercito as it were, then it is easy enough to see that it can readily come to an end. I think that one may perhaps be puzzled as to how a beginningless series could ever get started in the first place; but this is to demand of a beginningless series that it have a beginning, which is surely unreasonable.

If it is intelligible, then, that ‘an infinite time ... be viewed as having elapsed’, what relevance does this principle bear in regard to Kant’s argument that the world cannot be infinite? It will be recalled that Kant insists that ‘in order ... to think, as a whole, the world which fills all spaces, the successive synthesis of the parts of an infinite world must be viewed as completed, that is, an infinite time must be viewed as having elapsed in the enumeration of all co-existing things’. Kant is demanding of us that we actually specify, one by one, everything that exists; he is demanding an ‘enumeration of all co-existing things’. I think that we are now in a position to satisfy his demand, on his own terms. Let it be the case that today I am at point A (see Fig. 1), that yesterday I was at point B, the day before at point C, &c. ad infinitum. Having always been on the move, I have succeeded in travelling throughout the entire infinite universe spiral-fashion.

I have been everywhere and seen everything. You may indeed choose to doubt my claim, and though you may be not able to verify it (assuming it to be true), you can certainly falsify it if it is false. You have only to prove that forty years ago I was not even born. How, then could I have actually been travelling throughout an infinite lapse of time? But my claim is not only falsifiable, it is also verifiable—assuming fairly privileged conditions. Thus you may confirm my claim by asserting that today you stand at my side at point A, that yesterday you were at my side at point B, the day before at point C, &c. Your insistence that you have been my companion in my travels throughout all of the infinite past is, again, not only falsifiable by a third party, it is also verifiable.

Even Kant does not wish to say that it is a matter of a priori fact that I must have come into being at some time in the past, but this is equivalent to admitting that I may have been always in existence, which in its turn entails that I may have actually visited an infinite number of stars in my world tour. It means also that I could have spent each night of the past lodging in a different room in Hilbert’s Grand Hotel. It will be evident that we have now an alternative route into hyper-mathematics. Let it be the case that today I have established that the number 2 is the sum of two prime numbers, that yesterday I established that the number 4 is the sum of two prime numbers, that the day before I verified this property of the number 6, &c. ad infinitum. Such being the case, it is surely a hard fact that I have
actually tested every even number in turn. I have thus succeeded in verifying Goldbach's hypothesis (if indeed it is true), and should it be false I am now in a position to specify precisely which numbers defy it: I have been keeping a record of all counter-examples. One of the nice things about this procedure is that on any day whatever in the past only a finite number of operations remained to be executed. I was thus free to discharge the whole process at any time I saw fit.

Have we any right now to congratulate ourselves on having supplied a general solution to the problem of the infinite regress? It is known that this problem, by intruding itself into the most diverse contexts of philosophy, operates to bedevil one issue after another. Two such issues may be here considered, one in epistemology, the other in metaphysics. In the formidable battery of arguments that the sceptic employs to prove that knowledge is impossible, the εἰς ἀνεπίπτον argument has always been felt to be especially irritating. If I am to establish the truth of some proposition \( p \), I must back it up with evidence, say \( q \); but if I am to establish the truth of \( q \), I must in turn back up \( q \) with further evidence, say \( r, \&c. \) ad infinitum. It would appear, then, that nothing less than an infinite argument can ever succeed in establishing the truth of any proposition. We may distinguish conditional and unconditional arguments. Any finite argument or 'proof' can be only conditionally binding: it is binding on the condition or assumption that the terminal premises are allowed to be true. Man then will be said to be capable of attaining only conditional knowledge which, of course, cannot be accredited as knowledge in the strict sense. Knowledge in the strict sense is indeed possible (the sceptic notwithstanding), but only for a god. Let it be the case that a god has today deduced \( p \) from \( q \), that yesterday he deduced \( q \) from \( r, \&c. \) ad infinitum. Is not the god entitled to assert that he has unconditionally demonstrated the truth of \( p \), without relying on any first principles whatever? The infinite regress having been exhaustively discharged, has not the god succeeded in consummating an infinite argument? Have we not found here the secret of that highest kind of knowledge—knowledge absolute that is beyond hypothesis—which Socrates dreamed of on the fourth level of the Divided Line? Alas. The god has proved only that \( p \) is true if \( q \) is true, that \( q \) is true if \( r \) is true if \( s, \&c. \) He has not proved that \( p \) is true. It is quite possible that \( p, q, r, s, \&c. \) might all be false. The infinite argument would in that case remain deductively valid but it would certainly fail to constitute a proof. I am afraid that if the challenge of the sceptic is to be met, it must be along very different lines from those we have suggested.

In metaphysics the problem of the infinite regress is most notably bound up with what Kant styled the cosmological argument for the existence of God. There are two versions of this argument. In the first it is argued, on the strength of the principle of causality (i.e. whatever has a beginning in time has also a cause of existence) that since any event \( a \) must have a cause \( b \) which in turn (if it, too, is an event) must have a cause \( c, \&c. \) it is necessary that there be at last a First Cause. Itsel not an event, this first cause is clearly without any beginning in time and hence requires no cause. The crucial assumption in this version of the argument is that an infinite regress of causation must be ruled out as unintelligible. God, the first cause, is seen to have created the world at some finite time in the past, and though the creation of the world constitutes the first event, it is preceded in time by the eternal first cause. If God is thus invoked to block an infinite regress, then God and the infinite are found to be metaphysically equivalent: the one does the job of the other. According to Kant, it is the empiricist who rejects, and the rationalist who accepts, the cosmological argument. The empiricist is held to rest content with the infinite whereas the rationalist is supposed to fight shy of it. Actually, in the history of philosophy it is the empiricist even more than the rationalist who is embarrassed by the infinite—we recall Locke's conviction that of the infinite we have only 'the indeterminate confusion of a negative idea'. The British empiricists Locke, Berkeley and Hume are all more or less hostile to the actual infinite, whereas the Continental rationalists Descartes, Spinoza and Leibniz are all more or less devoted to it. I would suggest that, owing to Kant's preoccupation with the moral pathos of the theological question, it was the atheistic tendency of empiricism that blinded him to some of the purely theoretical issues. Certainly, if one accepts the principle of causality as true and also rules out the actual infinite as unintelligible,
then one has no choice but to acquiesce in the cosmological argument.

The grounds of our own rejection of this first version of the cosmological argument should be evident enough. We are prepared to posit the existence of a god who has today established the cause of some current event $a$ to be $b$, who yesterday established the cause of $b$ to be $c$, &c. ad infinitum. Assuming that the god has on each day recorded the results of his investigation on a separate sheet of paper, we must say that he has actually composed a book of infinite length in which may be found the total explanation of why the event $a$ occurred. That every event requires an infinite ground for its existence, is seen to be quite intelligible.

It is at this point that the second version of the cosmological argument comes to sight. Reconciled to the intelligibility of an infinite regress of causation, Leibniz raises a further question. Why this infinite chain of causation and not some other equally conceivable? In order to raise this question Leibniz must appeal to a principle that lies beyond the principle of causality. The principle of causality requires merely that anything that comes into being must have a cause of existence. Seeing that the infinite chain of causation cannot itself be said to have a beginning in time, we can only ask Leibniz's question if we appeal to the wider principle of sufficient reason, which was cast by Wolff in the following form. Nihil est sine ratione cur potius sit quam non sit. On the strength, then, of the principle of sufficient reason Leibniz demands a transcendent ratio or ground which

1 Aristotle's position is somewhat perplexing. Although he will not allow the actual infinite, he insists on the eternity of the world. Infinitely many days having elapsed prior to the present, does not count for him as an actual infinite, i.e. a concurrent infinite. Which is understandable enough, but which raises some serious issues. Is that infinite sequence in the past to be viewed as a potential infinite? Kant with his idealism can answer this question in the affirmative but surely not Aristotle. There is a further difficulty. Is it not the case that the sense in which the infinite past may not be viewed as an actual infinite, is precisely the same sense in which the $Z$-series of non-continuous intermittent motion (which Aristotle tacitly rules out on the ground that the actual infinite is inadmissible) is also not an example of the actual infinite? If the actual infinite in the strict sense be a concurrent infinite, then the $Z$-series of non-continuous motion is no more an instance of the actual infinite than the infinite past. In the light of these considerations, we shall continue to subsume the infinite past under the rubric of the actual infinite, though not without the awareness of differences.

will explain why any one of the infinite range of possible infinite chains of causation should exist rather than some other. Is the principle of sufficient reason true? Can it be known to be true? Much admired in the past, the principle in our own day has been proscribed as positively meaningless. It has been ruled out on the ground that it is neither verifiable nor falsifiable. If the principle of sufficient reason has been decried as meaningless, the principle of causality has itself proved of some embarrassment. It, too, has been held to be neither verifiable nor falsifiable; nor verifiable for the very reason that any universal proposition such as 'All crows are black' is held to be unverifiable; not falsifiable for the very reason that any proposition of mixed quantification such as 'The world there exists an acid solvent' is held to be unverifiable. Merely because we have not been able to find the cause of some particular man's death, or an acid solvent for some particular bit of metal, cannot be said to falsify either proposition.

The principle of sufficient reason and the principle of causality are both cases of mixed quantification, and like the proposition 'For every metal there exists an acid solvent', they may be felt to be neither verifiable nor falsifiable. Although no one wishes to rule out as meaningless the hard scientific statement 'For every metal there exists an acid solvent', the principle of causality and the principle of sufficient reason have been more harshly treated, the one to a lesser, the other to a greater extent. It has been generally allowed that the principle of causality might possibly be true, but the principle of sufficient reason has been widely dismissed with scorn. Actually these three propositions under discussion here are rather more akin than has been commonly believed.

Returning to our thematic examination of Kant's proof that the world cannot possibly be infinite, I have pledged myself to detect in it five distinct errors, and as yet we have explored only one—Kant's premise that it is impossible that 'an infinite time ... be viewed as having elapsed'. My second objection to Kant's argument is very much more pedestrian, though perhaps no less important, than the first. Here I would convict Kant of a non-sequitur. Let it be granted that he has succeeded in proving that 'an infinite aggregate of actual things ... cannot be viewed as a given whole nor consequently as simultaneously given.'
Does it then follow that ‘the world is, therefore, as regards extension in space, not infinite’? Merely because it is impossible to ‘think as a whole the world which fills all spaces’, how does it follow that the world cannot be infinite? The underlying premise here is a perversion of the classic principle that to be is to be intelligible. If an infinite world is unthinkable, inconceivable, unintelligible, then an infinite world must be ruled out as impossible. In theory, I do not doubt the validity of a passage from inconceivability to impossibility, but in practice I am persuaded that the inference is fraught with difficulties. What precisely is to count as the criterion of conceivability and inconceivability? Kant is very exacting indeed. If we are to conceive of an infinite world, he requires of us an ‘enumeration of all co-existing things’. Relying on a utopian procedure, I have shown that such an enumeration is indeed possible; but I am eager to establish the intelligibility of an infinite world on entirely non-utopian grounds. To that end I would submit two logically independent arguments.

First, and most obvious, there is the force of the following self-evident analytic proposition to be reckoned with. The number of stars that exist is either finite or it is infinite. If by a tautology we choose to understand any proposition the truth of which is seen to follow simply from the meanings of the words in the statement, then the foregoing is surely a tautology. (It is necessary, indeed, to enrol zero as a number in this context so as to guarantee the truth of the statement even under the extreme condition that no stars whatever exist.) We are now to ask a peculiar question. How can a tautology become a subject of controversy? One can only marvel wide-eyed. And yet it might almost be said that the whole purpose of these studies of ours is to defend a self-evident tautology from the sophistical objections of such luminaries as Kant and Wittgenstein. A thankless task. It will be recalled that Russell has reluctantly been driven to the conclusion that mathematics consists entirely of tautologies. Why the reluctance? Because it has been felt that if mathematical locutions are merely tautologies, then they must be altogether trifling. To our astonishment we discover that there exists at least one type of tautology—those tautologies grounded in the finite-infinite dichotomy—which are capable of baffling the greatest minds. We must therefore introduce the concept of a profound tautology. I am not arguing that from the one tautology ‘there exist either a finite or an infinite number of stars’ it follows at once that the world might be infinite. No. We must add that it is not logically necessary that the number of stars be finite. Here we have, I submit, a second tautology. It is from these two tautologies, taken jointly, that we infer that it is logically possible that an infinite number of stars might exist; and it is this conclusion which entails that the world might be infinite. To conceive of an infinite world does not require, as Kant seems to suppose, some special mental act of prodigious scope. It is not at all a matter of racking one’s brains. One has only to recognize two tautologies as tautologies and then to perform a simple logical inference. That is all there is to it. I do not deny that in some unimportant sense of ‘conceive’ it is impossible for us to conceive of an infinite world, i.e. picture it as a whole. Kant is guilty of a non-sequitur in inferring the impossibility of an infinite world from this unimportant kind of ‘inconceivability’.

What Kant is to cosmology, Brouwer is to mathematics. Brouwer is indeed the Kant of mathematics, an almost identical strategy being employed by both. Nevertheless, I believe that there may be some special justification for Brouwer’s refusal to acquiesce in the tautology ‘Either the sequence 7777 occurs in the decimal expansion of π or it does not’. What precisely do we mean by the decimal expansion of π? Are we referring to an actual infinite or to a potential infinite? In mathematics, certainly, the question is left entirely open. If Brouwer wishes to argue that the tautology is meaningless in the absence of any specification as to which of the two, the actual or the potential infinite, is intended, I can only record my hearty agreement. Does this mean, then, that Brouwer is right, that the tautology is senseless? No. The tautology as it is expressed in pure mathematics is cast in a neutral mode of discourse; it is designedly ambiguous as between the actual and the potential infinite. But it is true in either case. It is thus seen to be a portmanteau or double tautology which may be cashed either in terms of the actual or in terms of the potential infinite. Viewed in terms of the actual infinite, π may be assigned the following cosmological model. Let it be the case that there exists an infinite sequence of stars such that the first star is purple, the second
star black, the third green, the fourth black, the fifth yellow, the sixth brown, the seventh blue, the eighth orange, the ninth yellow, the tenth purple, the eleventh yellow, etc. However much this sequence of colours may appear to proceed at random, it is to be understood as obeying an inner λόγος of its own. Let us adopt the following code: 0 white, 1 black, 2 blue, 3 purple, 4 green, 5 yellow, 6 orange, 7 scarlet, 8 pink, 9 brown. By replacing the sequence of colours by their respective code numbers, we find—lo and behold—that π is generated —3.1415926535... Now it is certainly an a priori fact that either the π sequence of coloured stars exists or it does not. Moreover, it is also an a priori fact that if the π sequence of coloured stars exists, then there either does or does not exist within that infinite sequence a group of four successive scarlet stars. Scarlet = 7.

Dispensing now with the actual infinite, the tautology is found to lose none of its force when it is understood purely in terms of the potential infinite (Brouwer notwithstanding). The following infinite succession of events either will or will not occur: later today a shower of purple sparks in the heavens, tomorrow a shower of black, the next day a shower of green, the next a shower of black, the next a shower of yellow, the next a shower of brown, &c. ad infinitum, as the π sequence of colour showers successively burst forth in the heavens. Assuming that the infinite succession of showers will in fact occur, then it is certainly the case that there either will or will not occur within that series four successive showers of scarlet sparks. Moreover, it is also the case that, given the π series of showers, there either will or will not occur within that infinite series a final shower of scarlet sparks; which is equivalent to affirming that there either is or is not a terminal 7 in the decimal expansion of π. In regard to the foregoing tautology, lest it be supposed (as Wittgenstein supposes) that it is altogether lacking in any use, it should be remembered that according to the sacred law of the Balubas, the high priest must on each day record one further digit in the decimal expansion of π (the institution was established at the founding of the society 811 years ago and has been religiously observed ever since) and on any day that a 7 is recorded a terrible human sacrifice must be performed. For the last 286 years no 7 has been encountered, and being a humane people the Balubas are convinced that the sacred law will never again require them to descend into the barbarity of human sacrifice. Are the Balubas mistaken? This is no academic matter, and how can it be denied that the Balubas either are or are not mistaken in their conviction?

Our first argument designed to show that the actual infinite is intelligible (eschewing all utopian procedures) has rested on the tautology that either a finite or an infinite number of stars exist. This first argument is purely logical in character. We have ignored here the whole question of verification, feeling quite free to insist on the intelligibility of the actual infinite without in any way linking it up to experience. Our second argument is calculated to cope with this challenge. Again eschewing all utopian adventures, I urge simply that the metaphysical hypothesis that the world is infinite must be intelligible seeing that it is logically entailed by the 'empirical' hypothesis that this wall before me is infinite, which itself is falsifiable. This second argument must not be confounded with the first: the two are radically distinct. What now of the hypothesis that there exist an infinite number of stars? Neither verifiable nor falsifiable by any ordinary method, this hypothesis can easily be shown to be logically entailed by the following hypothesis which is clearly falsifiable, namely that Sol is the first of an infinite sequence of stars each of which is not more than 1,000 miles removed from its immediate successor. And the hypothesis that there exists a terminal 7 in the decimal expansion of π? Albeit neither verifiable nor falsifiable, this hypothesis is logically entailed by the following falsifiable proposition—the 94th digit is the terminal 7 in the decimal expansion of π. This latter proposition, if it is false, can certainly be refuted by some finite procedure, and if it is true, then indeed there exists a terminal 7 in the decimal expansion of π.

Part 3: Order and Chaos

What now of the hypothesis that there exists a real number, an infinite sequence of digits, for which there is no finite rule that will generate it? Here we touch upon what seems to me to be the most beautiful problem in all of mathematics. Unrivalled in its simplicity, the problem is of very recent origin, having come
to sight only in the present century. It is because of its profound
cosmological import that I have chosen to linger over it in some
detail. Compared with the decimal expansion of any rational
number, the decimal expansion of any irrational number may
be said to be altogether random in character: the one always
exhibits a simple recurrent pattern (a regularity of sequence, in
Hume’s sense), the other does not. Irrational numbers are thus
invariably irregular in their expansions—there is no order in the
succession of digits. In another sense, however, many (perhaps
all) irrational numbers are certainly regular, for they are governed
in their expansions by definite rules. The decimal expansion of
π, however far extended, will never disclose any persistently
recurrent pattern in its formation, but despite the seeming
chaos, there is a clear and precise rule which dictates the in­
finite sequence. The decimal expansion of π may thus be said
merely to appear as a random sequence, actually it is not.
Let us define a random number in the strong sense as any
number so as to unpack the cosmological book. If there are
no random numbers, then in a not unimportant sense it may
be said that a universe of chaos is impossible.

Ordinarily, chaos is understood not in terms of random but
rather in terms of any irrational sequence. Let us suppose that
the π sequence of colour flashes successively burst forth in the
heavens. To the uncritical observer, applying the standard
principles of empirical induction, the series of colour flashes will
be regarded as utterly devoid of any rationale. He will be
mistaken. There is indeed no ‘regularity of sequence’ in the
sense in which he hopes to find it, but there is a rule which may
be employed to predict each successive burst of colour. Someone
might be lucky enough to hit on the right rule. What we
ordinarily call empirical induction—the hitting on the right
rational rule—is but a special case of a more general faculty,
the hitting on the right rule, be it rational or irrational. Hume’s
problem of induction may be cast in the following mathematical
form. For any finite sequence of digits, no matter how far
protracted, there exist an infinite number of diverse rules (some
rational, others irrational) which are capable of continuing
the series in very different ways. The sun has risen every day
for the last billion days. Will it rise tomorrow? This is equivalent
and it will then constitute a real number and, indeed, almost
certainly an irrational number. We may style this particular
Gödel number as the cosmological number. If for every real
number there exists a finite rule which generates its decimal
expansion, then there must exist a finite rule which generates
the cosmological number. This finite rule may be styled the
cosmological rule, and it will constitute the master λγος which
presides over the universe. Given the rule, we can generate
the cosmological number and, as we proceed, we can decode
the number so as to unpack the cosmological book. If there are
no random numbers, then in a not unimportant sense it may
be said that a universe of chaos is impossible.

Let us suppose that there are no random numbers, what follows?
Specifically, what follows for cosmology? Imagine a book that
contains an exhaustive description of everything that exists in
the universe, a history of everything that has happened and
everything that will happen. Assuming that the book is not only
infinite but denumerably infinite, there will then exist, following
Gödel, an infinite sequence of digits which will correspond,
letter for letter and word for word, to the complete text of the
book. Let a decimal point be prefixed to that infinite sequence

1 There is a striking scholium to Bk. x of Euclid (ed. Heiberg, p. 417) in
which the mathematicians of the irrational is held to find its ontological
counterpart in the chaos of the Heraclitean flux. ‘The story is that the first
Pythagorean who brought into the open the theory [about irrationals]
suffered shipwreck, and perhaps they meant [in the story] to say enigmatically
that everything which is irrational in the whole is wont to hide itself
without λγος and without φίλει, and if any soul should assault such an
image (? of life (έφος τούτου τελέστω) and make it publicly known, he is carried
away into the sea of becoming and overwhelmed by its restless currents.’
One is reminded of Heraclitus’ dictum, ‘Nature is wont to hide itself’,
φίλος κρύπτει ταύτα φιλει.
to asking what the billion-and-first digit will be, following upon a billion successive 1’s. Let 1 represent the rising of the sun on schedule, let 0 represent the failure of the sun to rise. Will the billion-and-first digit (in a binary notation) be 1 or 0? We can imagine a rational being who will insist that the billion-and-first digit must be 0! According to the finite rule which he has been following with a success equal to ours, the sun will not rise tomorrow. It may be noted that his rule need not generate an irrational sequence; it may be quite rational. If there are no random sequences, then there must indeed be a rule which is the right one. But which is the right one? Both his claim that his rule is the right one and our claim that our rule is the right one, are falsifiable, and equally falsifiable. Man, we may suggest, is an animal innately endowed with what we might call the ‘1111…’ rule. Built into his genetic make-up, this rule has served him quite well, despite its evident crudity. By some strange mutation, however, a super-man might evolve who would be endowed with a very different and very superior rule or complex of rules which would enable him to predict the future even more effectively. Like the dinosaur we might become extinct.

From our point of view, √2 might be called a rational number in an extended sense of the term. For it admits of a rationale, a λόγος. The distinction between rational and irrational numbers is superseded, in our eyes, by the far more profound distinction between random and non-random numbers. This distinction is no less profound even if it can be shown that there are no random numbers. In the strong sense of the term only random numbers are really irrational, only they lack a rationale. If we could prove mathematically that there are no random numbers, then we would have an a priori proof guaranteeing to us that there is always a right rule for any infinite sequence (be it an actual or a potential infinite). There would thus be an a priori guarantee for the existence of a ‘regularity of nature’ in a non-trivial sense. In these circumstances the great question would be, not whether there is a right rule, but how it might possibly be discovered.¹

¹ Cf. Leibniz, Discourse on Metaphysics, § 6. ‘That which passes for extraordinary is so only with regard to a particular order established among the created things, for as regards the universal order, everything conforms to

We are now in a position to view Peirce’s question in a new light. Peirce, it will be recalled, asks whether it is conceivable that by sheer chance, in the repeated tossing of an unbiased die, a 6 might turn up at every throw throughout an infinite sequence. He is reluctant to believe that such a prodigy is possible, but he fails to grasp precisely why it is impossible. Anything that happens persistently in accordance with a rule, cannot occur by chance. This is true by definition. Something is said to be a matter of chance if and only if it occurs independent of rule. Thus the π sequence of colour flashes cannot possibly occur by chance. We should refuse to call it chance. Assuming that there are no random numbers, then a die being cast throughout an infinite sequence the resulting series of digits will necessarily occur in accordance with some rule and hence can never be a matter of chance. Anyone fortunate enough to hit on the right rule will refuse to call it chance, and he will not be mistaken. If the principle of causality may be interpreted as asserting that everything that happens, happens in accordance with a rule, then a mathematical demonstration of the non-existence of random numbers will constitute an a priori proof of the principle of causality.

Prima facie it may seem wildly implausible to suppose that there are no random numbers. There being no immediately evident necessity ensuring the existence of a finite rule which is to govern the formation of any infinite sequence, we may be strongly tempted to regard it as virtually certain that there are random numbers. This temptation must be resisted. In matters of this kind intuition is notoriously unreliable. I should be surprised, but not very much surprised, to be confronted with a strict demonstration of the non-existence of random numbers. What renders the problem peculiarly challenging is that we have no assurance that the problem is at all solvable, one way or the other, by any method whatsoever, even by utopian it. This is so true that not only does nothing occur in this world which is absolutely irregular, but it is even impossible to conceive of such an occurrence. Because, let us suppose for example that some one jots down a quantity of points upon a sheet of paper helter skelter…; now I say that it is possible to find a geometrical line whose concept shall be uniform and constant, that is, in accordance with a certain formula, and which line at the same time shall pass through all of those points, and in the same order in which the hand jotted them down…’ (Translation by G. Montgomery and A. R. Chandler).
procedures. It is quite true that the class of finite English sentences being denumerable, the class of finite English sentences which are real number generators is also denumerable. We are thus entitled to conclude that there exist real numbers which cannot be generated by any finite English rule, and if it were only the case that every finite rule were expressible in English, it would at once follow that random numbers exist. Unhappily, we know that there exist finite real number generators which are incapable of being expressed in English. One such exotic rule may be exhibited as follows. Let us embark on the programme of listing all finite English sentences in turn. Some of these sentences will constitute real numbers generators. Consider now the first digit of the real number generated by the first real number generator that we encounter in our list, then the second digit of the real number generated by the second real number generator in our list, &c. _ad infinitum._ By means of a diagonal procedure we can progressively generate a real number which will necessarily be different from any real number generated by any finite English sentence. In generating this exotic sequence, we shall indeed be following a finite rule but it will not be a rule which can be expressed in English.\footnote{Modelled on the Richard paradox, this exotic rule can certainly be expressed in English in a loose sense: we have already done so (in an abbreviated form). In the strict sense, however, it fails to measure up to the standard of a well-formed English rule, being by its very nature parasitic on all of the English real number generators. There are other difficulties. Are we quite sure that, in mobilizing all of the English generators, we are capable of distinguishing between genuine and spurious rules? Thus there will be found in English the putative rule which asserts the existence of a real number which will necessarily be different from any real number generated by any finite English sentence. In generating this exotic sequence, we shall indeed be following a finite rule but it will not be a rule which can be expressed in English.1}

1 What holds here for English holds equally for any formal system that is designed to express the mathematics of the real numbers. A formula can always be written outside the system that is not to be found within it. Quite apart from Gödel, then, any such system is necessarily incomplete.

Are there any random numbers? The crux of the problem lies in the fact that not only is it the case that the class of all real numbers is non-denumerable but it is also the case that the class of all finite real number generators is non-denumerable. Does there exist in the first class a real number which is not to be generated by any rule in the second class? In order to answer that question, it may well look as if we are obliged to conduct an exhaustive search throughout both classes. Even hyper-mathematics proves inadequate for this formidable undertaking. In the circumstances it is not extravagant of us to surmise that the question as to the existence of random numbers may, quite possibly, be _absolutely_ undecidable—by any and all methods. At this point there will doubtless be those who will wonder whether the question itself, granting its absolute undecidability, may be accredited as meaningful. Brouwer, certainly, is unwilling to allow the possibility of random numbers. It is his conviction that the very concept of a real number is inseparable from the concept of a rule. I submit that Brouwer may be shown to be mistaken even if we limit ourselves to the potential infinite. A lottery being employed, let a digit be recorded today, another digit tomorrow, a third the next day, &c. _ad infinitum._ No one will contest the empirical possibility that this infinite sequence—we may style it the lottery number—may be successively recorded. We have now to ask whether the lottery number constitutes a random or a non-random number. Consider the following proposition _p_. 'The lottery number will prove to be the decimal expansion of π.' The proposition _p_ may not be verifiable; it is, however, certainly falsifiable. Hence _p_ must be accepted as meaningful. Now consider the following proposition _q_. 'The lottery number will prove to be a random number.' It is clear that _p_ logically entails _not-q_. _Not-q_ must thus be allowed to be meaningful, and since the denial of any empirically meaningful proposition must also be meaningful, _q_ must be accepted as meaningful. But _q_ logically entails _r_, namely that there exists at least one random

It may be noted here that in our discussion of the metaphysical library we tacitly assumed that since we are capable of scanning all finite English discourses we must be capable of scanning all finite discourses. This assumption is now seen to be a mistake.
number. The hypothesis that there exist random numbers is thus seen to be logically linked up with experience, even through it may be absolutely undecidable. It may be of some interest to note here that, granted the existence of at least one random number, it is easy to prove that there must be infinitely and, indeed, non-denumerably many random numbers.

If the thesis that there are no random numbers be both true and unprovable, then the thesis must be a synthetic a priori truth; and if it is equivalent to the principle of causality, then the principle of causality must also be a synthetic a priori truth. The denial of either one could never lead to a contradiction.

Part 4: Infinite Vision and the Paradox of the Transfinite

Resuming our protracted investigation of Kant’s ‘proof’ that the world cannot possibly be infinite, it is my present object to detect a third error in Kant’s argument, and in this connexion I propose to explore the actual infinite afresh in terms of a new utopian concept. Vital to Kant’s argument is the assumption that ‘the magnitude of a quantum which is not given in intuition as within certain limits can be thought only through the synthesis of its parts’. Kant is unwilling to admit that ‘the magnitude of a quantum which is not given in intuition as within certain limits’ may be given in intuition free of all limits whatever. Very simply, he denies that the actual infinite can be imagined totum simul. But why not? That the human mind is incapable of such a feat, is true enough; but is this incapacity built into the very nature of mind as such? I do not believe that it is. If I am able to conjure up before my mind’s eye the image of three or four or five distinct stars, then is it too much to suppose that some other mind—doubtless divine—might be able to conjure up the image of an infinite number of distinct stars in one boundless panorama? Almost staggered by the thought, we are engaged in envisaging a possibility that touches Kant to the quick. If an infinite universe is to be intelligible, it must be, according to Kant, a possible object of experience. What he is really demanding is not merely that the infinite be imaginable; even more to the point, he is insisting that it be seeable at once. So be it. Put the case of a god (with eyes in the back of his head) who actually sees everything that exists through-
In acknowledging the possibility of an infinite panorama, we are enabled to supply an additional warrant for hyper-mathematics. Let it be the case that the following infinite sentence actually exists writ large across the heavens. 'Every even number is shown to be the sum of two primes, viz. \(2 = 1 + 1, 4 = 1 + 3, \ldots\). This infinite sentence need not be regarded as necessarily the product of some mind, be it human or divine. Nature is seen to exhibit so much design in her works already that I am tempted to believe that the wide open sky, being almost a *tabula rasa*, is virtually inviting Nature to inscribe upon it infinite discourses of the most wondrous kinds. I do not believe that there is any *a priori* necessity that only finite discourses can exist writ large upon the heavens. The intelligibility of infinite proofs is thus found to be altogether independent of the question as to their origin or source. It is not necessary to specify any effective procedure, even of a utopian kind, in order to recognize the intelligibility of infinite proofs. Such proofs may simply exist as natural phenomena. Understood in this way, infinite proofs are invested not merely with mathematical 'existence' but with possible physical existence as well. I wish now to entertain the possibility of a god scanning the whole universe in an infinite panorama. 'Yes,' he says, 'Goldbach’s hypothesis is quite true. I can see at one glance every even number decomposed into two primes.' 'But precisely where in the universe is that spectacle to be found?' we ask, and the god in reply will be able to supply us with exact directions to the scene. To Hume the god may say, 'All crows are black. I can see them all, and if you wish I will specify precisely where each one is.' Finally, it will be evident that we are now in a position to encompass Hilbert’s Grand Hotel in one all-embracing experience. ‘Ah, yes,’ says the god, ‘I can see every room in the hotel, they are all occupied, and I have each lodger under close scrutiny. Look. A traveller has come to the hotel. He is asking for a room. The manager is now instructing each lodger to vacate his present quarters and remove himself to the next room along the corridor. There they are now, the lodgers, moving each out of one room into another. Room § is now vacant, and the traveller is being conducted to his quarters.’

In his third *Critique* Kant identifies the sublime with that which is great, not that which is merely relatively great but that which is absolutely great. But only the infinite is absolutely great, and hence Kant links the sublime to the infinite. The sublime is thus found by Kant to transcend all possible experience. By subjecting the infinite—at any rate, the *denumerably* infinite—to a panorama, we have succeeded in rendering it as a possible object of experience. If the infinite is to transcend all possible experience, it can only be in the form of the non-denumerably infinite. It is then, not the infinite as such, but rather the non-denumerably infinite which must be identified as the locus of the sublime, certainly of the mathematical sublime. Although the non-denumerable may transcend all possible experience, it is Cantor’s profound achievement to have demonstrated that it falls within the compass of the intelligible, and though the intelligible may transcend the experiential, we have shown that there is a logical link between the two.

That we have been indulging ourselves almost shamelessly in a flagrant exercise of the logical imagination, is too evident to be denied. If there be some who should stiffly decline to be transported by our metaphysical afflatus, I am not without a certain hidden sympathy for their resistance. In my defence, let me plead the inherent reasonableness—not perhaps of my methods which are in large measure certainly quixotic—but of the goal to which they are directed. Granting the existence of an objective world, it must either be finite or infinite, and this means that the objectivity of the external world is intimately bound up with the intelligibility of the actual infinite. With empiricism and idealism rampant in modern philosophy, it has been widely felt that nothing can be intelligible—even more, that nothing can so much as be at all—unless it is a possible object of experience. If we are then to preserve the objectivity of the external world in the teeth of the empiricist and idealist, we find ourselves obliged, under duress, to establish the possibility of the scope of experience being so much enlarged as to be at last accommodated or ‘adequated’ to the full sweep of the infinite. However grotesque our myth of the cosmological octopus may be, it must be seen as being elicited from us for the eminently sober purpose of saving the objectivity of the natural world.

Our fourth objection to Kant’s argument will be found to be even more wildly quixotic than our third. Kant tacitly assumes...
in his 'proof' that there is and can be only one Space. It is to Kant's great credit that this assumption—it has every plausibility in its favour—is expressly recognized by him as a synthetic principle in his doctrine of the transcendental aesthetic. It may well be that Kant was the first to take note of this primitive assumption, the truth of which (so far as I know) has never been doubted. What does it mean to say that there is and can be only one Space? It means simply that if any two bodies exist, say two stars, then there must be a spatial distance between them. It means that all bodies exist and must exist in a common spatial order, that there is a spatial connectedness or togetherness binding them all together, that the universe is indeed a universe and not a pluriverse. Prima facie self-evident, this assumption has not been challenged even by the non-Euclideans—philosophers, mathematicians, and physicists—who merely refuse to concede that space must necessarily satisfy the axioms of classical geometry. All geometries, both Euclidean and non-Euclidean, may be supposed to admit of a metric that relates every point in the space to every other. Thus Euclid postulates that between any two points a straight line, i.e. a finite straight line, can be drawn. It is this general axiom that I wish now to question. I am suggesting that there may exist two distinct stars which are not separated from each other by any spatial distance whatever. Why must there always be a spatial route that would enable one to effect a passage from any one star to any other? Perhaps we ourselves actually live, not in a universe, but in a pluriverse consisting of more than one Space. One star may exist in one Space and another star in another Space, there being no spatial distance between them. In thus shattering the unity and integrity of the universe, I propose to explore a possibility with some fairly far-reaching cosmological and cosmo-mathematical implications.

In raising the possibility of a pluriverse, I am eager to vindicate my suggestion in terms of the most exacting standards. Is there any empirical evidence that might lead us to suspect that ours is a pluriverse and not a universe? From the very nature of the hypothesis it may appear that no evidence could possibly be forthcoming in its support. If there exists some other Space distinct from our own but not separated from it by any spatial distance, what possible access could we ever have to it, being invincibly sealed off from it? On the other hand, the hypothesis that there exists no spatial distance between Mars and Venus is certainly capable of an empirical refutation. Does this fact of falsifiability suffice to sustain the intelligibility of the hypothesis?

Let us enlarge upon this theme in some detail. It is not altogether extravagant to suggest that physics in the distant future may advance to the stage of enabling us to subject an object—a table, say, or even a man—to some high-powered electro-magnetic charge which would result in the table or man vanishing from its place in New York and re-appearing in London—without passing through the intervening space. In the light of angelology and quantum mechanics, such a 'passage' is, if utopian, not entirely inconceivable. Granting this lesser oddity, let us now entertain the greater. We are to suppose that a man, wired for passage, disappears altogether from our own Space and re-appears in another. There he is now roaming a beautiful meadow—we are following him closely on a television screen. I have no doubt that we should have every right, at first, to assume that our traveller is surely at some definite spatial distance from the earth. We may indeed find that the beautiful meadow is a billion miles from us. But is it absurd to suggest that it may perhaps be removed from us at no spatial distance whatever? I think not. With the return of our traveller, we can imagine a fairly extensive commerce being conducted to and from earth and meadow, pending the settlement of the question as to whether or not the meadow is at any spatial distance from us. It should be emphasized that we are not exclusively relying on the private testimony of one man, to sustain our conviction that the beautiful meadow does in fact exist. In addition to the television screen, we have the general testimony of as many men as we might wish; indeed, we may choose to join the expedition ourselves. In the absence of any evidence to the contrary, it will always remain an open question whether in fact the meadow may not be removed from the earth at no spatial distance. We may suppose that, in these circumstances, a new concept of 'distance' would be introduced. The meadow will be said to be, say, 20 electro-units 'distant' from New York, and if there should be a valley 813 electro-units from New York, it will then be said that the valley is more
There will thus be a concept of 'distance' in such a pluriverse, but this is not to be confounded with a spatial distance. If the 'passage' from New York to London should require 250 electron-units, we should be tempted to say that, though the valley lies at a much greater 'distance' from New York than London does, the meadow is actually very much 'closer'. This 'proximity' can certainly be verified to be not a spatial proximity. Notice that we are able to construct a common standard of 'distance' which will univocally apply both intramurally and extramurally.

If I am right in my suggestion that ours may possibly be, not a universe, but a pluriverse, it is quite intelligible to suppose that not merely two distinct Spaces may exist but three or four or five or, indeed, infinitely many. And if there are infinitely many co-existing Spaces, may they not be of an order beyond the denumerable? May not the number of Spaces be non-denumerably infinite? And if there are non-denumerably many Spaces, then it is possible that the decimal expansion of every real number might actually exist writ large, one in each Space. Moreover, if nothing in our universe escapes the notice of the cosmological octopus with his infinite panorama, may we not enlarge his powers so that he is engaged in simultaneously beholding everything everywhere, in all the Spaces? In which case it would follow that the octopus is the witness, the eye-witness, of every real number. And if the witness of every real number, why not the witness of every finite real number generator as well? Endowed with a range of intellect equal to his range of sensibility, will not the octopus be able to see at a glance whether there exist any random numbers?

Flushed with a heady draught of vicarious power, let us coolly reconstruct the serious purport of our latest extravaganza. The nice thing about our own parochial universe is that it admits of a natural itinerary whereby one is enabled to effect a passage from any point in it to any other. An especial beauty of this Space lies in the fact that even if it be infinite each point in it is separated from every other point in it by a finite spatial distance. How convenient if all reality—certainly all material reality—should be boxed into this one receptacle, the Platonic χώρα. There being this ready accessibility of all parts of our

Space, we should have great cause for gratitude if there is no overflow of reality into other Spaces. Although an infinite universe is certainly an open system contrasted with a finite universe, when we contrast a universe with a pluriverse, then even an infinite universe may be regarded, comparatively speaking, as a closed system. It is this very closure of our universe that afforded us, in our myth of the cosmological octopus, the prospect of a synoptic vision of the Whole. For what we mean by the closure of our universe is simply the fact that a geometrical rule exists which enables us to negotiate our way throughout—our initial point of exploration being any point in the Space whatever. The dazzling beauty of this natural itinerary is lost upon us if its vulnerability is not seen. I am suggesting that it is quite possible that Reality, instead of being distributed throughout a single Space, may rather be parcelled out over many Spaces, in which case it becomes far more difficult to envisage a panorama.

It is important to distinguish two very different empirical claims. One is the claim of a god who insists that he is engaged in seeing everything everywhere; the other is the claim of him who insists that he is engaged in seeing everything that is at any spatial distance from the earth. The two claims should not be confused. In order to falsify the first claim it is enough to establish the existence of a meadow of which the god is ignorant. Such evidence will not suffice to refute the second claim. We must also show, in this case, that the meadow is at some spatial distance from the earth. It is on the strength of this distinction that we are entitled to affirm the intelligibility of a pluriverse. If ours is a pluriverse, then the general accessibility of all reality proves to be far more problematic than even Kant supposed. Kant was persuaded that there was an inherent lack of 'adequation' between the scope of all possible experience and the range of the Whole, but for Kant the difficulty was limited to the inaccessibility of the infinite qua infinite. Although we, for our part, have succeeded in showing that the infinite may indeed be an object of experience, Kant's problem of the Whole breaks out anew for us in connexion with the possibility of a pluriverse. No matter how many Spaces one might have access to, how could one ever be sure that there did not exist some Space which escaped his notice? Even God might be puzzled by
that question. Within our own Space it is easy to establish that one enjoys comprehensive access to all the sub-spaces that compose it, for they succeed one another in accordance with a strict rule. Thus the cosmological octopus, simply through his act of witnessing (in all directions) a successor-star that lies beyond each star in his field of vision, knows that all of our Space is available to him. Not so with the pluriverse, for in this case there is no rule which dictates the 'succession' of one Space to another, and this is especially true if the number of Spaces should be non-denumerable. Nevertheless, although even God might not be able to prove that he has access to all Spaces, there is clear empirical evidence which would convince him that bodies exist of which he was previously ignorant. Anyone—God included—who claims to be omniscient is engaged in advancing a thesis which even the positivist will allow to be empirically meaningful, for the claim of omniscience is always falsifiable.

Although reality would be very much tidier if it were all wrapped up in one package within our universe, there is at least one bright note in the possibility of an overflow. Our own universe, be it finite or infinite, would seem to be too small to accommodate all of the real numbers and, a fortiori, all of the discourses (I am thinking primarily of those of infinite length) which are capable of being written in English. Both being non-denumerable, I doubt if they could all be packed into our universe. But if there be non-denumerably many Spaces, then there is room enough in reality (if I may so express myself) for their accommodation. With one proviso. We must limit ourselves to infinite discourses which are denumerably infinite in length. I have already raised the possibility of infinite proofs of non-denumerable extent. A proof of the continuum hypothesis or of the non-existence of random numbers might very well require that scope. Now if there exist non-denumerably many Spaces, then even such proofs might be accommodated, not in any one Space certainly, but by spilling over into all of them. Unfortunately, it is impossible that all such discourses could be accommodated in any pluriverse, no matter how extensive. Here we touch upon the grave cosmological import of the paradox of the transfinite. It is easy to prove (with complete generality) that for any class of transfinite numbers there exists a transfinite number which does not belong to the class. The consequences are momentous. It means that there is no class of all transfinite numbers and, indeed, the very notion of a totality of all transfinite numbers would seem to be exploded. We must then confess that no matter how many transfinite numbers (let them even be non-denumerable) may be accommodated in our pluriverse, there must necessarily be a transfinite number which has been refused admittance. A corollary to this theorem is that for any class of discourses (allowing for discourses which are of non-denumerable extent) which have been enacted in our pluriverse, there must necessarily be discourses which cannot be accommodated. All discourses which are merely of denumerable infinitude can certainly be installed. It is only when we push on to those of non-denumerable scope that our aggrandizement is arrested. I am strongly tempted to argue that the paradox of the transfinite constitutes a strict a priori proof of the impossibility of omniscience.

In a curious way the finitist has his vengeance in the end. How many Spaces exist altogether? One? Two? Three? . . . or? or? or? . . . . Is it possible that for each and every cardinal number a, be it finite or transfinite, there exist a Spaces in our pluriverse? No. For the paradox of the transfinite necessitates that for any class of cardinal numbers there must be a cardinal that lies outside the class. This means that no matter how many Spaces our pluriverse may contain, a greater pluriverse can always be conceived. In a wildly extended sense of the term, we may say that the cosmic Whole must necessarily be 'finite'. Adopting an extraordinary trope, a class will be said to

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1 The paradox of the transfinite rests on the following considerations. For any finite class $K_1$, it is evident that the class $K_2$ of all sub-sets of $K_1$ contains a greater number of members than $K_1$. This is entirely straightforward. Now Cantor extends the concept of a cardinal number in such a way that he is able to prove that for any infinite class $K_1$ the class $K_2$ of all sub-sets of $K_1$ contains a greater 'number' of members than $K_1$. It is shown that all of the members of $K_1$ can be placed in one-to-one correspondence with a proper sub-set of $K_1$ but that all of the members of $K_2$ cannot be placed in one-to-one correspondence with a proper sub-set of $K_1$. Hence, by a persuasive trope, it is said that $K_2$ contains a greater 'number'—a transfinite number—of members than $K_1$. In general, then, for any class $K_1$ be it finite or infinite, there exists another class $K_2$ which contains as its members all of the sub-sets of $K_1$. It follows that, given any cardinal 'number' $a_0$, be it a finite or a transfinite cardinal, there is another cardinal $a_1$ such that $a_0 < a_1$. 
be ‘finite’ or ‘limited’ if another class with a greater ‘number’ of elements could possibly exist, and by ‘number’ here we mean any Cantorian cardinal, be it natural or transfinite. In this super-sense every actual class of entities must be ‘finite’, and though the actual ‘infinite’ is now impossible, the concept of the ‘infinite’ is not denied all application—it persists in the form of the potential ‘infinite’. Granting that this super-concept of the ‘finite’ can only be understood in terms of a utopian flight of hyperbole, it may be seen to reproduce the original spirit of the standard concept. In the standard sense a class is finite if another class with a greater number of elements (number meaning natural number) can be conceived, and a class is infinite if for any $n, n$ being a natural number, there exist $n$ elements in the class. According to the finitist, every class of entities that exist in concreto must necessarily be finite, and the infinite has meaning only as a potential infinite. In constructing a super-concept of both ‘finite’ and ‘infinite’ we are not departing from the point of the standard distinction, and though we are certainly not finitists, there is a sense in which we must be ‘finitists’. For though there is no logical contradiction in the standard concept of the actual infinite, there is indeed a logical contradiction in the super-concept of the actual ‘infinite’. It is self-contradictory to assert the existence of an ‘infinite’ pluriverse such that for any $a, a$ being a Cantorian cardinal, there exist $a$ Spaces in the pluriverse. In a very transcendental sense, then, the paradox of the transfinite in its cosmological import constitutes an a priori proof of the ‘finitude’ of the world.

Let me restate the burden of my argument as follows. Do we have any a priori guarantee that the number of stars is finite? No. Assuming that the number of stars is infinite, do we have any a priori guarantee that they must be denumerable? Again, no. But if there are non-denumerably many stars, it is evident that they cannot all be packed within our universe. Nor will any finite or even denumerably infinite number of Spaces be sufficient to install them. There must then be non-denumerably many Spaces. But if there are non-denumerably many stars, have we any a priori guarantee that their number is $\aleph_0$? Here also we must answer in the negative. In the end the only a priori guarantee we have is that, whatever the number of stars might be, a greater number in the Cantorian sense can certainly be envisaged, and this means that the Cantorian cardinals are now understood as full-fledged numbers. For only if we can speak of $\aleph_n$ stars, can $\aleph_n$ be accepted as a number on a par with 5 and 186. Otherwise we are forced to say that the higher transfinite cardinals are applicable only to entia rationis and not to entia naturae. It is seen to be impossible to understand the higher transfinite cardinals as full-fledged numbers unless one projects the concept of a pluriverse.

Part 5: The Well-Ordered Pluriverse

Our fifth, and last, objection to Kant’s argument is derived from our Zeno procedure. We have been exploring the large infinite in radical independence of the small, as if there were no connexion the one with the other.

Let us accept Kant’s position that if an infinite world is to be envisaged as a whole there must be a ‘repeated addition of unit to unit’. Furthermore, we may allow that ‘an infinite time must be viewed as having elapsed in the enumeration of all co-existing things’. But we must ask here: what kind of infinite is required? The large infinite or the small? Must it be the large infinite, as Kant assumes? Why not the small? Indeed one minute will suffice to enable us to exhaust an infinite world. We have only to launch a rocket and travel one thousand miles into outer space during the first $1/2$ minute, another one thousand miles after that during the next $1/4$ minute, still another one thousand miles during the next $1/8$ minute, &c. ad infinitum. At the end of the minute we shall have succeeded in travelling an infinite distance. Travelling in a spiral, our metaphysical rocket may visit an infinite number of stars—indeed all of the stars—that may exist within an infinite Euclidean space. The ‘enumeration of all co-existing things’ (assuming with Kant that they are denumerable) is thus seen to require no more than one minute.

At the end of the minute we find ourselves an infinite distance from the earth, and it is here that a new vista is opened up for our investigation. Where in the world are we? On the beautiful meadow? Why not? Although we have established the existence of a non-spatial ‘route’ from the earth to the meadow, this does not preclude the existence of an alternate route, indeed a
spatial route, which will serve the same purpose, particularly as the concept of an infinite spatial distance has now been rendered intelligible. Assuming that ours is a Euclidean Space, the meadow must lie in a Space different from our own. For although any Euclidean Space is infinite, each point in it is a finite distance from every other. There is no contradiction now in asserting both (1) that the meadow lies in a Euclidean Space distinct from our own, and (2) that the meadow lies at a spatial distance from the earth, albeit a very peculiar spatial distance, an infinite distance. It will be evident that any star which is at any finite spatial distance from the meadow must be at an infinite spatial distance from any star in our Space. All stars, then, at a finite spatial distance from the meadow lie within Space § 1, as distinguished from our own Space, Space § 1. There is thus seen to be a spatial route from Space § 1 to Space § 2.

Let us now resume our flight. In the next 1/2 minute we shall travel a second infinite distance. Exhausting all of Space § 2 within that 1/2 minute, we shall find ourselves—where? Back in Space § 1? Perhaps. In which case we can entertain the falsifiable hypothesis that the Whole consists of two, and only two, Spaces which comprise a cyclic unit. The hypothesis is clearly falsifiable because it is quite possible that, instead of returning to Space § 1, we may push on to Space § 3, landing perhaps in that very valley which was previously determined to be removed from the earth by a ‘distance’ of 813 electro-units. Dallying in Space § 3 for 1/2 minute, let us now return to our rocket and travel a third infinite distance, this time within the span of 1/4 minute. Having exhausted Space § 3 in 1/4 minute, we may find ourselves back either in Space § 2 or in Space § 1, and again the falsifiable hypothesis that ours is a cyclic pluriverse—in this case, of three Spaces—may be mooted. For it is quite possible that we may find ourselves in Space § 4. Dallying in Space § 4 for 1/4 minute, we shall travel another infinite distance, this time within the span of 1/8 minute at the end of which we may find ourselves in Space § 5. It will now be evident that in a very short finite interval of time we may be capable of exhausting an infinite sequence of distinct Euclidean Spaces. The hypothesis that we live, not in a universe, but in a pluriverse and, indeed, in a pluriverse containing an infinite number of Spaces, is seen to be precisely verifiable.

Having exhausted an infinite sequence of Spaces, shall we find ourselves back in Space § 1? This is quite possible. In which case we shall have some grounds for asserting that ours is a cyclic infinite pluriverse containing denumerably many Spaces. But there is always the possibility that after exhausting an infinite sequence of Spaces we may find ourselves in a Space which is entirely new to us, and it is this possibility that suggests the hypothesis that the pluriverse may contain non-denumerably many Spaces. Not that the mere fact of descending into a new Space after exhausting an infinite sequence, in itself entails that the number of Spaces is non-denumerable. By no means. For that new Space may be the last Space in the pluriverse, in which case the pluriverse will be of order-type \( \omega + 1 \) (following Cantor’s notation in regard to the transfinite ordinals), and hence denumerable.

If the World be constituted by two infinite sequences of Spaces, then it will be said to be of order-type \( \omega + \omega \); if by three infinite sequences plus a sequence of 86 Spaces, then it is of order-type \( \omega + \omega + \omega + 86 \). In general, the geometry of a pluriverse will be defined, in part at least, by its order-type. Now let us suppose that in 1/2 minute we explore one infinite sequence of Spaces; in the next 1/4 minute, a second infinite sequence; in the next 1/8 minute, a third infinite sequence; &c., ad infinitum. At the end of the minute, assuming that we find ourselves back in Space § 1, we may reasonably suppose that we have visited all of the Spaces in the World, in which case the pluriverse will be said to be of order-type \( \omega^3 \) (neglecting the cyclic character of the Whole); but if at the end of the minute we find ourselves in an altogether new Space, then (assuming that we have reached the last Space in the World) the order-type of the pluriverse will be \( \omega^3 + 1 \).

The most interesting case is the pluriverse of order-type \( \omega_1 \). Here we have the simplest well-ordered pluriverse which is non-denumerable. In this case there are so many infinite sequences of Spaces which succeed one another that it would be impossible in any finite interval of time (or indeed in all of time, past, present and future) to explore them all by means of our metaphysical rocket. A pluriverse of this kind may be said to be Infinite in a strong sense, meaning by Infinite neither the infinite of the standard sense nor the ‘infinite’ of the super-sense.
which we introduced recently, but meaning rather that property of \textit{actually} being so vast as to be incapable of being exhausted. Any pluriverse of order-type less than \(\omega_1\) will be said to be Finite in the sense that it is quite capable of being exhausted. The distinction between Finite and Infinite in this new super-sense is equivalent to the distinction between the denumerable and the non-denumerable. Again, this new super-sense of Finite and Infinite is found to preserve the spirit of the standard concept. For surely what we commonly \textit{mean} by the infinite is that which is so vast as to be incapable of being exhausted, and seeing that we have established that the whole sequence of natural numbers is indeed capable of being exhausted, albeit by metaphysical means, we are entitled to insist that it is not Infinite but Finite.

The World, then, is either Finite or Infinite, and though the hypothesis that it is Infinite cannot be verified (waiving any comprehensive panorama), it admits of a weak, though not a strong, falsification. The difficulty in regard to a strong falsification of the hypothesis stems from the possibility that there may be Spaces which can be reached only by a non-spatial ‘route’. In any event, even granting that proposition \(q\) ‘the World is Infinite’ is neither verifiable nor falsifiable, it may be shown to be meaningful as follows. The proposition \(p\) ‘the world is of order-type \(\omega + 8\)’ is certainly falsifiable. Hence \(p\) is meaningful, but \(p\) logically entails not-\(q\). Hence not-\(q\) is meaningful, and since the denial of any meaningful proposition must be meaningful, \(q\) must be meaningful. Furthermore, the proposition \(s\) ‘the World is well-ordered’ may be neither verifiable nor falsifiable but being logically entailed by \(p\), it must be allowed to be meaningful. In this connection, one may wish to speculate on the cosmological import of Zermelo’s ‘well-ordering theorem’. Zermelo having demonstrated that there is a well-ordering of any set, are we entitled to conclude that there is necessarily a well-ordering of all the Spaces in the World? Unfortunately, Zermelo’s proof is non-constructive in character: it fails to specify any general rule for well-ordering non-denumerable sets. In particular, there is at present no rule available for well-ordering the real numbers.\footnote{A set is well-ordered if (1) it exhibits linear order (e.g. the natural, rational or real numbers taken in order of magnitude) and if (2) every proper sub-set contains a first element \textit{viz.} \textit{uni-}
\textit{versus} the ordering relation. Zermelo’s well-ordering theorem presupposes, and in fact is equivalent to, the controversial axiom of choice which is accepted by Russell but rejected by Brouwer. What should be our own stand in regard to the axiom of choice? The axiom of choice asserts that for any collection of disjoint non-empty classes there exists a set—known as the representative set—which contains precisely one element arbitrarily selected from each of the classes. The bone of contention here is not the issue of the actual infinite as such but rather the question as to whether or not a set may be allowed to ‘exist’ in the absence of any effective procedure that ‘constructs’ the set, \textit{i.e.} which specifies precisely which elements belong, and which elements do not belong, to the set. Although finitism and constructionism have been widely felt to be all of a piece, they would seem to be logically independent of one another.}

to see how even the cosmological octopus, assuming that its tentacles extend into all the Spaces of an Infinite World, could convince itself that it has succeeded in writing down all of the real numbers. If the octopus has non-denumerably many hands (as it were), it could certainly write down in one minute a non-denumerable set of real numbers—one number per hand, all hands writing simultaneously. But how could the octopus be sure, at the end of the minute, that none of the real numbers has been omitted?

Passing beyond order-type \(\omega_1\), what is the empirical difference between a World of order-type \(\omega_1\) and a World of order-type \(\omega_2\), both being non-denumerable? In the first case, although the World as a whole cannot be exhausted by the metaphysical rocket, there does not exist any single Space which cannot be reached by it. In this respect, each Space being a Finite distance (so to speak) from every other Space, the World is very much like any Euclidean Space in its standard interpretation, each point of which is separated from any other point by a finite distance, although the Space as a whole is of infinite extent. In the case of a World of order-type \(\omega_2\) there exist Spaces so far removed from Space \(\Sigma 1\) that they could never be reached by the metaphysical rocket. Every Space in a World of order-type \(\omega_1\) is separated from every other Space by a denumerable set of Spaces. In a World of order-type \(\omega_2\) there exist Spaces which are separated from one another by non-denumerably many Spaces. There is thus an important sense in which the gap between order-type \(\omega_1\) and order-type \(\omega_2\) (despite the fact that both are non-denumerable) is ontologically
greater than the gap between order-type \( \omega_1 \) and all of the
denumerable order-types. For in any World of order-type \( \omega_1 \)
or less the principle of general spatial access is preserved, in
virtue of the free-wheeling negotiability whereby every Space
is within striking distance of every other. What now of \( \omega_2 \)?
Is \( \omega_2 \) the smallest transfinite ordinal which abrogates the principle
of general access? No. Consider a pluriverse of order-type \( \omega_1 + 1 \).
Here there is one (though only one) Space—the last Space—which is removed from Space \( \S 1 \) and, indeed, from
every other Space by non-denumerably many Spaces. The special ontological significance of order-type \( \omega_1 + 1 \) lies in the
fact that it is here, for the first time in a well-ordered World,
that the principle of general access is breached. It is then the
ontological gap between order-type \( \omega_1 \) and order-type \( \omega_1 + 1 \),
not the gap between order-type \( \omega_1 \) and order-type \( \omega_2 \), which is
of the deepest import. If ours be a World of any order-type
equal to or greater than \( \omega_1 + 1 \), then there exist Spaces which we
can penetrate only via a non-spatial 'route'. It may be noted
that if we succeed in entering any Space \( X \) by a non-spatial
'route', the hypothesis that Space \( X \) cannot be reached by any
spatial route from Space \( \S 1 \) is certainly intelligible, being open
to falsification. But what of the specific hypothesis that Space \( X \)
is the last Space in a World of order-type \( \omega_1 + 1 \)? What evidence
is relevant here? I suspect that only the cosmological octopus—
its tentacles reaching out into all Spaces—is in any position
to speak with authority on this matter. Enjoying a World-
wide panorama, the octopus may be supposed to see at a glance
that Space \( X \) is separated from Space \( \S 1 \) by non-denumerably
many Spaces. (We certainly could falsify its claim). But though
the octopus is doubtless able to falsify the hypothesis that the
World is precisely of order-type \( \omega_1 + 1 \), one may doubt whether
it is capable of decisively verifying it. There is always the
possibility that some free-floating Space exists of which the
octopus is ignorant. Indeed, we ourselves might succeed in
calling it to its attention.

Burali-Forti having shown that for any class of transfinite
ordinals there exists a transfinite ordinal that lies outside the
class, it is evident once again that the World must be 'finite':
a greater well-ordered World can always be conceived.
absolutely straight. Otherwise, the rocket-ship would never have landed in Australia. There must have been a slight curvature in its path. Now if we require of a straight line that it satisfy at least two conditions, (1) that it be a line which is the shortest distance between two points, and (2) that if it is indefinitely extended it will never come round full circle, if a straight line is defined in this way, than according to Einstein there are in fact no straight lines. Any line-segment which satisfies the first condition is held to be an arc of a great circle. That is to say, for any line-segment which is the shortest distance between two points there exists a finite distance such that if the line-segment is extended that finite distance, what is generated is a great circle. Hence the curvature of space.

It is important to realize that Einstein is making an ontological claim. During the past century the naïve approach to natural science as the ontological disclosure of reality has grown almost as suspect as the ontological approach to mathematics. Taking quantum physics as the model of all that is really interesting in science, it has become fashionable to argue as follows. We must not think of the electron as an ens naturae; it is rather to be understood as an ens rationis, a logical construct which enables us to systematize and predict a wide range of phenomena. We must not be so naïve as to ontologize this ens rationis and project it into reality as an ens naturae. Whatever merit this positivistic approach may possess in regard to the electron, it fails to do justice to Einstein’s ontological thesis that space is Riemannian. I think that one is liable to be led astray here by the following considerations. At the present time we are not capable of supplying a strong verification of Einstein’s hypothesis: our rockets are pitifully inadequate for the job. In the absence of such strong verification, if Einstein’s hypothesis is to be of any use to the physicist, it can only be as a schema which successfully organizes the limited astronomical data to which his present instruments have access. One is thus tempted to assimilate Einstein’s hypothesis to the false model of the electron, particularly since according to Einstein’s computations the universe is so enormously vast that no one supposes that our rockets and telescopes will ever, at any foreseeable time in the future, achieve sufficient power to enable those computations to be subjected to the decisive test of either a strong verification or a strong falsification. For all practical purposes, then, it may well be the case that Einstein’s hypothesis can serve no other function than as a schema capable of organizing present and future astronomical evidence. Granting all this, it remains true that Einstein’s thesis can only be partially understood in those terms. However much it flouts our natural pre-scientific understanding of the world in its naïve realism, theoretically it admits of both a strong verification and a strong falsification in accordance with the vulgar canons of that very naïve realism which it outrages. We have only to launch the rocket. Einstein is thus seen to be making an ontological claim as to the nature of the world, a claim that is not to be merely bracketed as a logical construction.

In our discussion of mathematics I have argued that, mathematical locutions being systematically misleading, a special hermeneutic is required which will serve both to sort out the literal from the tropological and to mine the tropological locutions of their ontological import. I should argue in the same way in regard to scientific locutions: they are no less systematically misleading. The naïve approach to natural science as ontology pure and simple cannot be accepted without many reservations. Here, also, a hermeneutic is required which will explore the ontological import of the wide range of technical tropes in which scientific discourse abounds. Even in regard to Einstein’s thesis, where the tropical element would seem to be at a minimum, there are serious difficulties. Our vulgar concept of a straight line is over-determined. We require of a straight line not only that it satisfy condition 1 but also condition 2. Any line, certainly, which on being extended can be shown to come round full circle, we refuse to call a straight line. This refusal is built into the very meaning of ‘straight line’ as it is employed in our mother-tongue. Does Einstein’s thesis entail, then, that there are no straight lines, meaning by ‘straight line’ what we commonly mean in our vulgar pre-scientific discourse? Are all the paradigm cases with reference to which the vulgar concept of a straight line is learned, shot through with error? If ‘straight line’ in vulgar discourse connotes a line which satisfies conditions 1 and 2, must we say that in fact (our Space being Riemannian) it denotes nothing?

These questions, in somewhat simpler form, are seen to
antedate Einstein. Centuries ago a wicked king banished a loyal baron from his realm, ordering him to walk due west in a straight line forever and ever and never to show his face again in the kingdom. Thirty years later the loyal baron, having travelled in a 'straight line' around the earth, found to his infinite distress that he was arrested at the eastern frontier of his homeland. No one, himself included, was willing to allow that he had really travelled in a straight line, though after it was decided (various evidence being digested) that the earth must be round, there was serious perplexity. What should the baron have done, tunneled his way through the earth? When the king ordered him to walk in a straight line due west, nothing of the kind was intended. In fact, if the baron had undertaken to dig his way west (assuming that it is possible to dig one's way west), the king would have executed him on the spot—for flouting the clear meaning of his order. The High Court, assembled to try the case, handed down the following decision. The king's order was declared null and void. Although the order was not logically self-contradictory, nevertheless, owing to the shape of the earth, it was found to be self-defeating.

I have insisted that in vulgar discourse any line which on being extended can be shown to come round full circle, we refuse to call a straight line. Is that quite true? Consider two ship captains, A and B. A believes that the earth is flat; B knows that the earth is round. Both A and B will have no hesitation in accepting a commission to pilot a boat in a straight line from San Francisco to Tokyo. It is altogether, and equally, clear to both what is meant by the terms of the commission. The point here is that we who know that the earth is round persist in using the concept of a straight line in precisely the same way as our ignorant ancestors. It may then be argued that, though A's use of the concept 'straight line' is certainly accompanied by a false factual belief, namely that the earth is flat, we must not incorporate that false belief into A's use of the concept. If it be acknowledged that when A says 'the boat has sailed in a straight line from San Francisco to Tokyo' he means precisely the same thing as what B means when he says 'the boat has sailed in a straight line from San Francisco to Tokyo', then it follows at once that our account of the vulgar concept of a straight line, as satisfying conditions 1 and 2, is grossly mistaken. I submit, however, that A and B do not mean the same thing. Moreover, the use which they make of the concept 'straight line', even in the present nautical context, is not identical. How is the difference to be shown? A and B have arrived in Tokyo. They are now ordered to continue sailing along the same course which carried them to Tokyo from San Francisco. They are thus required to dig a canal across Japan and, later, across the Asian mainland. Combining forces in the enterprise, B says to A, 'I wonder how long it will take us to return to San Francisco along this beastly route', 'Return to San Francisco? Impossible!' replies A. 'We are sailing in a straight line, and it belongs to the very meaning of a straight line that it can never come round full circle.' What the expression 'straight line from San Francisco to Tokyo' connotes to A is quite different from what it connotes to B, but the route which the expression denotes to A is the same route which it denotes to B. The concept of meaning is complex: there is denotative meaning and there is connotative meaning. Do A and B mean the same thing when they speak of sailing in a straight line from San Francisco to Tokyo? Distinguo. Connotatively, they mean the same thing. Connotatively, they mean different things. So long as A and B confine their discourse to the limited context of the San Francisco-Tokyo route, the difference in connotative meaning remains invisible. But this difference in connotative meaning must not be regarded as purely internal to the mind. Any difference in connotative meaning is, in principle, capable of being cashed in denotive terms, as the extension beyond Tokyo of the San Francisco-Tokyo route illustrates. Now it should be clear that, given what 'straight line' connotes to A, A is in error when he says that he has travelled in a straight line from San Francisco to Tokyo. The route which A's concept 'straight line' in fact denotes is a tunnel through the earth—waiving Einstein. But if space is Riemannian, then A's concept of a straight line is in fact devoid of all denotive reference.

Very well. Granting that what A connotes by the concept 'straight line' is a line which satisfies conditions 1 and 2, how can this account apply to what B means by 'straight line'? Surely B means by 'straight line' something quite different? No. Or, rather, yes and no. Distinguo. Although B says, as a matter of course, that he has travelled in a straight line from San
Francisco to Tokyo, he will freely admit that, in the strict sense, his route was not really straight: he has sailed along the arc of a terrestrial circle. *In the strict sense*. What does that mean? Is it like saying that, strictly speaking, the whale is not really a fish but a mammal? That Mars is not really a star but a planet? (With my own ears I have heard a zoologist say that most people do not realize that the fly is an animal.) No. It is not like that at all. The concept of the 'strict sense' covers a wide family of diverse usages. When the vulgar designate the whale as a fish, they mean scarcely more than that its normal habitat is the water. They are not mistaken. If the biologist were able to show that actually the whale is a land animal (living on remote, uninhabited islands) which comes to the attention of men only on those occasions when it goes for a swim in the ocean, then indeed the vulgar will have been convicted of error in their designation of the whale as a fish. The facts being what they are, when it is said that, strictly speaking, the whale is not really a fish, all that we have any right to mean by that locution is that the science of biology finds it convenient to assign a special technical sense to the vulgar word 'fish' and that, in this technical sense, the whale is misclassified as a fish, although in the vulgar sense of the word the whale is indeed a fish. When B admits that in the strict sense he has not travelled in a straight line from San Francisco to Tokyo, does he mean that in some technical sense of the word his route was not straight? Not at all. He means that it was not straight in the vulgar sense of the word, given what the word vulgarly connotes. And B himself continues to accept the vulgar sense as the primary, authoritative meaning of the concept 'straight line'. It is only in some secondary sense of the word that he feels himself entitled to say that he has travelled in a straight line from San Francisco to Tokyo. Better: we may regard his locution as elliptical.

In the light of our present discussion I think that the naïve approach to Einstein's thesis is in large measure vindicated. If the universe is Riemannian, then all of the local paradigm cases with reference to which the vulgar concept of a straight line is learned, are found to be infected with error: they fail to exemplify the very concept which they are expressly intended to illustrate. Which is not to deny that, in one sense, the scientists who speak of a rocket travelling in a straight line to the moon, mean the same thing as the vulgar who also speak of a rocket travelling in a straight line to the moon. Both mean the same thing in the sense that the same route is denoted by both. In another sense they mean something rather different. The vulgar mean a line which satisfies condition 2 as well as condition 1, whereas the scientists, since Einstein, mean a line which satisfies only condition 1. It is precisely by driving a wedge between denotation and connotation that the Einsteinian concept of a straight line succeeds in superseding the vulgar concept. But though the scientific concept supersedes, it does not annul, the vulgar concept. It is not merely that the vulgar locution 'the rocket travelled in a straight line to the moon' is nominally preserved in the scientific account. More than just the words are preserved. The meaning is also preserved—certainly part of the meaning is preserved. A very large part. For not only is the same route denoted by both. What 'straight line' connotes to the scientist is not altogether different from what it connotes to the vulgar. The scientific straight-line-to-the-moon is literally the shortest-distance-between-two-points as the vulgar understand it. It is only in lopping off condition 2 (which is by no means trifling) that the scientific concept diverges from the vulgar. There is thus seen to be a continuity between our vulgar understanding of the world and our scientific understanding of the world, and though the ontological import of the one is in part erroneous, it is at once perfected and in part preserved by the other.

I do not wish to leave the impression that here in Einstein's thesis is to be found the model par excellence of all that is of any philosophic significance in natural science. Science, like mathematics, comprises a 'motley of techniques'. Any comprehensive account of the ontological import of science must proceed by way of a detailed ad hoc examination of case after case after case, with scrupulous regard for the conceptual dynamics which are peculiar to each. It is unfortunate, however, that the naïve approach to natural science as ontology has been almost entirely crushed in modern philosophy. Those who have been most enraptured by the achievements of science, have been precisely those who have been most hostile to ontology—the positivists; and those who have been most sympathetic toward
ontology have been precisely those who have been most eager to cut down the pretensions of science. A deplorable state of affairs. Nothing has so much enfeebled the progress of metaphysics as its studied estrangement, since Leibniz, from natural philosophy.

Part 2: Measurement and the Absolute

We have distinguished two types of 'strict sense'. There is a third type of 'strict sense' which is very different from the other two. It is this third type which negotiates the passage from the vulgar concept of a straight line to the mathematical concept. Learning the vulgar concept of a straight line includes learning to distinguish between lines which are more or less straight. Some lines are said to be straighter than others. Moreover, one and the same line may count as straight and even as perfectly straight (vulgarily speaking) in one vulgar context which in another vulgar context will be dismissed as not straight at all but as very crooked. The criteria which govern the application of the concept 'straight line' vary according to our needs and purposes. The concept is thus seen to be context-determined. Nevertheless, despite this relativity of the concept in vulgar practice, implicit in this relativity there lies an absolute standard. This is to say no more than that even young children will understand how to play the straight line game. In this game each contestant is to draw on a sheet of paper the straightest straight line that he is capable of drawing. (The game may be played either with or without technical implements.) The one who draws the straightest straight line wins the game. The criteria here are absolute: there is no adaptation of the concept to parochial needs. At once the following question arises. Is it possible to draw a straight line which is so very straight that no straighter line could possibly exist? Such a line will be said to be straight in the strict sense. Although it is always possible, in principle, to falsify the claim that any given straight line is a straight line in that strict sense, it has been believed that it is altogether impossible, even in theory, to verify the claim. It is at this point that our suggestion of a metaphysical measure is indispensable.

One must not suppose that when B says that his nautical route from San Francisco to Tokyo is not really straight in the strict sense, one must not suppose that B means by the strict sense the mathematical strict sense. How do we know this? Because when A, on learning that the earth is round, comes to realize that his route was not really straight, he is not discovering that, contrary to his original belief, his route was not mathematically straight: he never believed that it was. What I have called the mathematical strict sense is understood by young children even before they have studied any mathematics. To understand the mathematical strict sense is simply to know how to play the straight line game, and to know how to play the straight line game does not require being taught how to play it. No new rules are learned. Any child who has learned the vulgar concept of a straight line has already learned how to distinguish between lines which are more or less straight: the one entails the other. The innocent child—trailing clouds of glory—may thus be said to understand or virtually to understand the concept of a line which is so very straight that no straighter line could possibly exist, and it is this line which we mean when, in our sophistication, we speak of a line that is absolutely straight in the mathematically strict sense.

Students of Plato and Wittgenstein will view this account of ours as an effort to combine the insights of both. It is not enough to say, with Wittgenstein, that the meaning is the use. We must attend to his more profound aphorism: 'What I have to do is as it were to describe the office of a king;—in doing which I must never fall into the error of explaining the kingly dignity by the king's usefulness, but I must leave neither his usefulness nor his dignity out of account.' ¹ Is it accidental that this aphorism was struck off in a study of mathematics? Hardly. What, after all, is the use of a mathematical locution? Is the high dignity of pure mathematics to be reduced to the usefulness of applied mathematics? No. But no account of mathematics is complete which, like Russell's, abstracts from its practical usefulness. Sometimes Wittgenstein speaks of language as a tool; other times he speaks of it as a game. But a tool and a game are quite different. The one must be understood in

¹ Foundations of Mathematics, V, § 3.
terms of its use, the other in a rather more gratuitous fashion. To understand the vulgar concept of a straight line requires that we understand it in both ways, both as a tool (and it is here that the relativistic aspect is crucial, where the criteria vary according to our needs and purposes) and also as a game, specifically in its role in the straight line game, where the absolute character of the straight line comes to sight. The game is not merely a game: it prepares the way for ontology or that understanding of things as they are in themselves independent of our needs and purposes.¹

There is a whole class of vulgar concepts that have this Janus-like character, one face being turned toward specifically in its role in the straight line game, where the relativistic aspect is crucial, where the criteria is not merely a game: it prepares the way for ontology or that absolute character of the straight line comes to sight. The game absolutely straight in the mathematical, i.e. Euclidean, sense, does he also any breadth? This is another matter. The child, in learning the vulgar understand what it is for a line to be characterized by pure length without are more or less thin, and concepts of thickness and thinness, immediately adjacent to another mathematical line. Not so with the child's deprived of all breadth. But it is not such a line which the mathematician has line of absolute thinness, which is rather to be understood by us, in our terms, as a line of infinitesimal thinness, so to speak. There is thus found to be a great difference between the following two properties of a mathematical straight line: (1) its absolute straightness, and (2) its lacking all breadth. The innocent child understands the first property; he does not understand the second, i.e. he does not understand it in the way in which the mathematician understands it. It is in principle impossible for any actual line to be characterized by the second property. Hence the property is a mere ens rationis. Very different is the first property which, in principle at least, is capable of characterizing an actual line. The mathematical straight line is thus seen to be compounded out of an ontological element, on the one hand, and out of a merely conceptual element, on the other. Distinguis. The innocent child in his wisdom understands the first and, with equal wisdom, shrinks from the second. To attribute so much insight to a mere child is doubtless implausible, and yet it follows at once from a behaviouristic account of what it is to understand something. There is certainly no explicit conscious awareness of these abstruse principles in the mind of the child, but if to understand is to possess a capacity or a disposition of a certain sort, a habitus, as the logical behaviourists insist, then the child is very wise indeed. One cannot but think of Meno's slave boy. It is curious that the theory of behaviourism, which we are prone to associate with a low view of man, should issue in such an elevated Platonism. It should be remembered that a behaviouristic epistemology as defended (say) by Kyle does not reduce the human person to a mere physical body in motion that is to be understood materialistically. It is very much a teleological behaviourism.

¹ Granting that the innocent child understands what it is for a line to be absolutely straight in the mathematical, i.e. Euclidean, sense, does he also understand what it is for a line to be characterized by pure length without any breadth? This is another matter. The child, in learning the vulgar concepts of thickness and thinness, is able to distinguish between lines which are more or less thin, and a fortiori he knows what will count as a falsification of the claim that some particular line is so very thin that no thinner line could possibly exist. Such a line might be described with propriety as a line devoid of all breadth. But it is not such a line which the mathematician has in mind. Why? Because the mathematical line is one which cannot exist immediately adjacent to another mathematical line. Not so with the child's line of absolute thinness, which is rather to be understood by us, in our terms, as a line of infinitesimal thinness, so to speak. There is thus found to be a great difference between the following two properties of a mathematical straight line: (1) its absolute straightness, and (2) its lacking all breadth. The innocent child understands the first property; he does not understand the second, i.e. he does not understand it in the way in which the mathematician understands it. It is in principle impossible for any actual line to be characterized by the second property. Hence the property is a mere ens rationis. Very different is the first property which, in principle at least, is capable of characterizing an actual line. The mathematical straight line is thus seen to be compounded out of an ontological element, on the one hand, and out of a merely conceptual element, on the other. Distinguis. The innocent child in his wisdom understands the first and, with equal wisdom, shrinks from the second. To attribute so much insight to a mere child is doubtless implausible, and yet it follows at once from a behaviouristic account of what it is to understand something. There is certainly no explicit conscious awareness of these abstruse principles in the mind of the child, but if to understand is to possess a capacity or a disposition of a certain sort, a habitus, as the logical behaviourists insist, then the child is very wise indeed. One cannot but think of Meno's slave boy. It is curious that the theory of behaviourism, which we are prone to associate with a low view of man, should issue in such an elevated Platonism. It should be remembered that a behaviouristic epistemology as defended (say) by Kyle does not reduce the human person to a mere physical body in motion that is to be understood materialistically. It is very much a teleological behaviourism.
The meaning is revealed by the use. I may then undertake to speak French myself, and though I may suppose that I understand the meaning of the word 'chapeau', I may be mistaken. How will a Frenchman determine whether or not I understand the meaning of the word? He will look to see if I use the word correctly. If I use the word correctly, then ipso facto I understand its meaning. To understand the meaning of a word is to be able to use it correctly in accordance with the conventions of the language in which it plays a role. In the same way, to understand the meaning of the English word 'equal' in its diverse applications (e.g. equal weight, equal length, equal speed) is to be able to use it correctly in accordance with the approved conventions of the language. Plato's account is thus seen to be a travesty. He has assigned a super-sense of his own to the word, this factitious metaphysical equality of his invention. Nothing is more common than to say that two boys are of the same height or even that one is exactly as tall as the other. Even in work demanding high precision, we do not hesitate to say that two metal rods, say, are of the same length or that they are precisely equal in length. Examining the actual use of the concept, we find that it is context-determined. A time-piece which for vulgar purposes is said, and correctly said, to be 'accurate' may be dismissed as 'very inaccurate' in a context demanding high precision. It is doubtless owing to the contextually variable usage of these concepts (i.e. accuracy, equality, straightness, &c.) that Plato, in his effort to find a fixed meaning for these concepts, elevated them into a metaphysical realm. Hence he vainly sought to find a real meaning for these words hidden behind the actual usage; but there is no such real meaning: there is only the usage.

There is so much truth in both of these accounts, Plato's and Wittgenstein's, that we cannot but undertake to bring them into line with each other. Plato might reply to Wittgenstein as follows. Examine the actual usage we must, yes; but let us examine it with some care. When it is said in vulgar discourse that two men are of exactly the same height, is this not an elliptical locution, meaning that within the given context and relative to the demands of the occasion they may count as equal. You will admit that? Certainly. Now consider two time-pieces, the one counting as accurate for vulgar purposes, the other as accurate in a context demanding high precision. Do we not also say (and this, too, is part of the actual usage) that the second time-piece is more accurate than the first? Relative to what context is that statement made? The meaning of 'accuracy' may be context-determined, but these various contexts are not all on the same level; they are ordered in a hierarchy. One time-piece is thus found to be, absolutely speaking, more accurate than another, even though the less accurate time-piece may be described with propriety as very accurate indeed, relatively speaking, i.e. relative to a specific context. It is thanks to this hierarchy of contexts that we can distinguish one type of work as demanding a higher degree of accuracy than another. The distinction between what is true relatively speaking and what is true absolutely speaking is built right into our vulgar discourse. It is not accidental that the absolute sense of 'equal in weight' should come to sight in the course of a gratuitous inquiry conducted to resolve a frivolous contest between two fat men. Although the judges do not have at their disposal any set of metaphysical scales, they would eagerly welcome them for their purpose were they but available. What is their purpose? Simply to determine whether or not the two men are equal in weight, absolutely speaking. Their purpose, then, is disinterested here: it is an idle quest for the truth pure and simple (which is not to deny that the judges may not have ulterior motives of their own).

This particular context is privileged; it is not merely one context among others. It is the paradigm case: it brings to light the paradigmata, in the Platonic, not the Wittgensteinian, sense. The absolute or metaphysical sense of 'equal' is the privileged sense which is variously qualified in different need-determined contexts. When the mechanic says that the two metal rods are exactly equal in length, he is saying in an elliptical form that, relative to the degree of precision which his work requires, the rods qualify as equal. Implicit in this locution is a tacit reference to absolute equality, which only some metaphysical measure might succeed in verifying. The mechanic is thus implying a distinction between what it is for one metal rod to be in itself exactly equal to another, absolutely speaking, and what it is for that same metal rod, merely relative to his work, to be exactly equal to the other. Lest it be felt that we are reading too
much into the mere shop-talk of working men, these men themselves in a spirit of playful rivalry will on occasion compete with one another to see who can construct two metal rods most nearly equal in length. It is in this spirit of play that the absolute comes to sight. One must not suppose that we have here merely two types of contexts, the context of work and the context of play, the concept as a tool and the concept as a game, as if they were on a par. No. The mechanics themselves acknowledge the concept at play as cognitively authoritative. To understand the meaning of the word ‘equal’ is to be able to use it correctly both at work and at play and, not least of all, it includes the settled habitus of recognizing the concept at play as cognitively superior to the concept at work. Actually, the concept at play is inherently disposed toward a transcendent utility of its own. For if we were ever to succeed (per impossibile) in constructing two metal rods which were absolutely equal in length, they would also count as equal in all working contexts whatsoever. They would thus be supremely useful, being exchangeable at will.

The Platonic position is especially strong in connexion with concepts of measurement. By their very nature, these concepts are oriented toward an infinite sequence of decimal places. Thanks to this directedness toward the infinite, one type of measure is absolutely more accurate than another if it proceeds to a greater number of decimal places. Less convincing is Plato’s programme of assimilating all other concepts or, at any rate, many other concepts to the mathematical model. Justice, courage, knowledge, being—these also are to be seen as Janus-faced, one face being turned toward the absolute, the other toward the relative. Yet even here the conceptual dynamics may be in some respects analogous to the other. At any rate, when we inquire as to what it is for two bodies to be of equal length, we must distinguish between what it is to be, relatively speaking, and what it is to be, absolutely speaking. Two very diverse modes of being are thus brought to light. We may say that for two bodies to be equal in length, absolutely speaking, is for them to be such as to yield the same infinite sequence of digits on being subjected to metaphysical measurement—with the proviso that there are no infinitesimal quantities. On the other hand, what it is for two bodies to be equal in length, relatively speaking, will vary indefinitely according to specific human needs and purposes. Not surprisingly, there will be no science of this relative type of being. For science is the disclosure of what it is to be, not relatively, but absolutely speaking. Science is the uncovering of the absolute.¹

As in the case of being, so also in the case of knowledge, we must distinguish between what it is to know, absolutely speaking, and what it is to know, relatively speaking. Specifically, we must distinguish between what it is to know, absolutely speaking, that two bodies are equal in length (and here metaphysical measurement is required) and what it is to know, relatively speaking, that they are equal in length (and here there is such a copious diversity of contexts that no fixed formula can be assigned). Wittgenstein is right. To understand the meaning of a word, one must study it in actu exercito. What, then, is the meaning of ‘equal in length’? There is a whole family of usages all of which are various qualifications, diversely disposed in different contexts, of the one absolute sense.² This one absolute sense must not be seen as some mere Platonic construct. Even a young child knows that when he is solemnly assured that two sticks of candy, say, are exactly equal in length, they are actually equal only relative to the approved conventions of the nonce. We know that if the child were able readily to exercise the Zeno procedure, he would often insist on employing it, lest his brother receive the longer stick. Could he but have his way, the child would be satisfied with nothing less than absolute equality. No one need teach him this concept. It is already implicit in the vulgar concept of his mother-tongue. It is in an effort to secure that absolute equality that he frequently engages in some disgruntled, and ineffectual, measurement of his own. Again, it is not accidental that the child’s earnest demand for absolute equality should be essentially

¹ This is illustrated, at an advanced level, by Einstein’s proof that the space-time interval between any two events is an absolute for all cosmic observers, even though what will count, on the one hand, as the spatial interval and what will count, on the other, as the temporal interval, must (if taken in isolation) vary from one cosmic observer to another.

² I am waiving Einstein in this discussion. With Einstein, of course, even the ‘absolute’ sense of ‘equal in length’ proves to be relative, viewed on a cosmic scale, being superseded by the new absolute of the space-time interval.
frivolous. The pleasures of vanity gratified apart, his toothsome pleasures will not be at all diminished or enhanced by any jot or title of a difference between his own stick and his brother's. With what right, then do we call the absolute sense of equality the prerogative meaning of the concept, seeing that it is usually in abeyance? Ask the child: he will tell you. In all its other uses the concept is always qualified, if only tacitly, these other uses being always elliptical. The prerogative use of the concept is the use par excellence, not to be confused with any of the uses of the garden variety.

It must not be supposed that the absolute always lies beyond the horizon. Not at all. When we say that one stick is longer than another, no qualification is required. These two sticks are absolutely unequal in length; those two sticks are only relatively equal in length. The absolute, then, transcends the empirical at only certain critical points, being otherwise directly accessible. For there is seen to be a lack of cognitive parity between the logically correlative concepts of the equal and the unequal. This should not be surprising: the absolutely unequal is always defined with reference to a finite number of decimal places, whereas the absolutely equal is necessarily defined with reference to an infinite number of decimal places. As equal and unequal are correlative concepts, so also are finite and infinite, 'unequal' being parasitic upon the finite, 'equal' upon the infinite. The absolutely equal transcends the empirical precisely owing to the transcendency of the infinite, and the absolutely unequal is empirically available thanks to the availability of the finite.

The concept of measurement affords us a natural access to the existence of irrational magnitudes. If two metal rods may be said to be absolutely equal in length only if they yield the same infinite sequence of digits, it is wildly unlikely in any particular case that the sequence will be periodic in character. Precisely by anticipating a non-periodic sequence in any particular case of measurement, we are recognizing the irrational magnitude as the normal case. Rationals are seen to be very much the exception.

In the course of the present discussion I have been engaged, not like Wittgenstein in laying down some general theory of meaning, but in exploring the specific meaning of our concepts of measurement. Whether or not the peculiar dynamics which are operative here are also to be found elsewhere in other concepts, may be allowed to remain an open question. If this Janus-like character of the concepts of measure, one face being turned toward the absolute, the other toward the relative, should be a privilege exclusively enjoyed by the concepts of measure alone, then ontology in the Platonic sense may be presumed to be virtually equivalent to mathematical physics or to pure mathematics accompanied by mathematical physics. But not entirely equivalent. For we have seen that a distinction must also be drawn between absolute knowledge and relative knowledge, if only in regard to the concepts of measure. Knowing itself, be it only in this one domain, is seen to be Janus-faced. Hence not only the first but also the second of the two poles of Plato's thought—mathematics and the soul—must needs be accommodated in any ontology that is Platonically characterized as the science of the absolute.

**Part 3: Meaning and Teleology**

Seeing that Wittgenstein's is a general theory of meaning, there is a correspondingly general criticism to which it is peculiarly vulnerable, even on its own terms, indeed especially on its own terms. Wittgenstein insists that 'it is clear that every sentence in our language “is in order as it is”. That is to say we are not striving after an ideal'.¹ But if the meaning of a word is to be understood in terms of its use, its job, then like any tool or implement it will perform its job either well or poorly. What is the use of a shield? What is its job? To ward off the spears of the enemy. Is the shield ‘in order as it is?’ Does the shield succeed in doing the job for which it is designed? According to Homer, the shields employed by the heroes of the Trojan war were only too frequently penetrated by enemy spears: the heroes then would bite the dust. No wonder that these warriors should be ‘striving after’ the ideal shield. Hence the shield of Achilles fashioned by a god. For the ordinary shield simply fails to do at all adequately the job for which it is

¹ *Philosophical Investigations*, I. § 98.
designed. If words are tools, if concepts are implements (and I am highly sympathetic toward such a teleological or functional approach to language), then we have a standard by reference to which language may be criticized and even improved, namely the job of work that is to be done: the purpose of our activity. To use a word correctly is not—primarily—to use it in accordance with a convention; it is to use it effectively to do a job. What is it to use a buzz-saw correctly? To use it in accordance with a convention? Hardly. In regard to language there is seen to be a double standard of correct usage: correct usage vis-à-vis a convention, and correct usage vis-à-vis the job to be done. (Cf. the Grælus in this connexion.) The aborigines of Tierra del Fuego are said to be able to count only up to four. Anything beyond four is simply 'many'. Is it 'clear' that this primitive language of number 'is in order as it is'? How well does it do its job? If the shields of these natives are inadequate, why not also their system of number? Almost certainly, upon studying their 'form of life', we shall find that there are occasions when their impoverished system of number fails in the tasks that are set it.

It is not accidental that I have chosen the shield, a weapon of war, to illustrate what it is for an implement to be defective. For it is logically impossible for every weapon of war to be 'in order as it is'. One recalls the time when Zeus ordered Hephaistus to make himself available to Greeks and Trojans alike. ('Let us whip up this war,' said Zeus.) Quite naturally, the Greek and Trojan warriors imprompted the divine artificer to supply them not only with impenetrable shields but also with irresistible spears. Even an otherwise omnipotent god is thus seen to be incapable of answering all the prayers of men at war with one another.1 I do not wish to suggest that the logically defective character of military weapons may serve us as a

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1 Whenever warriors succeed in improving their shields, they set about at once to improve their spears, at least pro tanto. We can imagine (and not only imagine) a technological race, logically endless, between the frantic efforts of warriors to build the ideal shield and their equally frantic efforts to build the ideal spear. It is not surprising that war should be the most powerful spur to technological advance. In our own time, ever since the building of the atomic bomb, modern science (theoretical as well as practical, for the second presupposes the first) has been largely financed by government funds—for the express purpose of war. The scientists have been bought! Is this sinister alliance between science and war to be the reductio ad absurdum of man's destiny? This, then, being our form of life, there is a job to be done.

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model for all the inadequacies of language, though it may be supposed that some of these inadequacies are to be understood in that way.1 Many of the tools and implements even of peace are, if not logically, then certainly empirically less than satisfactory for their respective jobs. One might infer as much in regard both to vulgar and to scientific concepts. Couriers with urgent messages to deliver have been known to spur their horses unmercifully, flogging them sometimes unto death, in an effort to meet some imperative deadline. So, too, there may be a flogging of words and concepts in an effort to get the job done. Furthermore, as one may be led to replace the Pony Express by the telegraph, owing to its failure as an instrument to transmit messages with the dispatch which we require, in much the same way one may seek to replace an existing, but inadequate, language by an ideal one, one that is ideally suited for our purposes.

When Wittgenstein thinks of a language he is prone to think of it, primarily, as being embedded in a settled form of life in which the jobs to be done are relatively stable. But even in the most primitive society there is a flogging of horses, and mythological shields are projected by the poets. Later, when man will be aggressively on the move, he may come to project, not mythological shields, but metaphorical rockets and computers. Even Wittgenstein would admit that a language can be in order as it is only if the form of life in which it is embedded is in order as it is. But when is man's life, whatever be its form, so happily constituted? In Homer the heroes are interminably performing sacrifices to the gods. What is their purpose? To placate the enmity of the gods, ever vengeful and capricious. How successful are these propitiatory rites? How effective this religious language? What does Homer say? He shows us that they are almost useless. It was to be expected, then, that man should eventually undertake to replace these devices by others.

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1 Thus it is possible that Russell's paradox and, in general, all of the logico-mathematical paradoxes may arise in a comparable fashion, that is to say, within a language or system unwittingly designed to perform logically incompatible jobs. Discussing the paradoxes, Wittgenstein says, 'The fundamental fact here is that we lay down rules, a technique, for a game, and that then when we follow the rules, things do not turn out as we had assumed... things turn out otherwise than we had meant, foreseen', ibid. I, § 195. Here, then, is at least one language-game that is not in order as it is.
which promise (it is only a promise) to promote his security, and even his felicity, with greater success—the machines of a technological (and atheistic) modern science.

On a purely theoretical level we have seen that the language of measurement in science is necessarily oriented toward the ideal of absolute accuracy, and though that ideal will never be realized, it must ever remain regulative of the whole enterprise. No wonder that the real numbers should be decisive in science whereas the natural and rational numbers should be sufficient for ordinary life.

No one doubts that our vulgar language is inadequate for the purposes of science. Is it perhaps adequate for the vulgar purposes of life? That question cannot be answered a priori. What are those purposes? What do we want out of life, what are we seeking, what are we striving for? When Wittgenstein insists that the meaning of a word is its use, he thinks of the meaning as being conventional and of the use as being functional. But one cannot have it both ways. If language is understood in terms of its teleological import, we must then be 'striving after an ideal'. Only if language may be understood purely as a system of conventions, may it be regarded as being 'in order as it is'. Only then will it be self-validating.

Part 4: Science and Essence

Returning to our metaphysical rocket, it is evident that if our universe is Riemannian, finite but unbounded, we shall not be able to project ourselves an infinite distance from the earth. Less than one minute after launching the rocket we shall find ourselves in Australia, having come round almost full circle in a cosmic geodesic. This great circle that we have traversed, is it a circle in the vulgar sense of the word or is it a circle only by courtesy, so to speak, in virtue of a conceptual trope? We have already decided that our path cannot be described as a straight line in the primary sense of the term. Must it not then be described with propriety as curved? It is certainly not jagged or crooked. The difficulty with calling it curved is this. Any curved line, indeed any line which is not a straight line, can be stretched out so that it is straight. Not so with the cosmic geodesic. It is already perfectly straight, if I may so express myself. The only reason we have for refusing to call it straight is that its terminus a quo and its terminus ad quem are identical. I say the 'only reason', but it must not be supposed that the failure to satisfy condition 2 is not an altogether sufficient reason for withholding the vulgar predicate 'straight' from the cosmic geodesic. Neither straight nor curved (and certainly not jagged or crooked) ἀψηφωδής, the cosmic geodesic may be described as both 'straight' and 'curved' πρότειν τοῦτο. Such a description being adopted, Einstein's thesis is seen to require tropological analysis, and yet it would be a great mistake, in my judgement, to see in these tropes a refutation of the naive approach to Einstein's thesis as an ontological disclosure. The naive approach is right—in the last analysis. But it presupposes, for its proper appreciation, a sophisticated hermeneutic.

Wittgenstein tells the following story.

Suppose a physicist says to you, 'Physics has discovered how a man looks in the dark.' I tell you, do not be surprised until you understand him and see whether he is using these words in an ordinary way. Suppose you say to him, 'I don't understand you.' He may say, 'Don't you understand English—don't you understand "look like", "in the dark", etc.', and suppose he then shows you some X-ray photographs taken of a man in the dark. He has employed a sensational way of expressing what he has discovered, so that it looks like a different kind of discovery.

Although many things in science are like this, cast in the form of hyperbole, I submit that Einstein's thesis is not one of them. Wittgenstein was concerned, above all, with protecting our vulgar understanding of the world from the sophistical objections of philosophers, on the one hand, and scientists, on the other. A noble object. I am delighted with his versatile elenchus in many ways. But I am persuaded that he went too far. Physics and metaphysics are capable of enlarging and enriching—yes, and even amending—our vulgar understanding of the world, without subverting it altogether, as he seemed to fear. Once we have discounted the tropical element in the discourse

1 Unpublished typescript of Wittgenstein's lectures on mathematics in the spring of 1939, p. 1. This is but one of many important passages which are not included in the Remarks on the Foundations of Mathematics.
of Wittgenstein's physicist, the banality of our vulgar understanding is vindicated. How different it is with Einstein's discourse. Marvelous as it appears before the tropical element has been discounted, the true wonder of it all comes to sight only after it has been cashed in literal terms. No longer are we able to see an ordinary flag-pole as we have been seeing it prior to Einstein. We must see it now, not as a segment of an infinite line in Euclidean space, but rather as an 'arc' of a geodesic in Riemannian space. To see something as something is no mere private mental event. It is to be disposed to act in a certain way. The one who sees the flag-pole as a segment of an infinite line in Euclidean space is disposed, in certain circumstances, albeit circumstances which will doubtless never arise in vulgar practice, to act in a very different way from one who sees it as an 'arc' of a geodesic in Riemannian space. It is thus shown that even very humble objects, locally at hand, are only seen for what they really are in the light of an over-arching cosmology. To see something as something is to see it in the light, or darkness, of a horizon that transcends the merely local. It is the task of physics and metaphysics both, in diverse modes, to explore that horizon and hence to illuminate the familiar. The indigenus can only be seen for what it really is in the light of the exotic.

If Einstein's thesis smacks rather too much of the exotic to support the weight that we have laid upon it, there are other, more prosaic examples that will serve to point the same moral. Throw a stone up into the air; it will fall down. What could be simpler, more incontestable? Here we have a paradigm case of what it is for something to fall down. And yet do we mean the same thing as a Baluba when we both refer to the stone as falling down? Distinguo. We both denote the same event, certainly; but what it connotes to us is rather different from what it connotes to him. How is this connotative difference to be cashed in denotative terms? Let us both gaze down a bottomless ravine, only to descry an Australian bushman staring up at us, on the opposite side of the earth. 'You, down there!' the Baluba will shout. 'Are you chained to the earth or what? What keeps you from falling down into the void?' We, for our part, can scarcely repress a smile. When we refer to the bushman as 'down there', do we mean by 'down' what the Baluba means? No. We mean by 'down' 'down as relative to us'. The Baluba means by 'down' 'absolutely down'. When the Baluba says that the bushman is 'down there', he means—among other things—that if their positions were reversed, he would find himself looking up into the face of the bushman and that he would describe the bushman as being up above him. In terms of what 'down' connotes to the Baluba, it must be admitted that the Baluba is in error or partly in error when he insists that the bushman is 'down there'.

In regard to the stone, when we refer to it as falling down we mean that it is falling toward the earth or, more accurately, toward the centre of the earth. The Baluba does not see the falling of the stone as we see it. He sees the stone as falling absolutely down. There is nothing prima facie absurd in such a belief. (The recent researches of Lee and Yang suggest that there may be in nature an absolute right and left, at least in connexion with the 'weak forces' that operate among the so-called 'strange' particles, such as the theta and tau mesons.) The point is this. When the Baluba says that the stone is falling down, he means that it is falling absolutely down, and he is mistaken. What is true of the Baluba holds equally true of any young child among us. All of the paradigm cases with reference to which we learn the proto-concepts of up and down, are infected with error. For the proto-concepts of up and down are the concepts of an absolute up and down; but the cases themselves are cases of a relative up and down. Given what the paradigm cases primitively connote, they are seen to presuppose a cosmology, not a full-blown cosmology, to be sure, but the merest sketch of one. They presuppose a world, that kind of world which is characterized by an absolute up and down. To see the stone as falling absolutely down is to see it against a cosmological horizon; and to see that same stone as we see it as falling only relatively down, is to see it against a rather different cosmological horizon. Ontology is the study of what it is to be, and in examining the diversified modes of being, it must include a study of what it is to be a falling body. It is not enough here to supply a nominal definition of 'falling body' as the Balubas understand that expression. We must explore the

1 Cf. Philosophical Investigations, II, § 11, on seeing something as something.
possibility of a rift between denotation and connotation. Having exposed such a rift, we shall then construct a real definition of what it is to be a falling body. Thus we may say that to be a falling body is to be a body moving toward the centre of the earth, as distinguished from the primitive (and false) definition of a falling body as a body moving absolutely down. The ontological question as to what it is to be a falling body is bound up with the cosmological question as to the nature of the world at large.

Child and Baluba alike are certainly aware that the stone is falling toward the earth; but they see the stone falling toward the earth in the way in which we see the stone falling toward the grass. Just as, for us, the stone is falling toward the grass only per accidens, so, too, for the Baluba and the child, the stone is falling toward the earth per accidens. It is, then, not merely that we see the stone as falling toward the earth but rather that we see it as falling essentially toward the earth. What is merely accidental for the Baluba and the child is essential for us. Science is the disclosure of essence. What a falling body is essentially, is complicated for us by the fact that there are actually, not two, but three concepts of gravity that compete for our allegiance. There is, first, the proto-concept: bodies gravitate downwards—absolutely down. There is, second, the Aristotelian concept: bodies gravitate toward the centre of the earth, the earth being privileged. There is, third, the Newtonian concept: bodies gravitate toward one another with a force directly proportional to the product of their masses and inversely proportional to the square of the distances. As much as the Aristotelian concept is an advance over the proto-concept, so much is the Newtonian concept an advance over the Aristotelian concept. Hence it is a mistake to see the falling stone as moving essentially toward the earth qua earth. It is not the earth as such toward which the stone is moving. It is moving toward the earth only per accidens. Essentially, the stone must be seen as moving toward a neighbouring body the mass of which is enormously greater than its own: this body merely happens to be the earth. Against the background of Newtonian cosmology the innocent locution ‘Look at that stone there falling down!’ means, i.e. connotes, something quite different to us from what it connotes to the Baluba, on the one hand, and to Aristotle, on the other. Which is not to deny that in another sense the locution means, i.e. denotes, the very same episode to all of us. If the concept of meaning were reducible to denotative meaning, then indeed the paradigm cases with reference to which any primary concept is learned, would probably be unimpeachable.

We have advanced from gravity I to gravity II, and from gravity II to gravity III. (With Einstein there is a further advance from gravity III to gravity IV.) As we come to know more and more about the universe at large, we approach closer and closer to the essence of what is familiar and at hand. It is not enough to see something as something; one must see it as it is in itself, and this is possible only if the exotic is brought within the range of the understanding. ‘No one successfully investigates the nature of a thing in the thing itself; the inquiry must be enlarged, so as to become more general.’ 1 For the child and the Baluba the exotic comes to sight in the form of a question, a very deep question indeed, almost a paradigm of what we mean by a deep question. What holds the earth up? Why doesn’t it fall down into the abyss of bottomless space? If philosophy begins in wonder, then philosophy could as readily begin with this question as with any. In his perplexity the Baluba is pacified by a myth. The earth rests on the back of an elephant which rests on the shell of a turtle which swims in the sea of eternity. The ‘sea of eternity’ plays the role here of a sacred concept or pseudo-concept designed to block an alarming infinite regress. 2 As we advance from gravity I to gravity II, we learn the reason why the earth does not fall down. The reason why the earth does not fall down is the very same reason why stones do fall down. Astonishing. Stones fall down because they are attracted to the centre of the earth. The earth itself, like all other bodies, is also attracted to the centre of the earth—speaking in hyperbole. The earth, then, is seen as a degenerate case of a falling body. More accurately perhaps, it may be

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1 Bacon, Novum Organum, I, 70.
2 Cf. De Caelo, II, 19, 594b 12–25. ‘It would indeed be a complacent mind that felt no surprise that, while a little bit of earth let loose in mid air moves and will not stay still ..., yet here is this great weight of earth, and it is at rest. ... By these considerations some have led to assert that the earth below us is infinite, saying with Xenophanes of Colophon that it has “pushed its roots to infinity” ...’
suggested that the reason the earth does not fall down is the very same reason that stones do not fall down. There simply are no cases anywhere of anything falling down, i.e. falling absolutely down. No longer seen as exotic, the earth is now subsumed under the same universal to which the falling stone must be referred. The indigenous and the exotic are found to share a common nature, however much they may appear to the vulgar understanding as radically diverse.

The further passage from gravity II to gravity III proves to be another step in the accommodation of the exotic. If all bodies are naturally attracted to the earth, what holds the stars up? Why have they not fallen to the earth? They must be of a very exotic nature indeed: 'the superior glory of [their] nature is proportionate to [their] distance from this world of ours', for they must be of 'some bodily substance other than the formations we know'.

Travelling to Mars, we find that Aristotle was wrong: the stuff of which Mars is composed is no different from this mundane earth. Moreover, a stone being released on Mars, it is found to fall to the ground in much the same way as it does on earth. We are thus led (albeit more deviously) to gravity III, and again the exotic, on being understood for what it really is, serves to illuminate the local and familiar. Finally, there are anomalies which cannot be explained even by gravity III (such as the orbit of Mercury about the sun) and we are led by Einstein to gravity IV. As our cosmological knowledge of the universe grows ever more extensive, so does our ontological knowledge of what it is to be a falling body, close at hand, grow ever more intensive. What was previously seen as essence is now seen as accident, and what was previously seen as accident is now seen as essence. Whereas for Aristotle the falling stone is seen as moving essentially toward the earth but only accidentally toward a neighbouring body the mass of which is enormously greater than its own, for Newton essence and accident are reversed: the falling stone is now seen as moving essentially toward a neighbouring body the mass of which is enormously greater than its own but only accidentally toward the earth qua earth. Pushing on to gravity IV, we learn that the falling stone is only accidentally moving toward a neighbouring body the mass of which is enormously greater than its own but only toward a body qua body or even toward a body qua mass. Owing to the equivalence of mass and energy, it must now be seen as moving essentially toward a quota of energy which merely happens, in the present case, to be a physical body.

Have we any reason to believe that Einstein's is the last word on the subject? None whatever. A further advance from gravity IV to gravity V would not be surprising, and still another from gravity V to gravity VI, &c. ad infinitum. No account of what it is to be a falling body, no putative real definition of 'falling body', can be accepted by us as final and definitive so long as we fall short of an exhaustive and comprehensive knowledge of the whole world, and not only in the large but also in the small. If science is the disclosure of essence, and hence ontology, it is no less the disclosure of the Whole, or cosmology; and the one is seen to be logically bound up with the other. Understood in this fashion, science proves to be natural philosophy. Natural philosophy is natural science which has been mined for its ontological import through tropological analysis, seen against a metaphysical background. For it is impossible to raise the question of what it is to be a falling body without putting the world itself in question. Is the world an ens reale or is it, as Kant argued, a mere ens rationis, an illegitimate totality, which can only serve us as an ideal—a heuristic ideal—with reference to which our knowledge must always be viewed as incomplete?

Human reason is by nature architectonic. That is to say, it regards all our knowledge as belonging to a possible system. But the propositions of the antithesis are of such a kind that they render the completion of the edifice of knowledge quite impossible. They maintain that there is always to be found beyond every state of the world a more ancient state, in every part yet other parts similarly divisible, prior to every event still another event which itself again is likewise generated, and that in existence in general everything is conditioned, an unconditioned and first existence being nowhere discernible. Since, therefore, the antithesis thus refuses to admit at first or as a beginning anything that could serve as a foundation for building, a complete edifice of knowledge is, on such assumptions, altogether impossible.\footnote{De Cælo, I, § 2.}

\footnote{Op. cit. p. 233.}
Although Kant refuses to accept the propositions of the antithesis as ontologically true, and hence as *constitutive* principles of reason, he finds in them an important *use*: they are to be retained as *regulative* principles of reason. The world will be said to be infinite in the sense that it is always possible for us, in principle, to extend our empirical knowledge beyond any stage to which we may have advanced. On this view the real definition of what it is to be a falling body, must always lie beyond our grasp; and indeed the real definition will itself be a mere heuristic ideal without ontological standing. For we have seen that the real definition of what it is to be a falling body presupposes an exhaustive knowledge of the whole world. If there simply is no whole world as an objective thing in itself, then there is, equally, no real definition of what it is to be a falling body. Which was to have been expected. The rejection of an objective cosmological horizon entails, and is entailed by, the rejection of an objective ontology.

Now it would appear that with Einstein an exhaustive account of the world, at least in the large, is indeed feasible. If the world is finite in the Einsteinian sense, then it constitutes a closed system capable of being comprehensively scanned. Quite apart from any utopian procedures, we should be able to determine that all crows everywhere are black, and a real definition of ‘falling body’ lies within our scope. The hypothesis that the world is infinite, is now seen as empirically falsifiable. Unfortunately, Einstein’s closure of the world tacitly presupposes that the whole world is packed within a single universe. In suggesting the possibility of a pluriverse, we are opening up the world afresh, with the prospect of an infinite horizon once more stretching out before us. It is not difficult to imagine empirical evidence that would lead physicists to posit the existence of Space § 2, even before any actual entry into it had been effected. The physical presence of bodies in Space § 2 might very well be felt by us here in Space § 1, through various disturbances and perturbations which could not be explained or adequately explained on the assumption of a single Space. The actual discovery of a non-spatial route into Space § 2 might merely confirm suspicions already seriously entertained by working physicists. On effecting an entry into Space § 2, we shall find it easy to verify the absence of any spatial route from the earth to the Elysian meadow—assuming that our own Space is Riemannian. We have only to exhaust our own finite Space in a search for the meadow, failing which the independence of Space § 2 from Space § 1 will have been established.

Granting that our own Space is Riemannian, it is quite possible that Space § 2 will prove to be Euclidean. We may thus enjoy the opportunity of exploiting our metaphysical rocket at last. Launching the rocket from the meadow in Space § 2, we shall undertake to travel an infinite distance from Space § 2 to Space § 3 (or even from Space § 2 back into Space § 1).¹ Space § 3 in its turn will be found to be either Euclidean or, in one form or another, non-Euclidean. In general, the World is to be envisaged as an omniverse consisting of α Spaces (α being some cardinal number, finite or transfinite), some of which may be Euclidean, others non-Euclidean. In a curious way Kant’s position (waiving the idealism) is vindicated in the omniverse. No matter how many Spaces we may have access to, be they even non-denumerable, it is always possible that a Space exists which has escaped our notice. There being this lack of adequation between mind and world, the World is to be seen as inherently wide open to novelty. Does this mean that any exhaustive account of the World is impossible? I think not. It is precisely owing to the ever-present possibility that with further evidence our account of the World may have to be superseded—it is precisely this very threat of falsification that secures the intelligibility of the claim that some proffered account is truly exhaustive. Furthermore, no matter how many Spaces there may be, there must certainly be some account which would accommodate them all. If this master account should be non-denumerably infinite, as is quite possible, it can only be referred to the mind of God.

*Part 5: Infinitesimals*

Of the four ‘cosmological ideas’ it has been the first alone which has pre-empted almost all our attention. Absorbed in an investigation of the large infinite, which provides the substance

¹The absence of any spatial route, finite or infinite, from Space § 1 to Space § 2 does not preclude the existence of an infinite spatial route from Space § 2 to Space § 1.
of the first antimony, we have neglected the small infinite, which provides the substance of the second antimony. Taking an overall view of our programme, it will be evident that the small infinite is at least of equal interest to us as the large. For the world is characterized not only but its breadth by also by its depth.

Thesis and antithesis of the second antimony or, as Kant also styles it, the 'second conflict of the transcendental ideas' may be here recalled. According to the thesis, 'every composite substance in the world is made up of simple parts, and nothing anywhere exists save the simple or what is composed of the simple'. According to the opposing antithesis, 'no composite thing in the world is made up of simple parts, and there nowhere exists in the world anything simple'. In regard to the thesis there is some question as to what is to count as a 'simple part' or minim. Is it to be understood as an actual infinitesimal or is it rather to be taken as some rational quantity, such as one-billionth of an inch, which is held to be ontologically, if not mathematically, indivisible? We have had occasion, earlier, to touch on the actual infinitesimal but only in the most inconclusive fashion. It is my present object to execute a precise construction of an actual infinitesimal.

Take a stick of wood. In 1/2 minute we are to divide the stick into two equal parts. In the next 1/4 minute we are to divide each of the two pieces again into two equal parts. In the next 1/8 minute we are to divide each of the four pieces (for there are now four equal pieces) again into two equal parts, &c. ad infinitum. At the end of the minute how many pieces of wood will we have laid out before us? Clearly an infinite number. If the original stick was twenty inches in length, one inch in width, and one inch in depth, what are the dimensions of the metaphysical chips into which the stick has been decomposed? Each chip will be one inch by one inch by—what? So prodigiously thin must each chip be that its value is certifiably less than any rational (or irrational) quantity. Let us now take up one of the metaphysical chips and decompose it further into an infinite number of metaphysical splinters. Each splinter will be one inch in length. Let us now take up one of the metaphysical splinters and break it down into an infinite number of metaphysical motes. As each of the metaphysical motes is very much like a mathematical point we may call it a punct. As each of the metaphysical splinters is very much like a mathematical straight line we may call it a rect. As each of the metaphysical chips is very much like a mathematical plane we may call it a planum.

Seeing that the punct is an actual infinitesimal, are we entitled to conclude that it is also a minim, i.e. an indivisible entity than which nothing smaller can be conceived? No. For the composite which results from the packing together of, 1,000 puncts, is also an actual infinitesimal (the sum-total of any finite number of actual infinitesimals must be an actual infinitesimal), and yet this composite is, by definition, divisible. If a punct is indeed a minim, it is not so simply in virtue of being an actual infinitesimal. Have we any grounds for supposing that the punct may not in fact be a minim? Is there any way of verifying or falsifying this hypothesis? We have only to consult our metaphysical microscope. In 1/2 minute we shall double its power; in the next 1/4 minute we shall re-double its power, &c. ad infinitum. At the end of the minute the microscope will be so powerful that the punct will be rendered visible to us. Now, at the end of the minute, let us continue increasing the power of our microscope. In the next 1/2 minute we shall double the power—the infinite power—of our microscope; in the next 1/4 minute we shall re-double it, &c. ad infinitum. We shall thus be able to determine whether or not the punct is composed of parts. I am suggesting that any punct may be itself composed of infinitely many parts and perhaps even of non-denumerably many parts. It is quite possible that Kant's question as to whether there exist 'simple parts' or absolute minims may be undecidable even for us with our metaphysical microscope.¹

¹ Cf. Aristotle, De Generatione et Corruptione, I, § 2. "To suppose that a body (i.e. a magnitude) is divisible through and through, and that this division
There is seen to be a close connexion between the small infinite and the large. I do not mean merely that the one appears to be an image of the other. Consider two bodies A and A₁, say two chairs, which are separated from one another by an infinite spatial distance. It may be noted here that this hypothesis is intelligible quite apart from any utopianism. I am certainly able to falsify the claim that the Tower of London and the Taj Mahal are separated from one another by an infinite spatial distance. Now let it be the case that A and A₁ are connected to one another by a taut rope of infinite length. Let it also be the case that between A and A₁ along the rope, there are an infinite number of chairs which are ten feet apart. Each chair in the series is a finite distance from either A or A₁, and for any chair between A and A₁ there is a chair which is ten feet away from it on the left and another which is ten feet away from it on the right (see Fig. 1). We are now to undertake to walk from A to A₁. In 1/2 minute we shall walk from A to B, in the next 1/4 minute from B to C, in the next 1/8 minute from C to D₁, &c. ad infinitum. At the end of the minute shall we find ourselves at A₁? Not necessarily. It is quite possible that we may rather find ourselves at B₁ or C₁ or D₁, &c. For any one of the chairs in the right-hand series is an infinite distance from A. Let us assume that at the end of the minute we are at C₁. The great question that now arises is this. At what time in the course of

\[
\begin{array}{ccccccccccc}
\text{A} & \text{B} & \text{C} & \text{D} & \ldots & \ldots & \text{D₁} & \text{C₁} & \text{B₁} & \text{A₁}
\end{array}
\]

\text{FIG. 1}

is possible, involves a difficulty. What will there be in the body which escapes the division? If it is divisible through and through, and if this division is possible, then it might be, at one and the same moment, divided through and through, even though the divisions had not been effected simultaneously: and the actual occurrence of the result would involve no impossibility. Hence the same principle will apply whenever a body is by nature divisible through and through, whether by bisection, or generally by any method whatever: nothing impossible will have resulted if it has actually been divided—not even if it has been divided into innumerable parts, themselves divided innumerable times. Nothing impossible will have resulted, though perhaps nobody in fact could so divide it. Since, therefore, the body is divisible through and through, let it have been divided. What, then, will remain? A magnitude? No./*

"COSMOLOGY II"

the minute were we at D₁? Connected with this question is another question. How much time elapsed while we walked the distance of ten feet from D₁ to C₁? The answer here can only be: an infinitesimal interval of time.

The Z-series of temporal intervals must now be viewed as follows:

\[
\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots + \epsilon + \epsilon + \epsilon + \epsilon = 1
\]

For though the passage from A to B was discharged in 1/2 minute, and the passage from B to C in 1/4 minute, &c., it is clear that the passage from D₁ to C₁ was discharged in 1 epsilon minutes, as also the passage from E₁ to D₁ and the passage from F₁ to E₁, &c. ad infinitum. The Z-series is thus found to be a double series composed of an infinite series of rational quantities followed by an infinite series of actual infinitesimals. The sum-total of this latter series is seen to be itself an actual infinitesimal. These findings ought to come as no surprise to us. Earlier we saw that the sum-total of rational quantities in the Z-series cannot be proved to equal 1—unless we rule out the actual infinitesimal. Furthermore, it is generally accepted that the concept of an actual infinitesimal is free of self-contradiction if it is employed with prudence. Having actually succeeded in effecting a construction of actual infinitesimals (in the punct, the rect and the planum), we now find—on independent grounds—that the very concept of an infinite convergent series is unintelligible apart from an infinitesimal tail-end. Here we have an infinitesimal quantity of the first order which is composed of an infinite number of infinitesimal quantities of the second order. Consider now the distance of ten feet from D₁ to C₁. This distance may be sub-divided into an infinite number of rational quantities, viz. 5 ft. + 2 1/2 ft. + 1/4 ft. + ... The infinitesimal interval of time required to travel the distance of ten feet from D₁ to C₁ is seen to be itself composed of an infinite number of infinitesimal intervals each of which in turn is composed of an infinite number of infinitesimal intervals, &c. ad infinitum. In all of these cases our passage to C₁ along the infinite right-hand series of chairs could only have been conducted at an infinite velocity. Although any body travelling at an infinite velocity must traverse an infinite distance in any finite interval of time which is a rational (or irrational) interval of time, that same body is capable of travelling a finite rational (or irrational)
distance or even an infinite distance in an infinitesimal interval of time.

We are left with a final perplexity. Is the Z-series with its infinitesimal tail really intelligible? We are distressed by the absence of any cut between the first series and the second, between the series of rational quantities and the series of infinitesimal quantities. It makes no sense to speak of a point in time which separates the rational series from the infinitesimal series. For if there were such a point, we might break off our journey from A to $C_1$ precisely at that point. Where, then, would we find ourselves? At some chair in the left-hand series? No. At some chair in the right-hand series? Again, no. At some point in space which separates the two series? There is no such point, there is no actual cut which divides the two spaces. We may indeed introduce the negotiating concept of an imaginary cut but we must not confuse this imaginary point with the standard imaginary point of the mathematician. (The latter does not lie on the same straight line which includes all the rational and irrational points). What then of the infinite rope connecting A and $A_1$? There simply is no actual point on that rope which is not within ten feet of some chair either in the left-hand series or in the right-hand series.¹

This absence of any actual cut cannot but give us pause. You are at $A$, and I am at $A_1$. We are to walk toward each other. In $1/2$ minute you are to walk to B while I shall walk to $B_1$; in the next $1/4$ minute, as you proceed from B to C, I shall move on from $B_1$ to $C_1$, &c. ad infinitum. At no point in time and at no point in space will we pass each other! In fact we shall never come within any finite distance of one another. And yet we shall certainly pass each other by, if I may so express myself. Despite the fact that no logical contradiction has been detected in the execution of our programme, it is difficult to rest altogether easy in our quixotic flight. I am reminded of Edward Barksdale's maxim: who builds for eternity builds a house of cards. We should not be surprised if the whole thing came tumbling down.

Inarticulate as our present misgivings may be, they incite us on to re-open the whole question of finitism afresh. It is surely unlikely that such great finitists as Aristotle, Kant and Wittgenstein (otherwise so different from one another) should all be mistaken on this one issue. In the remaining two chapters we are to investigate the matter with fresh resources, from two very different points of view.

¹ Assume that there is such a mid-point $p_1$. Then there must be another point $p_2$ which is ten feet removed from $p_1$ on the hither side. How much time will elapse in our walking from $p_2$ to $p_1$? No rational (or irrational) interval of time, certainly. But, having lopped off the infinitesimal tail ex hypothesi, every distance of ten feet which we traverse must be discharged in a rational (or irrational) interval of time.
CHAPTER V

CONCEPT-FORMATION

Part I: The Vulgar Infinite

Once upon a time, long ago, a great controversy broke out among the Swahilis, plunging that ancient, benighted people into a state of confusion perilously close to civil war. A young hero had arisen to challenge some of the most deeply cherished beliefs of the tribe. With all the truculence of youth and ambition, he insisted that, contrary to received opinion, it must be admitted that there are a definite number of leaves in the jungle, a definite number of fish in the ocean, a definite number of stones in the valley. It was a profound mistake, he argued, to suppose that the stones in the valley were really innumerable, uncountable, numberless, indeed so plentiful as to be quite without number. The Swahilis at this time were fortunate enough to possess a decimal system of counting, but they rarely had any occasion to count beyond 100. Ancient records were on hand to prove that the highest that anyone had ever counted, in all the recorded history of the tribe, was to the number 488. This number was popularly regarded with almost sacred awe, it was chanted during the holy festivals, and it was held highly unlikely that anyone would ever count beyond it. It seemed to represent the very limit of human achievement.

To the horror of the old, to the delight of the young, our hero gathered the whole tribe together and undertook to break the spell of superstition under which they languished. In full view of all, he proceeded to count up to 200, then on to 300, 400, and as he moved on to 486, 487, 488! a great hush fell upon the tribe, 489!—the young burst forth with cheers, the old clapped their hands upon their ears, refusing to listen to this transgression. Our hero was unable to reach 500. Spears and rocks were being hurled in all directions. It was war. Happily enough, a venerable high priest intervened to compose the passions of the contending factions. He was wise and judicious. 'I am prepared to overlook this frivolous trespass,' he said charitably. 'A mere aberration of youth. But they are countable nonetheless. God is able to count them. Do you deny that God is able to count the stones in the valley?' All eyes were fixed upon the high priest. How would he answer this damaging question? 'God is able to do all things,' replied the priest unctuously. 'God then is able to count the stones in the valley,' our hero pressed on. 'But then there must certainly exist a definite number of stones in the valley if God is able to count them!' He paused to let this point sink in. 'I do not deny that in a very loose and lax manner of speaking the stones in the valley may perhaps be said to be innumerable, uncountable, numberless, indeed so plentiful as to be quite without number. Certainly, they cannot be counted by man. But they can be counted by God, and it is God, not man, who is the measure of all things. From God's point of view, there is a definite number of stones in the valley.' So beguilingly persuasive were our hero's words that not only did they succeed in restoring the young to confidence but even the old were visibly shaken in their faith. Murmurs rippled through the crowd. Of course, there is a definite
number of stones in the valley! They could almost see them all in their mind's eye, God having attached a number to each stone by a little tag.

Surely the ancestral faith was utterly exploded. What was there left for the old priest to say? 'Young man,' he spoke with surprising calm, 'you are doubtless familiar with the Great Rapids to the north. Are these Rapids navigable or unnavigable?' 'Everyone knows that they are savagely unnavigable,' replied our hero, puzzled by this odd turn in the controversy. 'And what of toadstools?' asked the priest. 'Are they edible or inedible?' 'They are inedible,' replied our hero uneasily. 'And what of tigers? Are they ridable or unridable?' 'It is impossible to ride a tiger,' said the young man testily. 'Really?' murmured the priest with evident irony. 'God is able to ride the tiger. Tigers must be ridable. God is able to navigate the Great Rapids. The Rapids must be navigable.' He paused to allow his young adversary an opportunity to speak, but our hero could only stammer in confusion. 'I am not surprised by your hesitation,' the old man said. 'When we say that the Great Rapids are unnavigable, are we so impious as to deny that God is able to navigate them? Certainly not. We mean simply that they cannot be navigated by man. When we say that toadstools are inedible, are we denying that they are edible for God? Certainly not. We mean merely that they are inedible for man. When we say that tigers are unridable... But I need scarcely continue. When we say that the stones in the valley are innumerable and uncountable, are we denying that they are countable for man? Of course not. We mean merely that they are innumerable and uncountable—for man! You do not suppose that it is a loose manner of speaking to say that the Great Rapids are unnavigable?' 'No,' said our hero weakly. 'No more is it a loose manner of speaking to say that the stones in the valley, the fish in the ocean, the leaves in the forest are all innumerable, uncountable, numberless, indeed so plentiful as to be quite without number,' boomed the old man. 'But I have already counted almost to 500,' protested our hero desperately. 'If I were to continue counting on and on, I would eventually reach a number equal to the number of stones in the valley.' 'Of course!' replied the priest with disdain. 'If! If! If you were to continue counting on and on...! If you were to succeed in navigating the Great Rapids, then you would prove that they are navigable after all. What good is this "if"? The "if" doesn't make the Rapids navigable, nor does your "if" make the stones in the valley numerable and countable.'

The old man grew suddenly gentle. 'My boy, you have allowed yourself to be transported by a fit of divine enthusiasm unsuitable to a mere mortal. From God's point of view, the stones in the valley are indeed countable, the Great Rapids are indeed navigable, toadstools are indeed edible, tigers are indeed ridable. All things are possible for God. But God's point of view is suitable only for God. Man is truly the measure of all things, not as they are in themselves (Heaven forbid!), but as they are for man. The great numbers that you envisage exist only in the mind of God; they do not exist for man; they are divine and holy; they are not to be profaned by human presumption. My boy, you must rest content with speaking the language of men: the language of God is his alone. You must speak as our fathers have always spoken. You must say that the Great Rapids are unnavigable, that toadstools are inedible, that tigers are unridable; above all, you must say that the stones in the valley are innumerable, that the fish in the ocean are uncountable, that the leaves in the forest are numberless, and that they are all so plentiful as to be quite without number.'

The young man bowed his head in a spirit of abject contrition, and the tribe of Swahilis, restored to their ancestral faith, returned to their dogmatic slumbers.

This fable may not be uninstructive; it is designed, above all, to illuminate the Swahili concept of the infinite. Although we have neglected to mention that concept expressly, it will not be difficult to reconstruct on the basis of the evidence presented. The Swahilis believe not only that the stones in the valley are literally uncountable and hence literally without number, they also believe that they are infinitely many. They are persuaded that they are so plentiful as to be literally infinite. Are they mistaken in their conviction? No. When they insist that there are infinitely many stones in the valley, they mean merely that if one (i.e. a human being) were to be so foolish as to attempt
to count all the stones in the valley, he would never reach the end of his task; he would die in the process. In that sense the stones are infinite, endless. If the literal meaning of a word is admitted to be the non-metaphorical meaning that it bears in common discourse, then it must be confessed that the stones in the valley are not only infinite but literally infinite, and not merely in Swahili but in English as well. It will have become only too evident that the Swahilis are no alien tribe of savages merely in Swahili but in English as well. It will have become evident that the Swahilis are no alien tribe of savages, or so it is said in English as well as in Swahili.

In a very crude sense, they are doubtless uncountable; the leopards roaming the vast forest might well be driven by their hunger to descend en masse upon the Swahilis and, being but few in number (if popular opinion is to be credited in this matter), they would then be readily available for counting. Never has a Swahili been heard to say that the leopards in the forest are innumerable, uncountable, numberless, indeed so plentiful as to be quite without number. Countable they are in principle only in regard to number, being quite content to preserve our human perspective in regard to the other concepts? Whatever the reason, it is evident that a divine dignity attaches to our standard concept of number which is altogether absent elsewhere. It is not surprising that, ever since Pythagoras, there have always been philosophers who have attempted to model their metaphysics (with its own divine pretensions) upon mathematics. Indeed our own metaphysical explorations in this book may be described as Pythagorean in inspiration. Three concepts of number may be distinguished. First, the informal, vulgar proto-concept of the Swahilis. Second, the formal standard concept which is itself so primitive that it underlies all our school arithmetic (not to mention all advanced mathematics). Finally, our own super-concept of number which is found in our hyper-mathematics. I believe that the leap whereby we transcend the standard concept so as to advance on to our own super-concept may be seen to be quite negligible in comparison with the truly profound leap required to supersede the proto-concept by means of the standard concept. The passage from the proto-concept to the standard concept—from the human to the divine perspective—is so deeply buried in our 'form of life' that it is no longer visible to the uncritical observer. It is imperative that we re-awaken that first startled flush of wonder that must have greeted this passage at its origin if we are to understand our own further passage on to the super-concept. A kind of conceptual archaeology must be undertaken to excavate this new altogether hidden passage that is so primordial for us that it almost certainly antedates all our recorded history.

At the very outset there are certain misconceptions that must be anticipated. One must not suppose that the Swahilis are ignorant of the distinction between what is possible in principle and what is merely feasible in practice. That distinction governs all their thought. The Swahilis are persuaded that there are a definite number of leopards in the forest, but no one supposes that it is at all possible, practically speaking, to count them. Slinking about most elusively, the leopard is believed to be almost extinct in those parts. In the event of famine all of the leopards roaming the vast forest might well be driven by their hunger to descend en masse upon the Swahilis and, being but few in number (if popular opinion is to be credited in this matter), they would then be readily available for counting.

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innumerable, uncountable, numberless, even infinite. But this is mere hyperbole. What is meant is that it must be difficult for the Great King, as for his subjects, to keep track of them—they are so many. Thus a distinction is drawn between what is infinite in the strict sense, the literally infinite, and what is infinite in a loose sense, the metaphorically infinite. Whereas the leaves of the jungle are supposed to be infinite in the strict sense, literally infinite, it is only in a loose, metaphorical sense that the wives of the Great King are ever said to be infinite. It ought to come as no surprise that the Swahilis insist that the leaves of the jungle are literally infinite. If the distinction between finite and infinite, between the countable and the uncountable, is to be recognized in vulgar discourse, if these concepts are to arise at all in mufi, it is imperative that they have a use in the language: they must be negotiable and cashable in terms of the actual experience of the Swahilis. What possible use could they have for the distinction between finite and infinite, between the countable and the uncountable, as that distinction is enshrined in our standard concept? The standard concept of the infinite simply has no application in the form of life that constitutes vulgar practice. Not that a Swahili may not be moved to remark that the stars in the heavens are infinite, but he has no occasion to mean anything more by his remark than that they are infinite in the sense that the leaves of the jungle are known to be infinite.

It would be a great mistake to appeal to logical possibility in an effort to prove that (the Swahilis notwithstanding) it is a hard fact that there are a definite number of stones in the valley. It may not be humanly possible, but it is surely logically possible, to count them all. Certainly. It is also logically possible for a man to navigate the Great Rapids, to dine on the toadstool, to ride the tiger. It is logically possible for gold to cure measles. But this is to say no more than that the statement ‘Gold has the power to cure measles’ is free of self-contradiction. From this mere logical possibility we are not entitled to conclude that gold has the power to cure measles. So, too, though it is logically possible that the stones of the valley might all be counted, we have no right to suppose that they are really countable after all. We might as well say that toadstools are really edible. Logical possibility is no measure of what is really the case.

Adopting another tack, one is tempted to adduce the following ‘proof’ in support of the passage from the proto-concept to the standard concept. Look! Here are five stones. If this pile contained one more, how many stones would then be in the pile? Six. And if still another one were present, how many then? Seven. You see: there is no limit to the number of stones that the pile might possess. Think of the pile growing larger and larger one at a time. Eventually there would be as many stones in the pile, the exact number, as there are stones in the valley. It is thus as plain as day that there are a definite number of stones in the valley. How absurd to say that they are really innumerable, indeed so plentiful as to be quite without number. This ‘proof’ is mere patter. It may be forcibly employed as a lure designed to effect a passage to the standard concept. But as a ‘proof’ it is no more than a sophism. Seductive, yes; binding, no. If this pseudo-proof is likely to strike one today as altogether cogent, it is only a measure of the profound estrangement that separates us from the depths of the proto-concept. One might as well argue as follows. You see this man, he is quite bald, isn’t he? Yes. Now if he had one additional hair on his head, would he still be bald? Of course. What if he had still another one? He would still be bald. In other words, the addition of a single hair will never suffice to rid any bald man of his baldness? True. That man over there, he looks as if he has a full bushy head of hair, doesn’t he? Yes. He is not bald then? Of course not. My dear fellow, you are quite mistaken: he may not look bald, but he is really bald nonetheless. Can’t you follow an argument? So much for the notorious sophism of the slippery slope. As a proof it is fallacious, but as a conceptual lure it has been known to serve, in less transparent a form, the most honourable purposes.

Lest it be supposed that our concept of baldness is some outrageous anomaly, other examples of the same sort are readily available. If a man X is very poor and a man Y is also very poor. Are we to infer then, that Croesus is very poor? If a man X is very thin and a man Y (of the same height as X) weighs one ounce more than X, how can it be denied that Y also must be very thin? Does it then follow that all men of the same height as X are very thin? The Swahilis are perfectly prepared to admit that if a collection
of objects X is countable, then another collection of objects Y, which contains but one element more than X, must also be countable. Does it then follow that the leaves of the jungle are countable? Shall we say that they are in principle countable? Then we must equally say that Croesus is in principle very poor. We must say that all men of the same height as X are in principle very thin. What could be more ridiculous?

Although one may be tempted to rule out all of these vulgar concepts as being radically illogical, it is the part of wisdom to leave them well enough alone. If they are illogical, it is only in the sense that they violate our standard concept of logic. This is not to deny that they obey a vulgar logic of their own. The distinction between our vulgar concept of number and our standard concept of number finds its counterpart within logic itself—in the distinction between our vulgar logic and our standard logic. That there is a close connexion between Mathematics and Logic, is certainly true enough; but even if one were to succeed in vindicating the programme of Frege and Russell (by deriving our standard concept of number from Logic), the foundations of Mathematics would remain in large measure obscure. Our standard concept of number and our standard concept of logic are so much on a par that it is not surprising that the one should be grounded in the other. But if we are to justify our passage from the proto-concept of number to the standard concept, we must equally justify our passage from our proto-logic to our standard logic. Unfortunately, it will not be sufficient for our purposes simply to justify in general our standard concept of logic and then, on the strength of that justification, execute the passage from our vulgar concept of number to the standard concept. In the name of Logic, one might as well supersede our vulgar concepts of baldness, poverty, and thinness in the same way. Shall we say that a man is very poor unless he has an infinite sum of money? No one is tempted to adopt such an exalted manner of speaking, exalted and fatuous at once. Number is evidently a special case: it must be understood on its own terms, quite apart from any appeal to Logic in general.

What of the following suggestion? The slippery slope argument designed to prove that all men are bald is fallacious because it disregards the qualitative distinction between those who are bald and those who are not bald; it insists on concentrating on the purely quantitative. Perhaps this is the essence of all slippery slope arguments—the suppression of the qualitative in favour of the quantitative or the presumption that all qualitative distinctions can be translated, without remainder, into exclusively quantitative terms. Thus the slippery slope sophism calculated to prove that the Swahilis are mistaken in their conviction that the leaves of the jungle are uncountable and infinite, rests on the premise that the distinction between the countable and the uncountable, between the finite and the infinite, must be purely quantitative in nature—this is in the teeth of the fact that the Swahili concepts of finite and infinite cannot be exhaustively defined in strict quantitative terms. Now let us suppose that we were to undertake to construct a science of pure quantity. Let our express programme be to abstract from all qualitative differences so as to concentrate on the purely quantitative. What was previously a sophism—the slippery slope—now becomes the very raison d'être of our programme. It lies in the very essence of a science of pure quantity that it disregard the qualitative. Alas. This explanation itself rests on a fallacy. How are we to understand the purely quantitative? In terms of the proto-concept of quantity or in terms of the standard concept? To insist on the standard concept here is simply to beg the question.

The vitality of the proto-concept comes through to us most strongly when it is seen to satisfy all of the Peano postulates. Not certainly in the precise sense that Peano intended, for the Peano postulates are designed to express our standard concept of number; but at least nominally, these postulates are satisfied by the proto-concept of number as well. Consider the five Peano postulates. (1) o is a number. (2) The immediate successor of any number is a number. (3) There are no two numbers with the same successor. (4) o is not the successor of any number. (5) Every property of o, which belongs to the successor of every number

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1 Our vulgar logic is not without its own rationale. Thus it is to be characterized, formally, not as a two-valued but, at the very least, as a three-valued logic. Some men are definitely bald; other men are definitely not bald; still others fall betwixt and between. The proposition “Caesar is bald” admits of at least three truth values, viz. true, false, and so-so. The law of the excluded middle fails to hold here.
with this property, belongs to all numbers. All five postulates are accepted by the Swahilis, but they refuse to admit that these five postulates logically (i.e., in terms of their proto-logic) entail an infinite progression—infinite being here understood in the standard sense. That there are infinitely many numbers, they admit, but only in the proto-sense of infinite: no man can exhaust all the numbers by counting, just as no man can count all the leaves of the jungle.

If the proto-concept of number nominally satisfies all of the Peano postulates, may we not expect that same nominal compliance from our other vulgar concepts as well? When we undertake to adapt these postulates to our concept of poverty, say, we find that there are two stipulations that must be made if the programme is to succeed. First, we must assume that there exists at least one man who does not have a single penny; and second, we must assume that for any very poor man \( X \), there exists at least one man \( Y \) who has but one penny more than \( X \). Neither of these conditions is actually entailed by our concept of poverty, but it is not impossible that they in fact obtain in some country, which is enough for our purposes. Positing these conditions, all of the Peano postulates are satisfied. (1) If a man is without a single penny, then he is very poor; also, there exists at least one man without a single penny. (2) If a man has but one penny more than some very poor man has, then he too is very poor; also, for any very poor man \( X \), there exists at least one man \( Y \) who has but one penny more than \( X \). (3) If a very poor man \( X \) has a different number of pennies from a very poor man \( Y \), then there does not exist a man \( Z \) who has but one penny more than both \( X \) and \( Y \). (4) No very poor man has fewer pennies than the man who is without a single penny. (5) Every property that belongs to every man without a single penny, which also belongs to every man who has but a single penny more than any very poor man with this property, is a property belonging to all very poor men.

On the basis of these five postulates, above all the second, the mathematician feels entitled to assert the existence of an infinite progression in the standard sense. There must then be an infinite number of very poor men. Furthermore, the only men who will not be very poor will be those who possess an infinite sum of money and, since in fact no such men exist, all men must be very poor. Does our standard mathematics rest on a fallacy, specifically the fallacy of the slippery slope? Is that fallacy to be found enshrined in Peano’s second postulate? No one can deny that if a man \( X \) is very poor—by which I mean clearly and indisputably very poor—and a man \( Y \) has but one penny more than \( X \), then \( Y \) also must be very poor—clearly and indisputably. But this principle must be employed with discretion. We may apply it once, twice, three times, four times, five times, in succession: our proto-logic assures us that our inferences will be valid. It is only when we undertake to abuse the principle—as the mathematician may be said to do systematically—by subjecting it to the escalator effect in a spurious effort to prove that Croesus must be very poor, that our proto-logic steps in and convicts us of a fallacy, indeed of a logical fallacy. Does our standard mathematics rest on a logical fallacy?

It may be of some interest to note that the Peano postulates do not always entail an infinite progression, even in the proto-sense of infinite. Consider our concept of thinness. Let us assume that the thinnest man in some community weighs 100 lb. Let us also assume that for any very thin man \( X \) in the community, there exists another man \( Y \) in the community who weighs one pound more than \( X \). Neither of these assumptions is implausible. Now since it is true that any man who weighs but one pound more than any very thin man, must be himself a very thin man, it will be evident that all of the Peano postulates are satisfied, mutatis mutandis, by the concept of thinness as it obtains in this particular community. (The man weighing 100 lb. counts as the zero element in the system.) But there is no infinite progression, even in the proto-sense of infinite. For no man whose normal weight is 180 lb., say, would ever be described as very thin. Hence it is quite possible that in all of the community there are not more than fifty or sixty or seventy or eighty men who are deemed very thin, and even the Swahilis admit that any collection of 50 or 60 or 70 or 80 elements is a finite collection. Viewed from the perspective of proto-logic, the Peano postulates do indeed entail an infinite progression in the proto-sense of infinite when they are fleshted out with the concept of number, but it may be doubted whether they entail an infinite progression in any other case whatever. That there is no infinite progression in the case of thinness has already been shown, but
what of the concepts of poverty and baldness? The Swahilis certainly will never describe any man who is merely well-to-do (and hence clearly not very poor nor even poor at all) as being possessed of an infinite sum of money (infinite being meant here in the proto-sense), nor will they describe any man who is only moderately hirsute (and hence clearly not bald) as being endowed with infinitely many hairs. In the case of the richest or hairiest man in the tribe, the word 'infinite' might indeed come to their lips quite naturally, but even here it would probably be used only by way of hyperbole, very much in the sense that the Great King is said to have infinitely many wives. Are we to conclude, then, that the vulgar concept of number is unique among vulgar concepts in generating an infinite progression, albeit only in the proto-sense?

In the midst of these uncharted perplexities, I am reminded of Wittgenstein's suggestive remark, 'The limit of the empirical—is concept-formation'. Die Grenze der Empirie—ist die Begriffsbildung.¹ There are no hard empirical facts that coerce us into rejecting the proto-concept of number as being empirically false: we cannot say, the Swahilis are mistaken, the leaves of the jungle are really countable. In the same spirit we must add, 'The limit of the logical—is concept-formation'. For, again, we cannot justify our passage to the standard concept on the ground of any logical defect in the proto-concept (unless we beg the very question at issue by appealing to our standard logic). Does this mean that the passage is quite without justification? I should be most reluctant to acquiesce in this conclusion. And yet what is there beside the empirical and the logical? Is there perhaps some tertium quid that may serve to negotiate the passage? The great question before us is, why are we so insistent on regarding the numerable from a trans-human, divine perspective (God is able to count the leaves of the jungle) even though we refuse to adopt the divine perspective in regard to the navigable, the edible, and the ridable. Why should the numerable be assigned a dignity all its own? Once we are prepared to accept the leap from the proto-concept to the standard concept (this is the big metaphysical leap), then the further leap on to the super-concept will be easy to negotiate.

I have suggested that the actual historical passage from the proto-concept to the standard concept must surely be so profoundly ancestral, so deeply buried in remote antiquity, that it would be unreasonable to expect to find any written record of it at its origin. Fortunately, there does exist a valuable document which recapitulates, even though it certainly does not originate, the grounds which may be supposed to have prompted the passage in the first place. In his treatise 'The Sand Reckoner' Archimedes expressly undertakes to refute the vulgar opinion that the grains of sand in the desert are infinite in multitude.

There are some, King Gelon, who think that the number of sand grains is infinite in multitude; and I mean by sand not only that which exists about Syracuse and the rest of Sicily, but all the grains of sand which may be found in all the regions of the Earth, whether inhabited or uninhabited. Again there are some who, without regarding the number as infinite, yet think that no number can be named which is great enough to exceed that which would designate the number of the Earth's grains of sand. And it is clear that those who hold this view, if they imagined a mass made up of sand in other respects as large as the mass of the Earth, including in it all the seas and all the hollows of the Earth filled up to the height of the highest mountains, would be still more certain that no number could be expressed which would be larger than that needed to represent the grains of sand thus accumulated. But I will try to show you, by means of geometrical proofs which you will be able to follow, that, of the numbers named by me and given in the work which I sent to Zeuxippus, some exceed not only the number of grains of sand which would make a mass equal in size to the earth filled up in the way described, but even equal to a mass the size of the universe.

Assuming the number of grains of sand that would fill a poppy seed to be not more than 10,000, Archimedes concludes as follows. 'It is evident that the number of grains of sand that could be contained in a space as large as that bounded by the stellar sphere as estimated by Aristarchus, is not greater than one thousand myriads of units of the eighth class.' The visible universe as limited by the fixed stars had been estimated by Aristarchus to be of a diameter equal to 10,000 earth-diameters, and the diameter of the earth was calculated as not more than 10,000 miles. Replying on these figures, Archimedes is able to

¹ Foundations of Mathematics, III, § 29. As a general principle this maxim is certainly false. We have seen that the proto-concept of up and down, being that of an absolute up and down, is simply incapable of standing up to the empirical facts.
infer that the visible universe as a whole cannot possibly contain more than \(10^{63}\) grains of sand (in our notation). Hence the Swahilis must be mistaken in their conviction that all of the grains of sand 'which may be found in all the regions of the earth, whether inhabited or uninhabited' are of a truly infinite multitude.

How successful is Archimedes' 'proof'? There is some reason to believe that what passes for a proof here is merely a shift to a new concept. On the other hand, it may be objected that if Archimedes' 'proof' is not to be accredited as a genuine proof, then what the devil is a genuine proof expected to be? In defence of the Swahilis, it must be said that Archimedes has certainly failed to show that the grains of sand in the desert (not to mention the earth or the universe) are really countable ('countable' being understood in the vulgar sense): they remain uncountable in the sense that no one, not even Archimedes, presumes himself able to count them. Instead of showing us that they are countable, he shows us something rather different: they remain uncountable in the sense that no one, not even Archimedes, presumes himself able to count them. Instead of showing us that they are countable, he shows us something rather different, but this new thing that he shows us is not unrelated to the 'countable'. To exaggerate, we may say that Archimedes shows us that, though the grains of sand in the desert, may be uncountable, this does not mean that they are uncomputable. The uncountable may yet be computable. But even this is inaccurate. Archimedes has equally failed to compute the number of grains of sand in the desert (or anywhere). Strictly, he has succeeded only in computing a number which is demonstrably greater than the multitude (we are not allowed to say 'number' here) of grains of sand anywhere on earth. But surely that is quite sufficient. If \(10^{63}\) is a number certifiably greater than the multitude of leaves in the jungle (the Archimedean number applying equally to this case), then how can it be maintained that the leaves in the jungle are really numberless? Uncountable they may be allowed to be, in the vulgar sense, but how can it be maintained—in the teeth of the Archimedean number—that they are innumerable, quite without number, and infinite? Very easily. When the Swahilis say that the leaves in the jungle are innumerable, they mean merely that no man is able to numerate them, i.e. number them one by one in serial succession; and the numberless and the infinite are understood to be synonymous with the innumerable.

The fact remains that Archimedes has shown us something important. But what? It is easier to divine what that something is than to state it clearly. One is tempted to argue as follows. We know that there are more than 100 leaves in the jungle, and the calculations of Archimedes prove that there are less than \(10^{63}\) leaves in the jungle. Hence it follows that there is a definite number of leaves in the jungle, a number which is somewhere between 100 and \(10^{63}\), I do not doubt that this argument is a very model of logical propriety. It is precisely the kind of thing we expect of a proof. Unhappily, it is only a proof within the context of our standard logic, and that very context itself has been put into question. Have we not also proved that Croesus must be a very poor man? Laying aside, then, the question of proof, we may rest content with saying that Archimedes has exhibited to us a number which exceeds the vulgar infinite. There is thus a whole range of large numbers that may be shown to transcend the vulgarly numerable. But even this account is not quite right. Are the Swahilis prepared to admit that \(10^{63}\) is a number? It certainly looks very much like \(27\) or \(10^9\) both of which the Swahilis are happy to accept as bona fide numbers in their sense: they are both countable in the primary sense of 'countable'. What now of \(10^{63}\)? Here I think the Swahilis are likely to be embarrassed. At first they will have no hesitation in accepting \(10^{63}\) as a bona fide number. A simple function of 10 and 63, themselves bona fide numbers, the meaning of the exponential notation employed in the expression \(10^{63}\) will be rendered plain enough to them by appealing to such innocent paradigms as \(27\) and \(3^5\) and \(10^8\). They will thus be seduced into acquiescing in \(10^{63}\) as a bona fide number. Let them now, as a game, undertake actually to count one by one up to \(10^{63}\). Nothing would seem simpler or less controversial. Alas. Once they get the hang of the enormity that is expected of them, they will recoil with horror. 'You tricked us!' they will protest. 'This pseudo-number \(10^{63}\) is not really a number at all. It is certainly not numerable or countable one by one. Why, it is actually infinite!' And they will have some justification for their protest. \(10^{63}\) is not a number in the proto-sense of number. Nevertheless, it may be expected that some at least of the Swahilis will be so charmed by the 'big number' game that they will insist on playing it. Not that they will actually persevere in their project...
of counting one by one up to $10^{63}$, but they will learn to execute a whole range of computations in which such 'numbers' as $10^{63}$ will be accepted as a matter of course. The high priest himself (out of vanity perhaps) may undertake to compute the 'number' of days that he has lived: $70 \times 365 = 25,550$. Or even the 'number' of hours: $25,550 \times 24 = 613,200$. Or the 'number' of minutes: $613,200 \times 60 = 36,792,000$. Or the 'number' of seconds: $36,792,000 \times 60 = 2,207,520,000$. A heady draught of power, that! Originally anchored to the practice of counting one by one, the concept of number, set free of its moorings, is now enlarged through computation. $10^{63}$ may not be countable but it is certainly computable.

It must be confessed that this shift from the countable to the computable—from the primary operation of serial addition to the secondary operation of shorthand multiplication—remains exposed to serious difficulties. The rules of computation cannot be established in their unrestricted generality without recourse to mathematical induction! And we have seen that mathematical induction smacks desperately of the slippery slope fallacy. Not directly, perhaps, but in so far as it presupposes an infinite progression in the standard sense. We are thus very far from having demonstrated the existence of the Archimedean 'number' $10^{63}$. It is not enough here to affirm, with Poincaré, that the principle of mathematical induction is a synthetic a priori truth, nor can we rest content with Russell's position that 'natural number' is simply to be defined, by fiat, as that kind of concept which satisfies Peano's fifth postulate. We might as readily define the slippery slope fallacy as a valid type of argument. Moreover, it is Peano's second, not his fifth, postulate that is crucial in this discussion.

Fallacy or no fallacy, the 'big number' game stands, be it only as a game. Actually, it proves to be more than a game. If the high priest in his seventy years has lived through $2,207,520,000$ seconds of time, then it will be true to say that precisely seventy years would be required for someone to count from 0 all the way up to the 'number' $2,207,520,000$—counting at the rate of one number per second (or is it 'number'?) per second. More than that: we can compute precisely the 'number' of years which would be required for a god—again counting at the rate of one number per second—to count one by one all the way up to $10^{63}$.

Although $10^{63}$ remains uncountable in the proto-sense, it is no longer disconnected from the primitive process of counting one by one. Invoking the trope of hyperbole, we may say that $10^{63}$ is countable in a utopian sense: it can be counted (meaning the primitive process of counting one by one) by a god. Above all, it can in fact be computed (though not counted) by us! This last is crucial. We are thus able to simulate the power of God. In the absence of computation, I suspect that God's being able to count the leaves of the jungle would be regarded by us as strictly on a par with God's being able to navigate the Great Rapids. The one would no more entitle us to insist that the leaves of the jungle are really countable or countable in principle than the other entitles us to insist that the Great Rapids are really navigable or navigable in principle. What is peculiar to the concept of the countable (and which sets it off from the navigable) is that it lends itself (granting a suitable notation) to being simulated through computation. Not that we are actually able even to compute the exact 'number' of leaves in the jungle, but we are certainly able to provide more or less rough approximations of that 'number' (and here the techniques of geometry and trigonometry come into play), and the very concept of an approximation presupposes that which is being approximated.

I offer these suggestions in a spirit of great diffidence. Many will doubtless feel that instead of affording a justification of the passage from the proto-concept of number to the standard concept I have merely supplied a genetic account of how in fact that passage is transacted. Furthermore, it may be felt that the purity of pure mathematics has been contaminated by our grounding it in its techniques. I am not sure but that these charges, if they are intended as criticism, may not be ill conceived. Each, in its own way, presupposes a standard of explanation that may itself have to be revised in the light of our present problem. For it is the problem, far more than any putative solution, that I am eager to emphasize. How are we to transact the passage from the proto-concept of number to the standard concept?

This problem acquires special urgency in the light of Wittgenstein's downward dialectic: we may style it the Antaeus principle. Antaeus, it will be recalled, was a giant absolutely invincible in strength so long as he remained in contact with his
mother, Earth. Even the great Hercules was unable to defeat him until, by lifting him off the ground, he strangled him in mid-air. According to Wittgenstein, philosophy is one protracted strangulation of our natural concepts which can only succeed if they are pried loose from their 'original home', our mother-tongue, being otherwise invincible.

When philosophers use a word—'knowledge', 'being', 'object', 'I', 'proposition', 'name'—and try to grasp the essence of the thing, one must always ask oneself: is the word ever actually used in this way in the language-game which is its original home? What we do is to bring words back from their metaphysical to their everyday usage.1

In the present chapter I have been engaged in exploring the infinite among the Swahilis, in 'the language-game which is its original home'. We have found that our standard mathematics is a half-way house between proto-mathematics, on the one hand, and utopian mathematics, on the other. I am persuaded that if one balks at the actual infinite in its standard interpretation (which leads at once to hyper-mathematics), if one insists on finitism, he will be driven to hold the line at proto-mathematics. Our standard mathematics is a mere way-station: one must either push on or retreat, there is no standing pat at this point. Although our own account of the advance beyond the vulgar concept of number to the standard mathematics is a mere way-station: one must either push on or retreat, there is no standing pat at this point. Although our own account of the advance beyond the vulgar concept of number to the standard concept may not as incisive and elegant as one might wish, I cannot believe that anyone will choose to take the barbaric step of retrenching all the way back to proto-mathematics and the Swahilis. Even Brouwer with his cyclopean finitism will shrink from such a move.

There is indeed a brief moment in which Wittgenstein veers very close to the Swahili concept of the infinite.

Suppose that children are taught that . . . God created an infinite number of stars . . . Queer: when one takes something of this sort as a matter of course, as if it were in one's stride, it loses its whole paradoxical aspect. It is as if I were to be told: Don't worry, this series, or movement, goes on without ever stopping. We are as it were excused the labor of thinking of an end. 'We won't bother about an end.' It might also be said: 'for us the series is infinite.' 'We won't worry about an end to this series; for us it is always beyond our ken.'2

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The consequences are serious. Pure mathematics as we know it, even the purest of the pure, which undertakes to abstract from all content and to limit itself to a formal study of abstract systems, presupposes our standard concept of the infinite, be it only the potential infinite. But this standard concept of even the potential infinite is no less dubious than our standard concept of the actual infinite, when called to account by proto-mathematics. How, then, is the potential infinite itself to be justified? It cannot be justified, I submit, on purely formal, or even logical, grounds.

Part 2: Teleology and the Absolute

We are now to make an altogether fresh start. Whatever merits our account of the passage from proto-mathematics to standard mathematics may possess, and I have no doubt that the shift from the countable to the computable is at least part of the full story, we have found that there is one decisive theoretical (it need not, however, be a practical) objection that invalidates it as the definitive solution. The rules of computation, if they are to enable us to transcend the vulgarly numerable, presuppose the principle of mathematical induction in the standard sense. Even more to the point, they presuppose an infinite progression in the standard sense. But to acknowledge an infinite progression in the standard sense is simply to beg the very question at issue. In the course of earlier discussions we have had occasion to indicate three different (though perhaps overlapping) procedures for superseding or transcending a vulgar concept. In the first place, a concept will be superseded if we are successful in driving a wedge between its connotation and its denotation. In the second place, a vulgar concept will be transcended, even on its own terms, if we can show that it is inherently Janus-faced, only one of its faces being turned toward the relative and the practical, the other face being turned toward the absolute and the theoretical. Finally, a concept may be amended or even totally replaced by another concept if it is shown to be less than adequate as a tool for the tasks that are set it, as the primitive spear made of wood was amended and improved on being cast in bronze and, still later, replaced altogether by the firearm. Have we any prospect that some one or more of these three procedures might empower us to transact the passage from our vulgar concept of number to the standard concept? I propose to argue that all three of these procedures, in one way or another, will be found to play a part in the total transaction, but it is the third procedure to which I am especially anxious to draw attention.

We have studied the concept of number both in its empirical import and in its logical import, even attending to its formal import as well, but we have failed to examine it in its teleological import, that is to say, in terms of its use, in terms of its role as a tool or instrument with a specific job of work to discharge. Is it possibly the case that there is a job to be done which only our standard mathematics—and I am thinking primarily of pure mathematics—is capable of performing? Apart from some fragmentary observations in Wittgenstein, a teleological account of mathematics has perhaps never been undertaken. Teleology and mathematics have been regarded as absolutely poles apart. Thus in his ferocious attack on teleology, Spinoza suggests that our innate propensity to explain the world teleologically 'might well have sufficed to conceal the truth from the human race for all eternity, if mathematics had not furnished another standard of verity in considering solely the essence and properties of figures without regard to their final causes'. I do not doubt that a mathematical account is never teleological, but I am suggesting that an account of mathematics may have to be teleological.\footnote{Cf. pp. 34–36 supra.}

When the Swahilis say that the leaves of the jungle are innumerable, uncountable, numberless, quite without number, indeed infinite, they freely admit that this locution is strictly on a par with many others, such as that the Great Rapids are unnavigable, that toadstools are inedible, that tigers are unridable. All of these locutions may be said to be on the same cognitive level: they are all judgments of practical reason and, as such, quite unexceptionable, \textit{in their place}. But none of these judgements would be found in a scientific or theoretical treatise, certainly not in any \textit{strictly} scientific treatise, for there is a whole continuum of more or less scientific and more or less vulgar discourses separating the strictly scientific from the utterly
vulgar. I am suggesting, then, that just as our vulgar concept of number is but one of many vulgar concepts which are all on the same cognitive level of practical discourse, so, too, our standard concept of number is but one of many scientific concepts which are all on the same cognitive level of theoretical discourse. The justification for our passage from the vulgar to the standard concept of number must thus be viewed as all of a piece with a more comprehensive justification of our passage from one entire cognitive level to another, namely from the plane of discourse suitable to practical reason to the radically different plane of discourse suitable to theoretical reason. We failed in our earlier efforts to justify our standard concept of number owing to the fact that we insisted on justifying it in isolation, independent of the entire cognitive level in which it is embedded. What requires justification is this entire cognitive level as a whole, for after we have succeeded in transacting the passage from practice to theory in general, we shall find that our standard concept of number has been justified along the way.

What, then, is the job of theory? What are we aiming to achieve when we engage in pure science? What is our purpose? In earlier discussions we characterized science, variously, as the uncovering of the absolute, on one occasion, and as the disclosure of essence, on another. Closely akin to these two accounts is a third which is peculiarly suitable for the nonce. What requires justification is this entire cognitive level in which it is embedded. What requires justification is this entire cognitive level as a whole, for after we have succeeded in transacting the passage from practice to theory in general, we shall find that our standard concept of number has been justified along the way.

Interestingly enough, the distinction between empirical and ontological predicates can be drawn, in part at least, on purely a priori considerations. The vulgar predicates 'cold' and 'hard' are foredoomed to drop out of any scientific account of the block of ice or, indeed, of anything else. We have only to examine the meaning, or use, of these predicates to detect their ontological inadequacy. It is 20° below zero. Bundling ourselves up in all the woollens available, we are still shivering with cold. A very cold day, certainly! But consider now the Hyperboreans who flourish at temperatures of 150° below zero. Here they are in our midst, sweating profusely, disposed to remove all their garments. A very hot day it is, they say, speaking in that flawless English which, by some odd coincidence, is their language as well as ours. Who is right? What kind of day is it, a very cold day, as we insist, or a very hot day, as the Hyperboreans insist? Clearly, we are both right in our different ways. The day is very cold as far as we are concerned, and it is also very hot as far as they are concerned. But it is neither the one nor the other, independent of our respective concerns. It is not surprising, then, that the vulgar predicates 'hot' and 'cold' should drop out of any account which is expressly designed to command the assent of all rational beings, regardless of their specific physical make-up. Such an account is expressly ontological or scientific in import.

'Hard' and 'soft' are very much like 'hot' and 'cold.' We call something 'soft' if it yields readily to our pressure and 'hard' if it stubbornly resists our pressure. How can anyone deny that a block of iron is hard? But the Hyperboreans actually employ blocks of iron as pillows to rest their weighty heads upon. They call them 'soft', and they mean by 'soft' what we mean by 'soft', namely 'liable to yield readily to one's pressure'. For the Hyperboreans are so powerful that a light tap of their fingers is quite sufficient to send a towering oak tree crashing to the ground. In one sense, certainly, the Hyperboreans do not mean by 'hard' and 'soft' what we mean by those predicates, seeing that we employ our, and they employ their, bodies as the standard of reference; though in another sense there is an identity of meaning. How are these senses to be sorted out? We say that the moon is high above the earth, but the moon people (speaking flawless English) say that the earth is high above the moon. Do we all mean the same thing.
by the expression 'high above?' Yes and no. Perhaps the wisest policy here is to insist that we all mean the same thing *mutatis mutandis*. This concept of 'identity of meaning *mutatis mutandis*' is useful in our effort to distinguish between empirical and ontological predicates. When we describe the block of iron as 'very hard' and the Hyperboreans describe it as 'certainly not hard at all, but actually very soft', is there an identity of meaning in the use of this predicate 'hard'? Yes, *mutatis mutandis*. We are, then, both right: the block of iron does in fact yield readily to their pressure, as in fact it resists our own most obstinately.

Although these predicates 'hot' and 'cold', 'hard' and 'soft', are out of place in ontological discourse, they have a job of their own to perform in practical discourse, and if we are tempted to style them as subjective rather than as objective predicates, it must not be in any invidious sense of the term. They are subjective only in the sense that they characterize the object in its specific relation to the subject, not indeed in its relation to any subject but only in its relation to the particular subject or race of subjects employing the terms. What shall we say of the block of ice? Is it really cold in itself? It is certainly cold, relatively speaking, i.e. relative to us; but it is no less hot, relatively speaking, i.e. relative to the Hyperboreans. Absolutely speaking, the block of ice is neither hot nor cold. We are here recalling the distinction between two modes of being, between what it is to be, relatively speaking, and what it is to be, absolutely speaking. If in practical discourse we are concerned with what is the case, relatively speaking, in theoretical discourse, in science, we are engaged in investigating what is the case, absolutely speaking. Science, then, is the uncovering of the absolute; and as for practical discourse it is not what is the case, relatively speaking, *in general* that concerns us: we are concerned with what is the case, relative to *us*.

There has often been expressed the fear that if 'hot' and 'cold', 'hard' and 'soft' are allowed to drop out of ontological discourse as subjective, then perhaps everything will prove to be subjective in the same way.¹ This fear is unwarranted. The subjective presupposes the objective, as the relative presupposes the absolute. If I insist that 'this is very cold' (pointing to a block of ice) and you protest, 'No, you are quite mistaken: *this is actually blazing hot* (pointing to a roaring fire), there is no disagreement between us; there is only a misunderstanding. Any disagreement presupposes a common ground of agreement. If I insist that this cube is very cold and the Hyperboreans insist that this *same* cube is very hot, our 'disagreement' (if I may be allowed to use this word here) presupposes a common ground of reference, namely the cube that lies before us. Two rational beings can differ in the subjective predicates which they assign to a body only if there is a fund of objective predicates which are presupposed by both. At the very least, there must be a substratum of agreement as to the *body* over which there are differences. One and the same spatio-temporal region must be designated as the common ground or substratum of all the various subjective predicates that different rational beings are free to assign, depending on their respective physical make-up.¹

¹ If on the basis of purely *a priori* considerations the vulgar predicates 'hot' and 'cold', 'hard' and 'soft' are found to drop in matter. . . . Cf. also Hume, *Inquiry*, XII, i, 'It is universally allowed by modern inquiries that all the sensible qualities of objects, such as hard, soft, hot, cold, white, black, etc., are merely secondary and exist not in the objects themselves. . . . If these be allowed with regard to secondary qualities, it must also follow with regard to the supposed primary qualities of extension and solidity. . . .' The *locus classicus* for these discussions is to be found in the *Theaetetus* in which Socrates explicitly says (152d) that all philosophers, with the exception of Parmenides, have insisted that everything is relative, on the ground that no predicate can be absolutely assigned to anything. 'Is it not true that sometimes, when the same wind blows, one of us feels cold, and the other does not?' "Certainly." "Then in that case, shall we say that the wind is in itself cold or not cold; or shall we accept Protagoras's saying that it is cold for him who feels cold and not for him who does not?" 'Apparendy.' . . . "And all the rest—hard and hot and so forth—must be regarded in the same way: we must assume . . . that nothing exists in itself, but all things of all sorts arise out of motion by intercourse with each other; for it is, as they say, impossible to form a firm conception of the active or the passive element as being anything separately. . . ." (152d and 157b, translated by H. N. Fowler).
out of ontological discourse, I submit that it is no less on the basis of purely a priori considerations that the objective substratum is found to be, at the very minimum, the spatio-temporal matrix. For though, with Einstein, even space and time are to some extent relativized, there emerges from that relativization the new absolute of the space-time interval which is common to all rational observers. All empirical predicates, then, cannot be subjective; some of them must be objective and, among these objective predicates, the spatio-temporal predicates of geometry (taken in a wide sense) are privileged. They provide the ontological frame of reference within which the very distinction between the subjective and the objective is capable of being drawn.

Dividing all empirical predicates into two classes, the objective and the subjective, we are now to divide all empirical predicates into two very different classes, the mathematizable and the non-mathematizable. By an extraordinary coincidence it is found that the supremely objective, indeed the very ground of all objectivity, namely the spatio-temporal matrix, proves also to be the supremely mathematizable. Why this should be the case; I do not know. Is it merely in virtue of some astonishing conceptual coincidence that the ontological substratum should prove equally to be the mathematizable par excellence? And hence the intelligible par excellence, on the assumption that the supremely intelligible is equivalent to the supremely mathematizable?

Oddly enough, this union of materialism with mathematics, this programme of a mathematically-materialistic ontology, escaped the attention of the ancient philosophers. The classical materialists grasped only one half the story: they saw that the ontological substratum is constituted by physical bodies moving through space and time; but they were radically indifferent, if not hostile, to the high conceptualizations of mathematics. The Platonists, on the other hand, grasped the other half of the story, namely that the idea of a science is realized most fully in mathematics, but thanks to their very preoccupation with the cognitive powers of the soul, they refused to accept a materialistic ontology. It was the privilege of Descartes to effect that union of materialism and mathematics which constitutes the inner core of all natural science, namely mathematical physics. We have seen that the ontological ground of mathematical physics is afforded by the happy intersection of the objective with the mathematizable. It is the intersection of the materialist’s insight with the Platonist’s, the one revealing to us the physical body as the ontological substratum, the other revealing to us mathematics as the perfected exercise of cognition. For it is the physical body, both at rest and in motion, which is the natural subject-matter of mathematics; and this means, finally, that it is the continuum or the theory of real numbers which is the heart of mathematics and indeed, if ontology be reducible to mathematical physics, then the theory of real numbers will be the heart of ontology as well.

The mere mention of the name of Descartes indicates at once the failure of mathematical physics, almost from the very beginning, to maintain its ontological standing in the face of a post-Cartesian modern philosophy tormented by the problem of knowledge. While science was to go the way of mathematical physics, philosophy was to go the way of epistemology, toward empiricism, idealism and even solipsism. The ontological underpinning of mathematical physics being (as Galileo was the first to see) the distinction between the primary and secondary qualities, Berkeley’s memorable attack on that distinction was to be the fateful move in the stripping of mathematical physics of its ontological dignity. Even today Berkeley’s attack on materialism is widely believed to be decisive. After Berkeley materialism was to be replaced by empiricism and positivism, and natural science was no longer to be understood ontologically.

The drift of our own inquiries in these pages will have become evident. In general, I welcome the Wittgensteinian elenchus in so far as it is designed to restore us to our natural birthright of common sense and to protect us against the flagrant subjectivity of Hume and Kant. But once we have recovered our natural pre-scientific understanding of the world, we are inevitably emboldened to test that understanding against the larger claims of the physical sciences. I am suggesting that our common sense merely opens up to us the Lebenswelt of practical discourse. In the press of life our primary object is not to know but to act, the one being subservient to the other. Such empirical predicates as ‘hot’ and ‘cold’, ‘hard’ and ‘soft’ play a vital role
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in the Lebenswelt, but they drop out of all specifically ontological discourse. The distinction between the primary and secondary qualities is seen to be equivalent to the distinction between ontological and practical predicates—with one important proviso. Such primary ontological predicates as 'cubical' and 'spherical' are also vital predicates in the order of practical discourse. This overlap should not be surprising. An ontological account of the world is one which is calculated to command the assent of all rational beings indifferently, independent of the specific physical make-up of each. Hence it follows that there must be a common ground between the life-world of the Hyperboreans and our own human life-world, and it is precisely this common ground which performs the double role of transcending each particular life-world and also of providing the substratum internal to each life-world in turn. The ontological order is thus rendered available to us within our specific life-world even though it transcends its parochial limitations.

In our effort to reinstate the primary-secondary quality distinction, we have hitherto evaded certain complications that may not be indefinitely deferred. All of the following vulgar predicates—'hot' and 'cold', 'hard' and 'soft', 'heavy' and 'light', 'rough' and 'smooth', 'sharp' and 'blunt', 'large' and 'small', 'thick' and 'thin', 'up' and 'down', yes and even 'countable' and 'uncountable', 'finite' and 'infinite'—all of these comprise but one type of secondary quality: they are all of the mutatis mutandis variety. There is, however, another type of secondary quality which resists being assimilated to that model. This second type of secondary quality is illustrated most notably by colours. The colour-words 'red', 'green', 'blue', &c., are certainly not mutatis mutandis concepts, and if they are to be shown as dropping out of ontological discourse, it must be for very different reasons. In addition, the colour-predicates are seen to drop out of ontological discourse quite easily. Let us raise the following question. Are the Hyperboreans colour-blind, either partially or completely? How shall we undertake to answer that question? Clearly, we shall employ the same behaviouristic criteria which we employ with a cat or a dog or a monkey or even a human being. Let it be established on behaviouristic grounds that the Hyperboreans are alive to the whole range of colours to which we have access. Convinced that they in no way blind to any of the colours, we are now—for the first time—to monitor their colour-language. Hitherto we have tested the Hyperboreans for colour-blindness in the same way that we would test a monkey or a kangaroo, quite apart from any reliance on speech. It is merely out of curiosity that we now undertake to monitor the colour-language of the Hyperboreans. I have already said that the Hyperboreans speak flawless English, but there is one difference between their language and ours that I have neglected to mention. They have no colour-words in their language. Instead, they speak directly of wave-lengths. 'Hand me that book which reflects a wave-length of 7,000 Angstrom units', a Hyperborean will say when we should say, 'Hand me that red book'.

Although we have already certified that the Hyperboreans are in no way colour-blind, we may be strongly tempted to retract that certification or at least to qualify it. If we engage them in conversation with the aim of determining whether or not they really see colours, they insist that they make their discriminations (their colour-discriminations) exclusively on the basis of wave-lengths; they profess to be aware of nothing else. When we wax lyrical about the greenness of the green grass—'a green thought in a green shade'—they can only blink down
at us bewilderedly. Furthermore, they are quite vehement in refusing to admit that there is anything in the grass of which we human beings are peculiarly aware. It is not difficult for us to prove to them that the word ‘green’, when we use it, is not understood by us to denote any wave-length of light, and one might suppose that that would be quite sufficient to establish that the word ‘green’ denotes some other property of the grass, not to be confused with any wave-length, to which they are denied access. But they do not accept that argument. They insist that the word ‘green’, as we use it, does not denote any feature of the grass at all. The word merely has a use. We call an object ‘green’ whenever the object reflects a particular wave-length of light, but when we call the object ‘green’ we do not mean by that word that the object reflects a particular wave-length of light. In fact, most human beings are altogether ignorant of the different wave-lengths. What, then, do we mean when we say that the grass is green? According to the Hyperboreans, we do not mean anything. There are certain specifiable physical conditions in which human beings say that an object is green, and that is virtually all there is to it. We may indeed distinguish those physical conditions in which human beings mistakenly call an object ‘green’ from those in which they veridically call an object ‘green’, but in making that distinction we are not entitled to assert that there is anything in the grass of which the Hyperboreans are unaware.

Still, there is the greenness of the green grass; is there not?, from which the Hyperboreans are invincibly shut out. So we are moved to argue. But what is this sheer greenness to which science is occupationally blind? Is it not ineffable? It is important in this discussion to sort out three different items: (1) the colour-impression, (2) the colour-word, and (3) the wave-length or whatever physical equivalent may actually obtain. On purely a priori grounds we are able to show that the ineffable colour-impression, the sheer greenness of green, not only drops out of all ontological discourse but also drops out of all practical discourse as well. Let us suppose that human beings were free to exchange their eyes with one another. You shall let me borrow your eyes, and I shall let you borrow mine. Looking at the world with your eyes, I find—mirabile dictu—that the grass no longer appears green and the sky no longer blue. Instead, there is a colour-reversal. The grass looks blue and the sky green. Looking at the world through some third person’s eyes, I find that the grass now looks red and the sky yellow. In general, the world will look different to me depending upon the particular eyes through which I view it.

What is the real colour of the grass? Green? Blue? Red? Yellow? Who can say? It looks different to me through each pair of eyes. No single pair of eyes is privileged: they are all on a cognitive par with one another. Granting this promiscuous use of eyes, surely it makes no sense to speak of the real colour of the grass. Which pair of eyes will disclose to me the real colour? I may indeed arbitrarily define the standard colour of the grass in terms of some one pair of eyes that I choose at random from the whole collection at my disposal, but such an arbitrary definition will be of no ontological import. The grass simply has no real colour.

Despite this promiscuity in our colour-impressions, we shall require in our practical discourse a settled language of colour-words. Thus the motorist must be enjoined to stop at the red light and to move on at the green. How, then, shall we fix the signification of our colour-words? There are various ways. The easiest is perhaps to insist that each person shall select from the public stock of available eyes one pair that pleases him. Henceforth he is solemnly prohibited from replacing his approved eyes by any others. There are to be more interchanges. Each man being committed to one pair of eyes, we shall all be drilled in the standard usage of colour-words. The colour of grass is called ‘green’; the colour of blood is called ‘red’; the colour of the sky is called ‘blue’. No one will be so naive as to suppose that the public colour-word ‘green’ designates some unique colour-impression. For we have all had the opportunity to view the grass through different eyes, sometimes seeing the grass in terms of one colour-impression and sometimes in terms of another. The private colour-impression of each man simply drops out as irrelevant in the give-and-take of practical discourse. The colour-word of practical discourse being divorced from the colour-impression of private experience, it is easy now to see that the colour-word itself may drop out of ontological discourse. Lacking all colour-words in their language, the Hyperboreans nevertheless insist that there is nothing in the grass of
which they are unaware. We, for our part, even though we certified on behaviouristic grounds that they were in no way colour-blind, came to have serious misgivings when we learned that they were quite without any colour-impressions. It was, then, the sheer greenness of the green grass that we regarded as being beyond the ken of the Hyperboreans. But the sheer greenness, we now find, is only a private colour-impression which drops out not only from ontological but also from practical discourse as well.

Dropping out of both ontological and practical discourse, how can the colour-impression be said to be, either absolutely or relatively speaking? Must we not then say that there simply are no private colour-impressions at all? This, certainly, is the conclusion of the radical behaviourist, and it may be felt that the logic of my argument forces me into that position. Wittgenstein himself shuffles on the issue. As to the private colour-impression he says: 'It is not a something but not a nothing either! The conclusion was only that a nothing would serve just as well as a something about which nothing could be said'.

However problematical the private colour-impression may be within the recesses of the mind, I believe that we have succeeded in showing how an ontological account of the material world at least may be free to dispense with all of the so-called secondary qualities. Not that these secondary qualities are to be scorned as illusory. I reiterate: they are certainly veridical on the plane of discourse where they belong, on the level of practical reason; for on that level we have at our disposal effective criteria that enable us to distinguish between mere appearance and reality. An object may look green under certain lighting and yet not actually be green. Conversely, an object may actually be green and yet under certain conditions (e.g. total darkness) fail to look green. On occasion one may even be deceived in the matter and, being deceived, one may also be set right. But though 'green' is certainly an empirical predicate and even a cognitive predicate, the level of cognition at which it functions is that of practical, not theoretical, reason. 'Green' is almost certainly not an ontological predicate. That is to say, an exhaustive account of any green object, say the green grass, may be given in which neither the word 'green' nor any synonym of the word 'green' is anywhere to be found. If a Hyperborean were to write a book on the nature and properties of grass, the word 'green' would be altogether absent from his account, and yet we should be unable to criticize his account as being incomplete, as omitting any feature of the grass. The converse does not hold. If a human being were to write a book on the nature and properties of grass, and if that book were to omit the fact that grass reflects light of a certain wave-length, then a Hyperborean would be entitled to criticize the account as being less than exhaustive.

Science is seen to be the ontological disclosure of the thing in itself, and as such it may be felt to be heartless and inhuman, being by its very nature trans-human. Within the human life-world it is man who is the measure of all things, not any man, certainly, but the normal man under normal conditions. It is he who provides the standard, in vulgar practice, both as to what is and as to what is not. Is there complete silence in the room? Who is to decide? Not any man picked at random. Not the man who is hard of hearing, but the man whose hearing is unimpaired. Let it be certified by this man that there is complete silence in the room. What now are we to make of the Hyperboreans who insist that the room is actually filled with loud noise? The English word 'noise' is found in their language, and like dogs they respond to high frequencies that are inaudible to us. Do the Hyperboreans mean by 'noise' what we mean by 'noise'? Yes, mutatis mutandis. We may even suppose them to have private noise-sensations in their minds. At any rate, when they lend us their ears with which we replace our own, we actually hear a great clatter of noise in the room. There is certainly noise in the room as far as they are concerned, but it is no less true that as far as we are concerned, with our native ears, there is no sound at all, only silence.

In transcending the human life-world, man qua man is no longer the measure of all things. It is no longer man with his specific physical make-up, with his specifically human body, that provides the standard. The human life-world and the Hyperborean life-world both fall within that over-arching common frame of reference which is the ontological order, and within that order it is reason, it is science, not man qua man,
that is the measure of all things, both as to what is and as to what is not. There is silence in the room, relatively speaking, i.e. relative to us in our specific life-world; but there is also loud noise in the room, relatively speaking, i.e. relative to the life-world of the Hyperboreans. Absolutely speaking, ontologically speaking, scientifically speaking, there is neither silence nor noise: there are only certain vibrations in the air. These vibrations are mathematically-physical entities that are beautifully measurable within the extensive continuum of the spatio-temporal matrix. For we have found that the primary qualities of physics in the very act of constituting the ontological substratum prove also to be the mathematizable, the quantifiable, the measurable, par excellence.1

Are we, then, to conclude that ontology is equivalent to mathematical physics? Although there are doubtless many difficulties with such an equation, there are two which are particularly germane to the present inquiry. They are summed up in the words 'subjectivity' and 'teleology'. Both of these difficulties are profoundly rooted in the peculiar status of man within the ontological order. By 'subjectivity' I am referring to the radical subjectivity of the private colour-impression and of private experience generally. If these things exist at all (pace the behaviourist), what is their ontological standing in a world otherwise characterized by the formulas of mathematical physics? Are we obliged to revert to the dualism of Descartes and Locke? The second difficulty resides in the fact that our whole approach presupposes a teleological account of concepts and, a fortiori, a teleological account of man as the user of concepts. But how is teleology to be brought into line with mathematical physics? Here, then, is a second dualism, mind being understood teleologically, matter mechanistically.

This second, and more profound, dualism is given its strongest expression by Leibniz.

It is true that according to me the soul does not disturb the laws of the body nor the body those of the soul; and that the soul and body do only agree together; the one acting freely according to the rules of final causes; and the other acting mechanically according to the laws of efficient causes.2

2 Leibniz, Fifth Paper in the Leibniz–Clarke correspondence, § 92.

Essentially, this is not far removed from the position of Gilbert Ryle in our own day. Ryle sees 'plenty of room for purpose where everything is governed by mechanical laws', and he is willing to allow that 'one and the same occurrence is governed both by mechanical laws and by moral principles'.1 Physics may succeed in supplying an exhaustive account of the human body and all its movements (including the wagging of the tongue), but Ryle insists that such an account can never be sufficient if we are to understand the human person and his mind.

In any case, whether or not we can succeed in bringing mind under the scope of mathematical physics, it must not be supposed that science can ever be the repository of all wisdom. Science may enshrine all ontological wisdom, but ontological wisdom is not the only wisdom: there is also practical wisdom. For we have seen that there are two kinds of reason, theoretical and practical, and if the former enjoys priority in the order of dignity, it is the latter which comes first in the order of urgency. Man is by nature amphibian. Exercising his theoretical reason, he is privileged to participate in the ontological order which transcends, even while it underlies, each and every life-world. Exercising his practical reason, he is enabled to prosper and flourish within his own life-world. There is no necessary connexion between these two types of reason. It is said of Thales, the first philosopher, that gazing up at the stars he fell into a well—to the infinite amusement of a Thracian slave-girl. Here was the wise Thales who did not know what was in front of his own nose. The rift between theory and practice need not always be comic. Hamlet reminds us of the tragedy that may await the speculative imagination challenged by the urgencies of the life-world; and in our own day the explosive entry of theoretical physics into the business of the life-world has taken a positively sinister turn. In the absence of practical wisdom, theory itself may be pressed into the service of folly.

Returning to the simple life-world of the Swahilis, we are regressing to a form of life in which philosophy and science are unknown. The faculty of theoretical reason lies dormant, it has yet to be awakened. If there is anything here beyond practical discourse, it is the language of myth which in its own confused way expresses the recognition of Powers that underly...  

and transcend the life-world. But the incantations that will raise these Powers from the depths are neither to be found in prayer nor in magic spells, neither in ritual sacrifice nor in purity of heart. The Powers of the ontological order will answer only to the formulas of mathematical physics.

The Swahilis being limited to the order of practical discourse, it is not surprising that the conceptual opposites 'finite' and 'infinite', 'countable' and 'uncountable', 'numerable' and 'innumerable' should be assigned a sub-standard interpretation in their life-word. If these concepts are to have a use in their language, they must be adapted to the limitations of their form of life. The inherent adaptability of these vulgar concepts has already been remarked: they are of the \textit{mutatis mutandis} variety.

According to the Hyperboreans, the leaves of the jungle are certainly countable. They are in the habit of counting collections quite as large almost every day in the week. The grains of sand in the desert—these (say the Hyperboreans) are uncountable, innumerable, numberless, quite without number, infinitely many. Do the Hyperboreans mean by these words (they speak flawless Swahili) what the Swahilis mean? Yes, \textit{mutatis mutandis}. In the same way the Hyperboreans call the Great Rapids eminently navigable. Unnavigable they call the western cataracts.

Although the 'navigable' and the 'countable', the 'unnavigable' and the 'uncountable' would seem to be strictly on a par, there is one striking difference that is big with ontological import. Are the Great Rapids navigable? Relatively speaking, they are certainly navigable, i.e. relative to the life-world of the Hyperboreans. Relatively speaking, they are no less unnavigable, i.e. relative to the life-world of the Swahilis. Absolutely speaking, they are neither navigable nor unnavigable. These predicates simply have no use in ontological discourse. What would it be like for a water-way to be \textit{absolutely} unnavigable? Unnavigable even by God. It is at this point that the parallel between the unnavigable and the uncountable breaks down. For the absolutely uncountable, the absolutely infinite is found to have a precise use in ontological discourse. The Swahilis say that the leaves of the jungle are infinite and that the natural numbers are infinite. The Hyperboreans say no, the leaves of the jungle are not at all infinite; they say that the grains of sand in the desert are infinite and that the natural numbers are infinite. God says no, the grains of sand in the desert are certainly countable, but he will not deny that the natural numbers are infinite. The absolutely uncountable comes to sight, then, in the natural numbers themselves.

All rational beings will agree (each speaking in the vulgar tongue) that the natural numbers are inexhaustible, on the ground that for every number \( n \) there is a successor \( n + 1 \) which is also a number, but this agreement may be felt to be scarcely more than nominal. Although Swahili and Hyperborean alike insist that the domain of natural numbers is inexhaustible, each has a different domain in mind. The domain of Swahili numbers and the domain of Hyperborean numbers are not identical. In fact, the domain of Swahili numbers is certainly not infinite in the Hyperborean (not to mention the standard) sense of infinite. Not that, even for a Hyperborean, there is a terminal Swahili number; but the whole series of Swahili numbers simply peters out somewhere within the domain of Hyperborean numbers. This is to say no more than that (1) every collection which the Swahilis call countable, the Hyperboreans also call countable, and (2) there exist collections which the Swahilis call uncountable that the Hyperboreans call countable, and (we might add) (3) any collection which the Hyperboreans call uncountable, the Swahilis also call uncountable. When we say that, ontologically speaking, the natural numbers are absolutely inexhaustible, it is imperative that we specify what domain of natural numbers we have in mind. Are the natural numbers \textit{qua} Swahili numbers absolutely inexhaustible? No, for they may be exhausted by the Hyperboreans. Are the natural numbers \textit{qua} Hyperborean numbers absolutely inexhaustible? No, for they may be exhausted by God. \textit{Qua} what, then, are the natural numbers absolutely inexhaustible? The natural numbers are absolutely inexhaustible \textit{qua} divine numbers. It is only when we undertake to envisage the natural numbers, not from the point of view of the Swahilis, but from the point of view of the Hyperboreans, that the absolutely infinite comes to sight. God himself will admit that even he cannot complete the task of writing down all of the natural numbers one after the other. There are always more of them than one has written down.
We are assuming indeed, in this discussion, that the Zeno procedure remains hidden from us. For after we have been introduced to its power, we shall withdraw our admission that even God is incapable of completing the task of writing down all of the natural numbers one after the other. No longer will the domain of natural numbers, taken in the widest sense, be regarded by us as absolutely inexhaustible. We shall be tempted to re-assign that dignity to the real numbers until we come to perceive that there is no logical contradiction or ontological absurdity in asserting that God has succeeded in writing down all of the real numbers during the past ten minutes. In the end the absolutely inexhaustible, the absolutely infinite will be found to reside in the cardinal numbers of Cantor. For it is logically self-contradictory to assert that God can ever succeed in writing down all of the Cantorian cardinals, both finite and transfinite, even with all of time—past, present and future—at his disposal. The very notion of all Cantorian cardinals is seen to be an illegitimate totality. Given any class of Cantorian cardinals, it is always possible to exhibit a Cantorian cardinal that lies outside the class.

Although this feature of his system originally came as a great shock to Cantor, I believe that the paradox of the transfinite, as it has been called, need not be at all embarrassing if it is embedded in a sufficiently rich ontology. For as soon as the natural numbers cease to represent for us the absolutely inexhaustible, we should be eager to find another type of number which will satisfy that concept, and it is precisely the Cantorian cardinals which we should welcome in that role even had we never been acquainted with the paradox of the transfinite. Instead of regarding the ‘paradox’ as at best a minor defect and at worst a formal reductio ad absurdum of Cantor’s whole system (as indeed it originally appeared within Cantor’s own formulation of the transfinite), I am suggesting that we rejoice in it as a positive merit. Finite and infinite, exhaustible and inexhaustible—these are correlative concepts, and though what appears to be inexhaustible on one level of discourse may prove to be exhaustible on a higher level, we should be reluctant to relinquish altogether the idea of the absolutely inexhaustible. The absolute inexhaustibility of the Cantorian cardinals is no longer to be viewed as an aberration for which the Cantorian

must apologize. I have already proposed that the transfinite cardinals be understood as potentially infinite in a highly exalted super-sense of the potential infinite.

Descending from these elevated regions, the finitist is right when he insists that our standard concept of the infinite is modelled upon the series of natural numbers and that this series is only potentially infinite: it may be indefinitely extended. But the finitist fails to see that, granting the intelligibility of the potential infinite, it now follows that there may exist an actual infinity of stars in the heavens—provided that for any \( n \), \( n \) being a natural number, there exist at least \( n \) stars in the heavens. It is not difficult to negotiate the passage from our standard concept of the potential infinite to our standard concept of the actual infinite. What is difficult to negotiate, is the prior passage from our sub-standard vulgar concept of the actual infinite to our standard concept of the potential infinite. This earlier passage I explain as all of a piece with the larger shift from the order of practical to the order of theoretical discourse. Speaking in the vernacular, as Swahilis, we say that the leaves in the jungle are numberless; but we cannot expect all rational beings to acquiesce in that location. The Hyperboreans insist that there is a natural number \( n \) which expresses the exact number of leaves in the jungle. Do the Hyperboreans mean by ‘number’ and indeed ‘natural number’ what we, as Swahilis, mean by those terms? Yes, *mutatis mutandis*. Now let us suppose that God insists that there is no natural number \( n \) which exceeds the exact number of stars in the heavens. God insists that for any natural number \( n \) there exist \( n+1 \) stars. Such being the case, the stars in the heavens must be allowed to be infinite in a peculiarly strong sense. For not only God but any rational being whatever, be he Swahili or Hyperborean, must admit that there are infinitely many stars, even in his own vulgar sense of ‘infinite’.

If the stars be infinite in this strong sense of the term, then they must be allowed to be *absolutely* infinite, not merely infinite relative to some parochial life-world. The domain of natural numbers as viewed by God thus proves to be the standard model of the absolutely infinite.

If we now say that the leaves of the jungle are *really* countable, we are not employing the word ‘countable’ in its literal, vulgar sense: we are employing it tropologically. What kind of trope
is it precisely? Broadly, it is one of analogy, but specifically it is
to be understood, almost in the Thomist sense, as an analogy
of proper proportionality. As any vulgarly countable collection
(e.g. our ten fingers) is to us Swahilis, so are the leaves of the
jungle to God. Seeing that the trope here is privileged, it may be
suitably styled a metaphysical trope.

The shift from the vulgar concept of number to the divine
concept opens the way, for the first time, to a science of mathe­
matics. Is there a greatest prime number? This question cannot
be answered—it scarcely makes any sense—in the vulgar
domain of Swahili numbers. One might as well undertake to
answer the following questions. Does there exist a poor man X
with \( m \) pennies (\( n \) being prime) such that there does not exist
any poor man Y with \( m \) pennies, \( m \) being also prime and
\( m > n \)? Does there exist a bald man X with \( n \) hairs on his head
(\( n \) being prime) such that there does not exist any bald man Y
with \( m \) hairs on his head, \( m \) being also prime and \( m > n \)? In
other words, is there a greatest prime poor man or a greatest
prime bald man? In raising these questions, I am assuming
(1) that there exists a poor man X with no money at all and a
bald man X with not a single hair on his head, and (2) that for
any poor man Y (or bald man X) there exists a man Y
with one penny (or one hair on his head) more than X. Even
granting these conditions, our questions remain not only
unanswerable but they fail so much as to achieve any clear
sense. Let us now but ask if there exists a greatest prime
number in God’s domain of natural numbers, and though we
are but humble Swahilis, we are able to answer that question
as though we were God himself. Hence the high exhilaration
that we experience in doing mathematics: we are playing at
being God.

The divine afflatus is ours, not only in pure, but also in
applied mathematics which opens the way to theoretical
physics. Archimedes is a case in point. Assuming that the visible
universe were packed solid with grains of sand, if God were to
undertake to count them all one by one, what number would be
finally assigned to the whole multitude? Although this question
may be too difficult for us to answer, there is another question,
closely related to it, that falls within our scope. Is there a num­
ber, in God’s domain of numbers, that we can name which is
greater than the number that God would assign to the totality
of grains of sand in the visible universe? The Swahilis in their
ignorance insist that this question also must be answered in
the negative—they are mistaken! I wish now to argue that,
even on their own terms, the Swahilis are guilty of an error
when they insist that the leaves in the jungle are innumerable,
uncountable, numberless, quite without number, indeed
infinite. If all that they meant were that no human being is
capable of counting them all one by one, their position might
be impregnable. But that is not all that they mean. A more
generous theory of meaning is required here. In their ignorance
of computational short-hand, the Swahilis are tacitly denying
the feasibility of a whole range of investigations which actually
lie within the scope of human ambition. ‘For us it is always
beyond our ken’, the Swahilis must be understood as saying;
and though in one sense they are right, there is another, more
profound sense in which they are wrong. After they have been
introduced to the computational techniques of applied mathe­
matics, they will say: ‘We never realized before that the
totality of leaves in the jungle does in fact lie within our ken—
in an unexpected way. Not that what we had uppermost in
our minds before has been shown to be erroneous. But much of
the point of our earlier insistence that the leaves in the jungle
are uncountable, has been exploded for us.’

We may put the matter in the following way. It is rarely the
case that anyone is primarily interested in whether or not a
particular collection of objects can be vulgarly counted one
by one. Indeed, once one has mastered the indirect methods of
computation, he finds that the kind of job which one by one
counting enables him to perform is usually achieved quite as
effectively, if not more effectively, by computational tech­
niques. This is no mere contingent fact which might be im­
agined otherwise. If there is a square array of soldiers, 100 by
100 strong, it will be difficult to count them all one by one but
very easy to compute the total number. Whether one follows the
first method or the second, the net result will be the same. The
Swahilis in their ignorance do not know that collections which
are vulgarly uncountable are frequently computable or, at
any rate, that a rough approximation may in fact be given.
On being enlightened, they learn to their surprise that what
they supposed to lie 'beyond their ken' can in fact to some extent be brought within their ken, albeit in an unexpected way.

To summarize our conclusions. There are at least three procedures that enable us to transact the passage from our vulgar concept of number to the standard concept. First, it is shown that the vulgar concept, even granting that it may be adequate within a primitive form of life, fails to perform another job that many come to command our attention. This new job is the job of theory, of ontology, of science, the job of transcending each and every parochial life-world in an effort to view the world from the perspective of universal reason. This new job opens the way to the second procedure, for it is found that both the concept of number and the concept of the infinite (they are bound up together) are inherently Janus-faced, one face being turned toward the practical and the relative, the other face (hitherto veiled from the Swahilis) being turned toward the theoretical and the absolute. Finally, in another vein, a wedge may be driven between what the Swahilis connote or mean in the deepest sense by the 'infinite' and what they denote by that concept. That is to say, we are able to show that what they believe to lie beyond their ken can in fact—contrary to their belief—be brought within it.

The foundations of mathematics are seen to be neither logical, in the style of Frege, nor empirical, in the style of Mill; they are neither formal, in the style of Hilbert, nor intuitive, in the style of Brouwer. They are rather to be understood teleologically, with reference to a specific job of work to be done. Although Wittgenstein divined this teleological import of mathematics, his inveterate distrust of metaphysics and the absolute prevented him from seeing that the specific job of mathematics, both pure and applied,\(^1\) is precisely the ontological job. Our passage from proto-logic to standard logic is equally—I submit—to be understood in terms of the shift from vulgar to ontological discourse. Why is our standard logic embarrassed by the vulgar concepts of baldness, poverty, and thinness? Because it is designed to monitor a very different type of discourse, the language of science and ontology. Logic can be two-valued only if it presides over a domain characterized by predicates which are absolutely exact: there must be no indeterminacy in borderline cases. But where are such predicates to be found? Only in the exact sciences; and even here they are to be found only in an ideal sense. For they presuppose a technique of utopian measurement that issues in nothing less than metaphysical accuracy.

\(^1\) By applied mathematics I mean here theoretical physics, not applied science. There are two types of applied mathematics, the one being found in theoretical, the other in applied, science.
CHAPTER VI

MINIMS AND THE CONTINUUM

Part I: The Serrated Continuum

Having wandered fairly far afield from our direct thematic examination of Zeno, I propose to return to the charge in this last chapter. Not that I can promise any definitive solution to Zeno’s challenge; indeed I shall doubtless be found merely to have aggravated it.

Looking back on our researches and setting to one side all of our copious digressions and excursions which, however interesting in themselves, ought not to deflect us from our principal inquiry, I am most struck by two things: among our earlier studies, the paradox of the alternating divergent series, and among our later studies, the paradox of the imaginary cut. Neither, to be sure, is a paradox in the strict sense; there is no actual logical contradiction involved in either; and it would not be unreasonable of us to argue that, seeing that no one has ever doubted that the concept of the infinite is at best a very odd notion, these ‘paradoxes’—not to mention others, such as Hilbert’s Grand Hotel—were only to have been expected. Furthermore, I may be forgiven for belabouring once again the point that I have belaboured so often, namely that it is finitism, initially so plausible, that is the metaphysically pretentious doctrine, with its tacit insistence that we know a priori, apart from all empirical evidence, that the sequence of stars must at some point come to an end.

Granting the deficiencies of finitism, one may yet be allowed to shrink from the enormities that the actual infinite entails. The paradox of the imaginary cut is perhaps the worst of the lot. In the course of travelling an infinite distance from A to \( A_1 \), we are to stretch a tightrope from the one to the other. We have found that a tightrope walker walking the full distance of the rope from A to \( A_1 \) will at no point intercept a tightrope walker walking the full distance of the same rope from \( A_1 \) to A.

They may indeed be described as intercepting one another at an imaginary point, but this location is a mere façon de parler that cannot be permitted to cloud the issue. If the rope were of any finite length, it would be easy to prove that the two tightrope walkers must at some point collide; but the rope being of infinite length, it is quite as easy to prove that no such collision is possible. Let us but reconcile ourselves to the fact that the infinite obeys laws radically different from those that govern the finite, and we may be able to yield to these paradoxes with good grace.

In regard to the paradox of the alternating divergent series, I think that it may be rendered somewhat more palatable if the infinitesimal tail of the Z-series is taken into account. As we travel in our metaphysical rocket from A to \( A_1 \), let us celebrate the paradox of the apple aboard ship. The table being initially empty, we are to place a single apple upon it after we have travelled ten feet from A to B; having travelled another ten feet from B to C, we are to remove the apple from the table, &c., ad infinitum. In general, at every ten feet of our journey we are either to restore the apple to the table or remove it depending on the previous condition of the table. At the end of the journey, landing at \( A_n \), where will the apple be? On the table or off? Let us suppose that at \( A_1 \) the apple is seen to be on the table. Then when we were at \( B_1 \) the table was empty; at \( C_1 \) the apple was on the table; at \( D_1 \) the table was empty, &c., ad infinitum. Viewed in this light the sum-total of Grandi’s series must be either 0 or 1—all other values are excluded. There is thus found to be a very restricted indeterminacy in the sum-total of the series. Although the infinitesimal tail of the Z-series cannot be lopped off in toto, any finite sequence of small infinitesimals can certainly be broken off at the far end. If at the end of the minute we find ourselves at \( A_1 \), then it was quite possible for us to have been shot down in full career at \( B_1 \) or \( C_1 \), or \( D_1 \), &c., ad infinitum, in all of which cases we should have travelled for an interval of one minute minus an infinitesimal duration.

That our approach to these two paradoxes—Grandi’s series and the imaginary cut—should all be of one piece, is only appropriate. For as the one arises out of the alternating, the other springs from the non-alternating, divergent series: in the one case the series \( 1-1+1-1+1 \ldots \), in the other case...
the series \(1 + 1 + 1 + 1 + \ldots\), neither of which can be said
to converge to a limit in the strict sense. As the former is
indeterminate in its issue, so also is the latter. We have seen that
as the metaphysical rocket leaps out of Space § 1 into Space § 2
(at an imaginary point) there are an infinite number of \textit{termini ad quem}
which are all equally eligible.

Lest it be supposed that it is only the divergent series (non­
alternating as well as alternating) that elicits our perplexity, we
are now to re-examine the convergent series itself. It, too, has
been found to be infected with indeterminacy in its tail; and it is \textit{that}
indeterminacy which has been the source of all our
befuddlement with the divergent series. For the divergent
series are in themselves innocuous: in themselves they are to be
understood in terms of the potential infinite. It is only when they
are grafted on to the Z-series (itself being enacted in the small)
that they inspire uneasiness. But the Z-series may be seen to be
in its own right a small horror. That horror comes to sight most
alarmingly in connexion with a paradox far more deadly than
any that we have hitherto considered. This new paradox we
may style the paradox of the serrated (or crenellated) continuum,
and I may say in advance that we shall be very fortunate
if we can provide anything remotely like a satisfactory
solution to it. In all previous versions of the ‘bisection’ paradox
we have been successful (I think) in administering place­
bos to allay our disquietude; but this new crisis will be
found to place our entire programme in the utmost jeopardy,
and I can only trust that out of all the wreckage there
may be some few fragments of our enterprise that may yet be
salvaged.

Here is a book lying on a table. Open it. Look at the first
page. Measure its thickness. It is very thick indeed for a single
sheet of paper—1/2 inch thick. Now turn to the second page
of the book. How thick is this second sheet of paper? 1/4
inch thick. And the third page of the book, how thick is
this third sheet of paper? 1/8 inch thick, &c. \textit{ad infinitum}.
We are to posit not only that each page of the book is
followed by an immediate successor the thickness of which is
one-half that of the immediately preceding page but also (and
this is not unimportant) that each page is separated from page 1
by a finite number of pages. These two conditions are logically
compatible: there is no certifiable contradiction in their joint
assertion. But they mutually entail that there is no last page
in the book. Close the book. Turn it over so that the front
cover of the book is now lying face down upon the table. Now—
slowly—lift the back cover of the book with the aim of ex­
posing to view the stack of pages lying beneath it. \textit{There is
nothing to see.} For there is no last page in the book to meet our
gaze.

Another version of the same paradox. Lying upon the
ground there is a slab of stone 1/2 inch thick, with the number 1
inscribed upon its top surface. Resting squarely on this first
slab is a second slab of stone, 1/4 inch thick, with the number 2
inscribed upon its top surface. Resting squarely on this second
slab is a third slab of stone, 1/8 inch thick, with the number 3
inscribed upon its top surface, &c. \textit{ad infinitum}. Once again
there is no logical contradiction in positing (1) that upon the
top surface of each slab there rests another slab the thickness
of which is one-half that of the slab upon which it lies, and (2)
that each slab in the pile is removed from slab § 1 (which rests
flat on the ground) by a finite number of slabs. Let us now
plant our foot firmly upon this infinite pile of slabs. But what
is there to plant our foot upon? \textit{There simply is no top surface
to the whole pile}. For the pile is constituted exclusively by
the slabs \textit{ex hypothesi}: absolutely nothing else has been
added.

A third, and final, version of the paradox. Here is a large
flat board of wood securely fixed in the ground. Five feet wide,
it stands ten feet tall above the ground, but it is only 1/2 inch
thick. Squarely parallel to this first board, widthwise, is a second
board of wood removed from the first by a distance of 1/2 inch.
Also five feet wide and ten feet tall, this second board is only
1/4 inch thick. Squarely parallel to this second board, again
widthwise, is a third board of wood removed from the second
by a distance of 1/4 inch (and from the first by a distance of
1/2+1/4+1/4 inches). Also five feet wide and ten feet tall,
this third board is only 1/8 inch thick, &c. \textit{ad infinitum}. Once
again there is no logical contradiction in positing (1) that each
board is followed by an immediate successor the thickness of
which is one-half that of its immediate predecessor, and which is
removed from its immediate successor by a distance equal to
its own thickness, and (2) that each board in the sequence is separated from board §1 by a finite number of boards. Whereas the combined thickness of all the pages in the infinite book or all the slabs in the infinite pile, taken together, could not have exceeded one inch, in the present case the complete sequence of boards (counting the intervening spaces) will not exceed two inches, i.e., \( \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{4} + \frac{1}{8} + \ldots \). Removing ourselves several feet from the whole sequence, we are now to walk straight on into the open end. Soon we must collide with the last board in the sequence. But there is no last board. Let us posit that each board is so securely fixed in the ground that no pressure of ours could possibly dislodge it. Any board with which we may collide will thus succeed in repelling us. What will happen? It will certainly be impossible for us to plough on into the open end.

Viewed in abstracto, there is no logical contradiction involved in any of these enormities; but we have only to confront them in concreto for their outrageous absurdity to strike us full in the face. Here for the first time, in the paradox of the serrated continuum, Zeno's message comes through to us loud and clear. All other formulations of the 'bisection' paradox (Zeno's included) admit of a wide range of prevarication whereby the full brunt of the issue may be deflected. Brunt, indeed! It is like staring into the mouth of a cannon, except that here there is no hole. It is like. . . . But how can we describe this horror, this ineffable entry of ours into the uncanny open end of the Z-series? It is difficult to resist the conviction that Zeno was right after all. The very notion of the continuum has surely received its decisive refutation.

It may indeed be objected that in all three versions of the serrated continuum it is only the spatial continuum which has been impugned. Is there any chance that the temporal continuum may escape untouched? Alas. The paradox is no less effective when it is applied to time. A god assures us that between twelve noon and one minute after twelve he will successively recite the names of all of the natural numbers, with a pause of silence between them. During the last half of the minute we are to hear him say 'one'; during the preceding \( \frac{1}{4} \) minute we are to hear him say 'two'; during the \( \frac{1}{8} \) minute preceding that, we are to hear him say 'three', &c. \textit{ad infinitum}. Consulting his timepiece, the god informs us that it is now ten minutes to twelve. His lips will remain tightly sealed for the next ten minutes, and he promises not to open them until the serrated sequence of numerals issues from his mouth. Can the god fulfil his promise? We have now only to keep our eyes fixed on his lips and wait for the first sound to be uttered. But there can be no first sound \textit{ex hypothesi}. As in the case of the spatial, so also in the case of the temporal continuum, being afforded direct access to the uncanny open end of the Z-series, we are utterly confounded.

Here I would counsel prudence lest we yield to panic and precipitately throw over all that we have accomplished. Every defeat need not turn into a rout. Granting that we have over-extended ourselves in our immoderate pursuit of the infinite, it does not necessarily follow that we must beat a hasty retreat back to finitism. There is indeed some question as to how we should proceed at this point, and I propose to show that the right way (if there is a right way) is far from clear. Doubtless the most obvious move is to interpret the paradox of the serrated continuum as constituting an \textit{a priori} proof of the existence of absolute minims. Space and time on this view are no longer seen as infinitely divisible: they are rather held to be composed of indivisible puncts which are, in magnitude, some rational fraction of the inch, on the one hand, and of the minute, on the other, say a billionth of an inch and a billionth of a minute, than which no smaller quantum is ontologically capable of existing. Ever since classical antiquity this retreat to the rational minim has been the favourite refuge of those who have been embarrassed by the 'bisection' paradox; but it has been widely felt that this 'solution' is no more than a \textit{pis aller} which has the further disadvantage of leading to difficulties at least as grave as those it is designed to overcome. I may say at once that I share these sentiments, and it will be my object in the sequel to substantiate them in detail. If I am successful in this project of exploding the concept of the rational minim, our perplexity will be very deep indeed. For if space and time are not infinitely divisible, then they must surely be composed of indivisible puncts, and if these are disallowed, where else have we to turn?
Part 2: The Geometry of the Minim

We are now to investigate the rational minim which, for purposes of discussion, we shall arbitrarily fix at precisely one-billionth of an inch, as regards space, and at one-billionth of a minute, as regards time. At the very outset there are certain specious difficulties that should be cleared away. It is no objection to say that it is logically possible to divide the inch into two billion equal parts. Given any rational fraction of an inch, it is always logically possible to divide it in half—no contradiction results. But if the paradox of the serrated continuum proves that the infinite divisibility of space and time is impossible, then there must be some rational fraction of the inch which is ontologically (though not logically) indivisible. No a priori proof is to be expected which will specify precisely which rational fraction constitutes the absolute punct: the actual specification is left open, and viewed aprioristically all rational fractions are equally eligible. Nevertheless, we are certainly able to rule out many fractions on the basis of our experience, e.g. 1/2 inch and 1/4 inch.

A second specious difficulty is attributed to Roger Bacon who argues (in Cajori's words) that 'the hypothesis of indivisible parts of uniform size would make the diagonal of a square commensurable with a side', 1 which the Pythagorean theorem proves to be impossible. If the sides of a square are each composed of 1,000 puncts, then it certainly makes no sense to allow the diagonal to consist of \( \sqrt{2000} \) puncts. Roger Bacon forgets that the Pythagorean theorem and indeed most, if not all, of Euclidean geometry presupposes that any line-segment can be bisected (which is equivalent to positing that any line is infinitely divisible). Assuming an a priori necessity for the existence of minims, then not only is it the case that all irrational magnitudes are ruled out tout court but most rational magnitudes are equally vitiated. It will no longer be true that, given a line-segment of one inch, there exists a line-segment of \( 1/n \) inches for any \( n \). In fact, given a rect of one billion minims, it is impossible to divide it into 16 equal parts. And a rect of a billion and one minims cannot even be bisected into two equal parts.

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sun. 1 Given our figures, all 'continuously' moving bodies must
creep along at the rate of one inch per minute. No matter
what the figures, there can be only one speed for all 'con-
tinuously' moving bodies. What follows? That there are no
minims? No. Merely that virtually every case of motion which
we encounter is a case of 'discontinuous' motion: there is either
a skipping of spatial puncts or of temporal puncts or both.
When Roger Bannister ran the mile in four minutes, for every
spatial punct that he advanced he must be understood to have
remained at rest for many consecutive puncts of time before
resuming his 'motion'. Unless he was lucky enough to have hit
on the one speed which all 'continuously' moving bodies share,
the only alternative is to suppose that he hopped from one punct
to another without passing through the intervening puncts
that separate them. This suggestion may not be as absurd as it
sounds: we are reminded of the quantum jumps of contemporary
sub-atomic physics.

The second argument is as follows: 'Suppose a distance made
up of nine indivisible places arranged in a row, and let two
indivisible bodies be moving over this distance from each of
its extremities, and let them move at equal speeds. Then, since
their motions are equal in speed, each of these bodies will
necessarily pass over four indivisible places. And on arriving at
the fifth place, which is midway between the one set of four
and the other..." 2 What could possibly happen at this critical
juncture? Sextus argues: 'That both should come to a halt is
extremely improbable; for when there is a place existing and
nothing repelling their motion they will not halt.' Improbable?
Not at all. Given the premises, it is absolutely necessary that
they come to a halt, repelling one another across the gap. It is
curious to find that the doctrine of minims logically entails an
a priori physics which is in surprising accord with our experi-
mental physics. 'The apparent simplicity in the collision of
billiard balls is quite illusory. As a matter of fact the two billiard
balls do not touch at all.' 3

1 Against the Physicists, II, § 154, commonly cited as Adversus Dogmaticos, IV,
or as Adversus Mathematicos, X. Translated by R. G. Bury in Sextus Empiricus
2 Ibid. § 144-8.
3 B. Russell, The ABC of Relativity (George Allen & Unwin, 1925), ch. 1,
ad finem. The Wittgensteinians protest as follows. The collision of two

Most powerful of all is Sextus' third argument. If the minim
is to be dissolved, I am persuaded that it is this third argument,
or some variant of it, that must be relied upon. 'Suppose there is
a certain small ruler which on one side is provided at intervals
with points, and let this be made to revolve, starting from one
of its extremities, over a certain plane surface in one and the
same time. Now as the extremity revolves, circles will be
described which differ from one another in magnitude, the
outermost which surrounds them all being the greatest and the
innermost the smallest, the intermediate ones in proportion,
becoming either greater and greater as we advance from the
centre or less and less as we recede from the outer circumference.' 1
If the circumference of the outermost circle consists of 10,000
minims, then the circumference of the innermost circle will
contain far fewer, say 1,000 minims. But for each minim that is
generated on the outermost circumference there is precisely one
minim that is generated on the innermost—hence they must
be equal in number, which leads to a contradiction.

A variant of this argument—we may style it the paradox
of the circle—is attributed to Roger Bacon. In Cajori's words,
'Bacon held that... if the ends of an indivisible part of a circle
are connected by radii with the centre of the circle, then the
two radii would intercept an arc on a concentric circle of smaller

billiard balls is a paradigm case of contact or touching in the vernacular.
This is precisely what we mean by touching. Russell is held to be guilty here
of the Pickwickian fallacy: the assignment of a super-sense to a word in our
mother-tongue even while cashing in illicitly on the vulgar sense. (Cf.
Wittgenstein's physicist, p. 175, supra). Now let us suppose that, actually
looking into a high-powered microscope, we come to see with our own eyes a
minute intervening space between any two bodies that are vulgarly described
as 'touching', e.g. my hand lying flat on the table. We should certainly
be surprised and astonished. Why? Because part at least of what we believe
when we believe that two bodies are in contact, is that no intervening space,
however small, would be disclosed even to the sharpest sight. Being surprised
consists (rougly) in discovering that what one previously believed to be
true is not true, and to believe something is to be disposed to act in certain
ways. Here we find that what 'touching' connotes to us—the absence of any
intervening space between two bodies as disclosed even to the sharpest
sight—simply denotes nothing at all. This is not to deny that, on being
enlightened, we shall continue to speak in the vernacular; we shall certainly
continue to say that the hand is touching the table; but we shall not mean
by that location all that we previously meant. Our habitus will be different.
We shall mean merely that no intervening space can be detected by ordinary
vision. This was not what we previously meant in our ignorance.

1 Sextus Empiricus, op. cit. § 49-53.
The use of the expression 'ends of an indivisible part of a circle' is somewhat unfortunate in this context. What is doubtless meant is as follows. Consider three consecutive puncts \( p_1, p_2 \) and \( p_3 \) on the circumference of a circle. (The circle, to be sure, is a regular polygon the sides of which are puncts all equidistant from the central punct.) Now consider two rects, one connecting \( p_1 \) with the central punct, the other connecting \( p_3 \) with the central punct. Here we have a triangle with the smallest possible base, three puncts if we include the termini, one punct if we exclude them. What now of a concentric 'circle' of smaller radius? The two rects must intercept an arc of this smaller circle, and this arc must be less than a single punct in magnitude, which contradicts the hypothesis.

The circle paradox is readily adapted to other figures, such as the triangle. Consider any isosceles triangle \( ABC \) the base \( AC \) of which is composed of 1,000 puncts. (see Fig. 1). Now consider a rect parallel to \( AC \) intersecting \( AB \) and \( BC \). This rect \( DE \) must contain fewer puncts than \( AC \). But for any punct \( F \) on the base \( AC \) there exists a rect \( BF \) which intercepts a unique punct on \( DE \). Hence the number of puncts on \( DE \) must be equal to the number of puncts on \( AC \). Again, a contradiction results. Although this argument, mutatis mutandis, is already familiar to us as proving a one-to-one correspondence between the points on any two line-segments (assuming the continuum), we are pleased to note that it may be employed as an a priori proof of the continuum itself, i.e. the rational (if not the real) continuum.

An a priori proof of the continuum! Surely this is unlikely. Have we really succeeded in deriving a strict logical contradiction from the hypothesis of the minim? Yes, but only on the assumption that any two puncts may be connected by a rect. Unfortunately, if it be posited that space is composed of minims, then the paradox of the circle, with all its variants, merely proves that there exist pairs of puncts which cannot be connected by rects! We have only to consider any solid block of puncts, a spatial mosaic, so to speak. Although certain pairs of puncts, e.g. \( a \) and \( a_5 \) or \( d \) and \( i_5 \),

\[
\begin{align*}
  &a \ b \ c \ d \ e \ f \ g \ h \ i \\
  &a_1 \ b_1 \ e_1 \ d_1 \ e_1 \ f_1 \ g_1 \ h_1 \ i_1 \\
  &a_2 \ b_2 \ e_2 \ d_2 \ e_2 \ f_2 \ g_2 \ h_2 \ i_2 \\
  &a_3 \ b_3 \ e_3 \ d_3 \ e_3 \ f_3 \ g_3 \ h_3 \ i_3 \\
  &a_4 \ b_4 \ e_4 \ d_4 \ e_4 \ f_4 \ g_4 \ h_4 \ i_4 \\
  &a_5 \ b_5 \ e_5 \ d_5 \ e_5 \ f_5 \ g_5 \ h_5 \ i_5 \\
\end{align*}
\]

certainly admit of being connected by rects, there are other pairs of puncts, e.g. \( a \) and \( e_5 \) or \( d \) and \( e_5 \), which cannot be connected by rects. We may indeed choose to distinguish between proper and improper rects. \( a \) and \( f_5 \) admit of being connected by a proper rect, i.e. \( a-b_1-c_2-d_3-e_4-f_5 \). But if by a straight line we are to understand merely the shortest distance between two points, then even \( a \) and \( e_5 \) can be connected by a straight line; though this is surely a very peculiar kind of straight line—we may style it as an improper rect. The shortest distance between \( a \) and \( e_5 \) being six puncts (including the termini) or four puncts (excluding the termini), there are more than ten different 'straight lines' or improper rects by which \( a \) and \( e_5 \) may be connected. Some of these improper rects are as follows:

\[
\begin{align*}
  (1) & \quad a-b_1-c_2-d_3-e_4-f_5, & (2) & \quad a-b_1-b_2-b_3-c_4-e_5, \\
  (3) & \quad a-b_1-c_2-e_4-e_5, & (4) & \quad a-b_1-c_2-b_3-b_4-e_5, \\
  (5) & \quad a-a_5-a_3-a_2-b_4-e_5, & (6) & \quad a-b_1-c_2-d_3-d_4-c_5. \\
\end{align*}
\]

All of these improper rects must be held to be equal in length—six puncts each. How does this account bear specifically on the paradox of the triangle? Consider the isosceles triangle \( a_4-e-i_4 \) the base of which contains precisely nine puncts, i.e. \( a_4-b_4-c_4-d_4- \)
$e_4 - f_4 - g_4 - h_4 - i_4$. Of these nine puncts on the base only three, namely $a_4$, $e_4$, and $i_4$, can be connected to the vertex $e$ by a proper rect. If we now consider the proper rect $c_2 - d_2 - e_2 - f_2 - g_2$ which lies parallel to the base, it is certainly true that for any punct on the base of the triangle which is connected to the vertex $e$ by a proper rect, there exists a unique punct on the rect $c_2 - d_2 - e_2 - f_2 - g_2$; but under the constraining conditions which have been found to obtain, there is no logical contradiction. More generally, it may be supposed that this particular non-Euclidean geometry, characterized not by any denial of the parallel postulate as in the case of Riemann and Lobatchevsky but rather by the denial of the ‘bisection’ postulate, is almost certainly consistent; and as the non-Euclidean geometries of Riemann and Lobatchevsky have actually been proved to be logically self-consistent, a proof of consistency in the present case ought to be feasible. There is, however, an important qualification to be noted. The denial of the ‘bisection’ postulate is not logically compatible with all the other Euclidean axioms. Even granting that we remain entitled to assert that between any two ‘points’ there exists a straight ‘line’ (if not a proper rect, then an improper one), it is no longer possible to maintain that between any two ‘points’ there exists only one straight ‘line’. The existence of the straight ‘line’ may be preserved (albeit in a bizarre sense) but not its uniqueness.

Contrary to received opinion, the concept of a non-Euclidean geometry is now seen to be very old indeed, for the doctrine of the minim has never been adequately spelled out. Despite all the deep ontological concern with the discrete versus the continuous, there has never been any corresponding mathematical concern with this issue, in the specific form that this concern ought to take, namely a systematic working out of the geometry of the minim. For it is only after the geometrical consequences of the minim have been comprehensively surveyed that we are in any position to understand its full ontological import. Two such consequences we have already noted: first, the absence of any proper rect between many, and probably most, pairs of minims, and second, the non-uniqueness of the improper rect. There are other consequences equally serious all of which spring from the fact that bare, empty space may no longer be understood as isotropic, no longer as uniform and undifferentiated. It is now to be seen as granular or cellular in character. Consulting our putative diagram of the spatial mosaic (there are other, alternative diagrams that may be mooted but they will all entail analogous difficulties), we are to consider what a ‘circle’ might look like, with a radius of five puncts and its centre at $a$. Consider the proper rect $a - b - c - d - e$. If $e$ counts as $p_1$ on the circumference, which punct will count as $p_1$ immediately adjacent to it? $e_1$? The ‘distance’ between $e_1$ and $a$ is certainly five puncts, but the radius here is no longer a proper rect. In general, we may say that if by a circle we are to understand the locus of all puncts equidistant from an assigned punct, then there are two possibilities. If each punct on the circumference is to be immediately adjacent to another punct on the circumference, some of the radii must be improper rects. Conversely, if all the radii are to be proper rects, then no punct on the circumference will be immediately adjacent to any other. It is in this way that the paradox of the circle is dissolved (‘aggravated’ may be the better word).

Here we are roused to protest as follows. Granting that this punctual geometry may be internally self-consistent, is it not.

1 Actually, there is no postulate or axiom in Euclid which explicitly affirms the continuum, i.e. the biseatability of any straight line. How, then, does Euclid rule out the minim? By introducing the mathematical point. Although the mathematical point has never been regarded as anything other than an ens rationis, it plays a substantive role in tacitly resolving the ontological issue of the minim versus the continuum in favour of the continuum.

1 Would Berkeley and Hume have persisted in their rejection of the continuum had they been aware of what the minim entails?
perfectly plain that it is not at all consistent with our actual experience of the world? Quite apart from internal consistency in abstracto, have we not an external check at our disposal on the strength of which this quantized geometry may be invalidated in concreto? This is a very treacherous question and by no means as easy to answer in the affirmative as one might suppose. It is not merely that we have no experience of the minim per se (that is the least of the difficulties), but even the large-scale consequences that are logically entailed by it, even these do not readily lend themselves to being falsified by our actual experience of things in the rough on the molar level.

We are reminded of Eddington's two tables. On the one hand, there is the table of our natural pre-scientific understanding, solid and compact; on the other hand, there is the table as it is disclosed to physics, mainly empty space, the atoms of which the table is composed being largely vacuous. How to bring these two accounts of the table into line with one another is a major philosophic problem; and though the first account is prima facie a refutation of the second, as the second is of the first, it is evident that each must be accorded at least some kind of validity. The school of Wittgenstein argues that the first account is to be accepted as literally true whereas the second is to be accepted only as tropologically true, very much as the medievals in their efforts to reconcile reason and Scripture, prima facie in conflict with one another, came to the conclusion (notably with Thomas Aquinas) that, since truth cannot conflict with truth, Scripture must be interpreted tropologically at every point that it flouts any clear disclosure of reason. Science being the modern scripture, we are faced with the analogous problem of reconciling our vulgar understanding of the world with our scientific. My own approach to Eddington's two tables is as follows. Our vulgar conviction is that the table is solid and compact simpliciter, without qualification, and owing to this conviction we are disposed to behave toward it, under certain specifiable, though highly exotic, conditions which never, or rarely, arise in common practice, in ways that are empirically false. Science is thus able to drive a wedge between what 'solid' connotes to us and what we mistakenly believe that term to denote. Consider the following related example. Sweeping the table clean with a brush, we

are confronted with a paradigm case of the-absence-of-all-living-creatures, but after studying the surface of the table under a microscope, we find that what absence-of-all-living-creatures connotes to us is not in fact exemplified by the case at hand. We admit, and must admit, that we were mistaken.¹ In brief, then, when we say that the table is solid, what we mean by that locution is literally false, though after we have discovered our error, we shall continue, in vulgar practice, to speak in the vernacular, only now with an arrière pensée. The same words will now be meant in a different sense; they will now be meant tropologically, i.e. in a manner of speaking. It will be seen that I am eager to maintain the position that, as common sense presents us with the appearance, so it is science that presents us with the reality, of things.

If our account of Eddington's two tables in terms of the distinction between denotation and connotation has been less authoritative than one might wish, there is an alternative account that may carry rather more of the ring of truth with it. This second account relies on the distinction between empirical and ontological predicates; that is to say, between the order of practical discourse and the order of theoretical discourse. It is imperative to note here that the empirical predicates 'solid' and 'compact' are concepts of the mutatis mutandis variety. One has only to consider Tom Thumb in this connexion. Although Tom's official size is that of the human thumb, he is free to assume whatever magnitude he pleases. Sometimes he inflates himself to the size of a Cyclops, other times he dwarfs himself to the size of an elementry particle of sub-atomic physics. Tom Thumb's proves to be the prerogative Lebenswelt, for what passes with him as the order of practical discourse is very much akin, if not equivalent, to the objective order of scientific, ontological discourse with us. That is to say, Tom Thumb's vulgar language of daily life may be almost identical with the technical language of mathematical physics. In any case, how will Tom Thumb describe the table? Will he describe it as 'solid' and 'compact'?

¹ This mistake may have serious practical consequences. What caused the disease? Some living creature, large or small? Certainly not, for we have removed all cats and dogs, all bugs and insects, from the house. But perhaps there is a living creature too small for us to see with the naked eye. We did not consider that possibility, and yet in our ignorance we tacitly ruled it out of account.
while he is travelling merrily through it at the size of, say, a billionth or a trillionth of an inch? He may very well describe it as perforated like a sieve or a sponge or a swiss cheese. And if he describes it as 'perforated', as 'mainly empty space', will he be employing those English words in the precise sense in which we employ them? Yes, mutatis mutandis.

I am suggesting, then, that our description of the table as 'solid' and 'compact' may be rather on a par with our description of a block of ice as 'cold'. When we say that the block of ice is cold, when we say that the table is solid and compact, we may be presumed to be right—on the level of practical discourse. But inasmuch as the Hyperboreans, speaking flawless English, describe the block of ice as hot, inasmuch as Tom Thumb describes the table as 'perforated like a sieve', our own account cannot be accepted on the ontological level of a theoretical mode of discourse which is designed to command the assent of all rational beings. At the very best, then, our description of the table as 'solid' and 'compact' enjoys the denotation and connotation. This second distinction comes into play as follows. When I say that a block of ice is cold, I am perfectly aware that some other animal might possibly find it hot. But when I say that the table is solid and compact, I find it wildly incredible that some homunculus, be he of any diminutive size whatever, might succeed in slipping through the 'empty spaces' in the table. What empty spaces? That is to say, in vulgar life we believe that the table is solid and compact, absolutely speaking, without qualification, and not merely solid and compact in a mutatis mutandis kind of way. That vulgar conviction science proves to be false. This double account of Eddington's two tables applies equally to Russell's statement that 'as a matter of fact the two billiard balls do not touch at all'. For 'touching' is also a concept of the mutatis mutandis variety. Witness Tom Thumb's insistence that an empty space of a trillionth of an inch separates the two billiard balls at the putative moment of contact.

This digression has not been unimportant for our theme. If the mosaic of punctual geometry is to be defended as literally true of actual space, then the vulgar circle which we apprehend in any wheel must have a character quite different from what we suppose it to have. In fact, given what 'circle' connotes to us, we shall have to admit that the term does not denote anything at all, strictly speaking. The relation between the punctual circle and the vulgar circle will be strictly on a par with the relation between the scientific table and the vulgar table. Perhaps something like this is what Zeno was trying to say, namely that what 'motion' connotes to us, in fact denotes nothing at all. Assuming the existence of minims, we have seen that when my arm moves, for every punct of space that it advances, it must remain at rest for many consecutive puncts of time before it resumes its 'motion'. Is that jerky progress to be described with propriety as motion? Not certainly in the unqualified sense that we originally had in mind, but although obliged to retrench, we shall not withhold the term altogether. Even granting this jerky progress, the arm certainly 'moves', if only in a qualified sense of the word. The motion picture provides us with an attractive model of the relation that obtains between the subterranean spatio-temporal mosaic and our specious experience of continuity. On the one hand, we have the illusion of continuous motion; on the other, we have the reality of a discrete succession of independent slides.

If the doctrine of the minim cannot be falsified by any gross experience, can it be falsified at all, under any conditions? Yes, certainly—by our Zeno procedure. If it be objected that our Zeno procedure has been annulled by the paradox of the serrated continuum, then with the destruction of the continuum, the minim is re-affirmed. In either case, whether or not the Zeno procedure is allowed, the intelligibility of the minim is sustained. Intelligibility is one thing, however; a priori necessity is another. Can we seriously argue that on the strength of the paradox of the serrated continuum an a priori science of space, time, and motion is at last vouchsafed us, an a priori geometry accompanied by an a priori physics? An a priori geometry,
moreover, which is non-Euclidean! These conclusions I should be reluctant to accept, and yet they would seem to follow at once from the paradox of the serrated continuum. Is there perhaps some tertium quid that would enable us to eschew both the minim and the continuum at once?

**Part 3: More Paradoxes**

Although we have failed in our express object of dissolving the minim, our investigation of it has not been in vain. We have established in some detail that if the continuum is a paradoxical concept, then the minim is scarcely less so. Oddity for oddity, there is little to choose between them; and if the minim be supposed to have the advantage on the ground of being free of self-contradiction, it must never be forgotten that no certifiable contradiction has actually been disclosed by us, or by anyone else, in the continuum, not even in the serrated continuum. The two would seem to be very much on a par. Logically consistent in abstracto, each is highly perplexing in concreto, the one perhaps as much as the other.

Embarrassed by both, we are eager to break the bind which oppresses us, if not in one way, then perhaps in some other. If we cannot unravel the knot on its own terms, let us adopt the Alexandrian solution and simply sever it with a sword. Alas. If wishes were deeds, even fools would be wise. But there is another maxim that emboldens us to push ahead. Philosophers rush in where even fools hang back.

Waiving all metaphysics, what are the crude facts in regard to our Zeno procedure? I mean the very coarsest, shabbiest facts of all. In 1/2 minute let us cut a stick of wood in half. This will not be difficult provided that utopian accuracy is not required of us. In the next 1/4 minute let us cut one of these pieces again in half. I think that we may succeed also in this, with the same proviso. In the next 1/8 minute we shall cut, or attempt to cut, one of these latter pieces again in half. Even if we succeed here, it is a harsh, even brutal, fact that after very few cuts the minute will have elapsed, all of our efforts to discharge the Zeno procedure ad infinitum being put to nought. And not only will we inevitably fail in this programme, but any machine that we might build will be no more successful.

**Why this failure? What is its cause? Is it owing to the existence of an absolute minim (be it of space or time) that arrests our progress? Not at all. How, then, is our failure to be explained? There are many factors at work: our lack of adroitness, the bluntness of the knife, the splintering of the wood, &c. In a word, sludge. There is always sludge that frustrates our efforts to implement the ideal in practice, not only in the present case but on all occasions. Sludge! It is what the Yankee calls the 'sheer cussedness' of things, that unruly, refractory element in the world that, assuming the form of mechanical friction in one case, renders the perpetual motion machine impossible and that, assuming the form of Gödel's theorem in another, renders the consistency of mathematics incapable of proof. Sludge is everywhere, it cannot be escaped, not only in the physical world but even in the realm of pure ideas. Although this principle of recalcitrance is familiar to all, it has been generally believed to be merely contingent in nature, a brute fact that has no intelligible warrant. Is it perhaps otherwise? Given the paradoxes of the minim, on the one hand, and the paradoxes of the continuum, on the other, may we not interpret them conjointly as constituting an a priori proof for the existence of sludge? We are allowed, in principle, to execute the Zeno procedure to any finite number of steps whatever but at some point the sludge sets in. It may be noted that with the minim there is some fixed point beyond which we cannot proceed. But to avoid the serrated continuum no such limit is required. Any finite sequence is allowable; it is only the infinite sequence that must be ruled out. The minim is thus seen to be not only perplexing in itself but gratuitous for the purpose at hand, being overly protective. Given the minim, on all occasions we are allowed to proceed only up to the same fixed point. Not so with the sludge. There is no fixed point that arrests our progress; we are free, in principle, to proceed any finite number of steps that we may choose. This freedom is not afforded by the minim. How many divisions do you wish to execute in a minute? Name any finite number whatever. Unlike the minim, the sludge factor does not prohibit us from building a machine which will succeed in that object. For the sludge can be indefinitely, though not exhaustively, diminished: we have only to build machines of ever finer precision. It is only the
perfect machine, free of all sludge and capable of discharging the Zeno procedure \textit{ad infinitum}, that is ruled out on a priori grounds. The sludge factor would seem to have every advantage over the minim. For the minim has never recommended itself on its own merits; it has always been invoked out of despair with the continuum. But the sludge factor, in performing the same job as the minim, performs it more economically, leaving us a wider berth for our operations. Lest it be objected here that the option of being able to execute the Zeno procedure any finite number of steps whatever necessarily entails the further option of being able to execute it \textit{ad infinitum}, one may be reminded not only of the slippery slope but also of the fallacy of composition.

There is a second objection that may be almost as readily dismissed. In walking a distance of one mile in the common course of life, do we not first walk a distance of \(1/2\) mile, then a distance of \(1/4\) mile, \&c. \textit{ad infinitum}? But if the sludge factor is always operative in any effort to exhaust the Z-series, how can we ever succeed in reaching the end of the mile? Will not the sludge factor play into Zeno's hands? \textit{Distinguo}. On the one hand, there is the continuous continuum; on the other, the serrated continuum. They are not to be confused. In regard to the first we shall follow Aristotle. When we walk a distance of one mile there is no actual division of the distance into an infinite sequence of articulated sub-intervals. Any proposed division of the continuous continuum will be as arbitrary as any other; and none has any claim to being recognized as authoritative. In the case of the continuous continuum the sludge factor simply has no application. It is only in the case of the serrated continuum, in our effort to break up our journey into an infinite sequence of actual sub-intervals, with a pause separating each from its successor, that the sludge factor sets in. Which, after all, merely describes the familiar facts with which we are all acquainted, only now they are to be regarded as no mere empirical but rather as metaphysical necessities.

There is—a third objection to the metaphysics of sludge which is far more damaging than either of the others, so damaging in fact that the sludge factor will be shown to be altogether disqualified as the solution to our problem. This third objection is very simple. Recalling the serrated continuum as it is bodied forth \textit{in concreto}, it will be found that in none of our examples, with the exception of the god reciting the series of numbers in reverse, is there a project which has to be executed. Thus in the case of the infinite sequence of boards nothing was said as to its origin. We are to assume merely that it exists, perhaps as a freak of nature, a \textit{lusus naturae}. All of the boards may have come into being simultaneously, in complete independence of one another, their respective positions in the sequence being a cosmic accident. Or perhaps the first board came into being earlier today, the second yesterday, the third the day before, \&c. \textit{ad infinitum}, in which case a serrated continuum would have been in existence on each and every day in the past. Finally, it is intelligible to suppose that none of the boards has ever come into being at all; they may have all been always in existence. The paradox of the serrated continuum is thus seen to be independent of any project to be executed in serial succession. Indeed, the paradox is independent of our Zeno procedure itself, as a procedure. Even if one had never thought of the Zeno procedure, whether in the small or in the large, the paradox of the infinite sequence of boards would suffice of itself to shake one's confidence in the continuum. Viewed in this light, the sludge factor is seen to be irrelevant to the deeper problem. As to our fourth version of the paradox, in its application to time, it too may be divorced from any project to be executed. Let the peal of a gong be heard in the last half of a minute, a second peal in the preceding \(1/4\) minute, a third peal in the \(1/8\) minute before that, etc. \textit{ad infinitum}. Each peal may occur in complete independence of all the rest, another \textit{lusus naturae}.

The sludge factor having failed, must we return to the minim, to the a priori spatio-temporal mosaic with its attendant a priori physics? I do not think so. There are other possibilities. What if all the pages of the infinite book were scattered helter skelter throughout an infinite universe? Would we have any cause for embarrassment? None whatever. It is, then, not the infinite \(Z\)-series of pages, slabs and boards \textit{as such}, independent of all other considerations, that perplexes us; it is rather their specific arrangement, with its open end, that constitutes the enormity. Very well. Why not allow the infinite \(Z\)-series of pages, slabs and boards—merely adding the proviso that there is an \(a\)
priori necessity that metaphysically de bars them from everfall ing together to compose a small open-ended sequence?

This proposal cannot but leave us breathless: it is so intransi
gently, so uninhibitedly barbaric. Even if one does not share the settled antipathy toward the ontological a priori which dominates contemporary philosophy, even if one is generously prepared to consider each case on its own merits; granting all this sweet reasonableness, one may yet be enticed to protest against the indiscriminate use of the a priori—first in the minim, then the sludge factor, finally the scattering to the winds of the infinite Z-series—in which we have been shamelessly indulging ourselves. Is there to be no self-restraint in metaphysics? Must it always squander itself in a positive debauch of the intellect?

Sharing these cautionary sentiments, I wish now to make a final suggestion. It is one which I should have blushed to have uttered earlier, so much would it have smacked of parti pris. But now that we have had an opportunity to explore the alternatives, in all their extravagance, this final suggestion of mine might almost be received as the voice of moderation itself. Why not simply acquiesce in the serrated continuum, even with its open end? Not, to be sure, with all the pomp of a priori-first necessity but merely as an ontological—for it is certainly a logical—possibility. We have seen that it is not the infinitude of the infinite open-ended Z-series that alarms us but rather its uncanny open-endedness. Might there not be some way of coming to terms with this open-endedness? If we could negotiate an accommodation here, we should be free to dispense with any kind of ad hoc apriorism expressly invoked to plug up the gap. We shall not deny the possibility of any of these—the minim, the sludge factor or even the scattering to the winds of the infinite Z-series; but we shall be released from the obligation to insist on their necessity. Perhaps in our pluriverse there may be one type of Space which is characterized by the minim (time as well as space may be punctuated here), another by the sludge factor accompanied by the scattering principle, and a third type by the unrestricted continuum. In these Spaces of the third type both the metaphysical rocket and the metaphysical computer will be operating in all their bravura.

In eschewing the a priori, this prospect may be felt to be especially attractive. But it presupposes the intelligibility of the open end of the serrated continuum. Can it be negotiated? Standing ten feet away from the sequence of boards, let us undertake to stare into the open end. What do we see? If each board be opaque ex hypothesi, then any board in the sequence, say the nth board, will be blocked from our view by the n+1th board. All of the boards will thus be blocked from our view, and we shall be seeing none of them. This is very curious—a body being blocked from view by something which we cannot see! This something being a sequence of ordinary material bodies each of which, in isolation, is perfectly visible. Here, then, is a method for making a man totally invisible: he has merely to be sheathed or incapsulated on all sides, from head to toe, in an infinite open-ended sequence of opaque capsules all of which, taken together, need not exceed one inch in thickness. We may style this device the shield of invisibility. Is this shield of invisibility to be credited as intelligible? This a priori procedure for rendering any material body invisible by means of an infinite sequence of ordinary material bodies all of which, in the result, prove also to be invisible?

It is important to realize that the presence of the invisible man is always easy to detect. His very presence blocks from our view everything that lies behind him, for each of his sheaths, if the whole sequence is to be effective, must block out all that lies behind it, and the total blockage will assume the shape of the man and will move about as he moves. Then we will see the man after all? Strictly speaking, no; but in a way, yes. Let me explain the nature of this unexampled visual experience. What is it like to stare into the open end of the sequence of boards? What is it like to gaze up into an absolutely clear, cloudless sky on a bright day? Our powers of vision being limited, we can only see so far and no further. There is perhaps the ‘illusion’ of an over-arching dome at the terminus of our vision. Imagine now a man who, for one reason or another, can only see ten feet ahead of him, though his vision over that distance is no less perspicuous than our own. (It should be understood here that there is no physical barrier ten feet removed from the man which truncates his field of vision.) That will be our kind of vision, standing ten feet away from the open end of the boards. With one important difference. This peculiar type of vision (it is perhaps not so peculiar, being exactly like
our own: it is merely somewhat more contracted in scope) will apply only to our glances directed at the open end. The rest of our field of vision, above and to the right and left of the boards will be extended its normal range. On any occasion, then, on which we are assured that our vision is otherwise unimpaired, we shall know at once when we are in the presence of one of these open-ended enormities, be it the invisible man, the sequence of boards, the infinite book, or the infinite pile of slabs. The shield of invisibility proves to be by no means as formidable as one might have supposed.

What happens now if we undertake to crash into the open end of the boards (or to pounce bodily upon the invisible man)? There are two cases that must not be confused. First, there is the case in which each board is so adamant that if we were to crash into it in isolation we should be effectively repelled. Second, there is the case in which each board beyond the thousandth, say, is so flimsy that, being taken in isolation, it would be easy for us to crash through it. In regard to the first case, seeing that each board is effectively shielded by its successor, it is evident that, although we shall be able to approach indefinitely close to the open end, there is a point beyond which we cannot pass: the ne plus ultra. Not that we shall be repelled by contact with any one of the boards. No. The infinite sequence logically entails what we may describe as a field of force which shuts us out from further advance. In invoking here the concept of a field of force, we mean no more than that there is a point beyond which we are assured that our vision is otherwise unimpaired, above and to the right and left of the boards.

The shield of invisibility proves to be by no means as formidable as one might have supposed.

The paradox of the before-effect may be generalized over a whole range of cases. A man is shot through the heart during the last half of a minute by A. B shoots him through the heart during the preceding 1/4 minute, C during the 1/8 minute before that, &c. ad infinitum. Assuming that each shot kills instantly (if the man were alive), the man must be already dead before each shot. Thus he cannot be said to have died of a bullet wound. Here, again, the infinite sequence logically entails a before-effect. Consider now the following even more radical version of this paradox. A man decides to walk one mile from A to B. A god waits in readiness to throw up a wall blocking the man’s further advance when the man has travelled 1/2 mile. A second god (unknown to the first) waits in readiness to throw up a wall blocking the man’s further advance when the man has travelled 1/4 mile. A third god . . . &c. ad infinitum. It is clear that this infinite sequence of mere intentions (assuming the contrary-to-fact conditional that each god would succeed in executing his intention if given the opportunity) logically entails the consequence that the man will be arrested at point A; he will not be able to pass beyond it, even though
not a single wall will in fact be thrown down in his path. The before-effect here will be described by the man as a strange field of force blocking his passage forward.\footnote{The details of the gods’ intentions are not unimportant. The first god vows: if and when the man travels \(\frac{1}{2}\) mile, I will throw down a wall \(\frac{1}{8}\) mile ahead of him. The second god vows: if and when the man travels \(\frac{1}{4}\) mile, I will throw down a wall \(\frac{1}{16}\) mile ahead of him, &c. \textit{ad infinitum.}}

Actually, there is no before-effect in this last example. The effect is the man’s being arrested at A; the cause is the sum-total of the gods’ intentions. All of these intentions temporally \textit{precede} the effect. Even in the other cases the before-effect may be said to precede the cause only by way of hyperbole. Lest we commit the fallacy of composition, it must be emphasized that, though we are in a state of deafness prior to \textit{each} peal, our hearing remains unimpaired at any assigned instant prior to \textit{all} of them. At twelve o’clock sharp, certainly, we were not yet in a state of deafness, though at any instant after twelve (waiving the infinitesimal tail-end) we are, and have been, deaf. As to the dead man, although he died of any \textit{single} bullet wound, his death was certainly \textit{caused} by the infinite fusillade of shots. Here, again, although he is already dead prior to each shot, he remains alive at any assigned instant which is prior to them all. Thus he cannot be said to have died at any moment of time whatever! Nor can we be said to have been struck deaf at any instant of time.

In regard to the paradox of the gods, the oddity here may be somewhat diminished if we replace each god by a law of nature. It is not, after all, the combined intentions of the gods \textit{as such} which block the man’s progress at A. It is rather the following sum-total of hypothetical facts, namely (1) if the man travels \(\frac{1}{2}\) mile beyond A, then he will be blocked from further progress, (2) if the man travels \(\frac{1}{4}\) mile beyond A, then he will be blocked from further progress, (3) ... \&c. \textit{ad infinitum.} It is not surprising that this infinite sequence of contrary-to-fact conditionals should logically entail the categorical fact of the man’s being arrested at A. For the cause of this arrest is simply the man’s encounter with a field of force, and this field of force is simply the physical equivalent of an omnibus law of nature which is compounded out of an infinite sequence of contrary-to-fact conditionals.

How successful are these placebos? Have we been reconciled to the open end of the serrated continuum? Alas. We ourselves would seem to be caught in a fusillade, albeit a fusillade of paradoxes. (We can only trust that it is not infinite.) If it is the case that none of the peals has been heard by us, then how gratuitous to suppose that we have been struck deaf. But if we have retained our hearing unimpaired, then we must have heard all the peals. But if we have heard all the peals, then we must have been struck deaf by each in turn, and hence we should not have heard any of them. Not having heard any of the peals, our hearing must have been retained unimpaired, in which case we should have heard all the peals. ... Once again we are caught in the circle of paradox the emblem of which is a snake with its own tail in its mouth.

Part 4: Mathematics and Ontology

Back to the minim? Or at least to the sludge factor and the scattering principle? If surrender we must, I do not think that we need surrender at this point; and I may add (though this is a purely personal conviction) that, however fecund the continuum (and, more generally, the infinite as such) may be in engendering paradox, it is no less vigorous in its resources for protecting itself against its own brood. It is the extraordinary fecundity of these concepts—that of the infinite in general and that of the continuum in particular—which is responsible for both the power and the glory of mathematics.\footnote{Cf. De Caezo, I, § 5, 271 b 13-15. ... a principle is great rather in power than in extent; hence that which was small at the start turns out a giant at the end. Now the conception of the infinite possesses this power of principles, and indeed in the sphere of quantity possesses it in a higher degree than any other conception.} Why is mathematics the only \textit{a priori} science we have? Of all of our concepts why are only some very few (primarily those of number and shape) capable of being systematically unpacked to yield an infinity of diverse theorems? If the output is infinitely rich, the input must be equally so; and in fact the infinite is found to be there at the start, in the iterative character of the natural, and the continuous character of the real, number concept. But if mathematics may be understood
at its core as the formal science of the infinite, it is also the formal science of the infinite, and it is here that tension arises. For in his effort to subject the infinite to a \( \omega \), the mathematician must be preoccupied not only with his subject-matter but also with his method. It is the very fecundity of the infinite which compels the mathematician to insist on the highest standards of rigour if he is not to be overwhelmed by all its lushness.

We have had occasion to note that the early pre-Socratics, in their preoccupation with natural philosophy (they were radical materialists), took the actual infinite in their stride\(^1\); but among the later pre-Socratics, above all with Zeno and Pythagoras, the infinite came to sight for the first time as a problem. Greek philosophy after Zeno and Pythagoras was to be characterized in large measure by a panic-stricken flight from the infinite; and if it be remembered that mathematics as a theoretical science was but in its infancy at this time, it is not surprising that the fateful decision was made to protect the infant science from the ontological problematic. Henceforth the sting was to be taken out of the infinite; it was to be in resourceful accommodation of an otherwise almost unbearably lush concept to the demands of rigour.\(^2\) That even in this reduced form the infinite should prove to be so enormously productive, as the history of mathematics (which has been largely finitist) demonstrates, is but a sign of the prodigious vitality latent in the unrestricted concept.

Actually, Greek mathematics was never able to divorce itself from the ontological problematic altogether. Aristotle himself says, ‘Admit... the existence of a minimum magnitude, and you will find that the minimum, small as it is, causes the greatest truths of mathematics to totter’.\(^1\) Classical geometry, then, presupposed the intelligibility at least of the potential continuum. The devotees of the minim, in their denial of the ‘bisection’ postulate, were unwilling to admit even that much. If the doctrine of the minim may be described as a hard finitism, then the finitism of Aristotle and Brouwer, of mathematicians generally, is merely a soft finitism.\(^2\) Aristotle may have succeeded in releasing the Greek mathematician from concern with the infinite qua actual infinite, but he could not release him from concern with the infinite altogether. Precisely by adopting a soft finitism, he was obliged to set his face against the hard. Although I have argued, contrary to Aristotle, that the geometry of the minim is a legitimate branch of mathematics, it is evident that such a geometry could never be more than a widely peripheral off-shoot of the science. It thus remains true to say that at its core mathematics is the science of the infinite.

I do not doubt that pinning mathematics down to the potential infinite is very much the safer course, but I fail to see that such caution precludes adventurous excursions into the actual infinite. If the mathematician feels free to investigate non-Euclidean as well as Euclidean geometries, if he is prepared (though not without some misgivings) to devise non-construcive proofs, why not also flights into hyper-mathematics? The parallel, admittedly, is not exact. The non-Euclidean geometries came to be accepted only after they were proved to be logically consistent. No such proof is available in the case of hyper-mathematics or the actual infinite generally. Since Gödel, however, the consistency of even finitist mathematics has been seen to resist proof, and in this one respect, at least, the actual infinite is on a par with the potential. In this connexion it may be recalled that even the proof of non-Euclidean consistency is only conditional: it presupposes the consistency of the natural (not to mention the real) number system and, with Gödel, a proof of that in finitist terms

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\(^1\) Ibid. 271b 1–10. ‘Is there an infinite body, as the majority of the ancient philosophers thought, or is this an impossibility? The decision of this question, either way, is not unimportant, but rather all-important, to our search for the truth. It is this problem which has practically always been the source of the differences of those who have written about nature as a whole. So it has been and so it must be. . . .’

\(^2\) Aristotle explicitly lends his support to this programme. Of the mathematicians he says: ‘In point of fact they do not need the [actual] infinite and do not use it. . . . Hence, for the purposes of proof, it will make no difference to them to have such an infinite.’ (Physics, III, § 7, 207\(a\) 30–34). For the purposes of proof! That is to say, the mathematician is entitled to pursue his science in relative independence of the ontological issue.

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\(^1\) De Caelo, I, 271b 10–13.

\(^2\) Proto-mathematics is another kind of hard finitism, though the Swahilis certainly do not regard themselves as finitists. They freely acknowledge the actual infinite, but only by assigning to it what we can only regard as a substandard interpretation.
is rendered unlikely. We, for our part, have shown that the consistency (or inconsistency) of the natural number system can be demonstrated through hyper-mathematics. Unfortunately, such a proof is reserved for the gods alone, and for mere mortals it may doubtless be felt to be a case of obscurum per obscurius. In addition, there is this further difficulty. Any proof of the consistency of our standard mathematics, above all of our standard concept of the infinite (let it be but the potential infinite), presupposes an adequate justification of our passage from the proto-concept of the infinite to the standard concept, and we have seen that any justification of that passage which appeals to our standard logic begs the very question at issue. Thus even if the consistency of our standard mathematics were in fact ‘proved’ (which means, proved by our standard logic), it would remain ultimately problematical in the absence of a specifically ontological approach.

Mathematics is perhaps to be understood less as the science of the infinite than as a game parasitic upon it, parasitic upon the ontological problematic. Formalism and all its variants represents but one side of mathematics, the emphasis on rigour. There is another side, the fecundity of the subject-matter itself. We cannot have too much rigour (one might argue), but lest that rigour degenerate into a mere rigor mortis it must be a rigour applied to a vital substance. Here—alas—is a serious problem, for a rigor vitæ would seem to be a contradiction in terms. Hence the demand for rigour, pushed to its logical extreme, might well entail the radical impoverishment of the underlying subject-matter. Nor may it be forgotten that the demand for rigour is itself problematical; it presupposes our standard logic which, as we have seen in our study of the Swahilis, is itself dubious from the standpoint of proto-logic. The demand for rigour makes sense only after the ontological approach to number has been adopted, and it is only at the ontological level of cognition that our standard logic is free to operate.

That mathematical rigour when ruthlessly pursued for its own sake independent of all ontological considerations, is ever liable to degenerate into rigor mortis, is strikingly brought home to us by the Löwenheim-Skolem theorem. Second only to Gödel’s in significance, the Löwenheim-Skolem theorem raises in a peculiarly trenchant form the whole issue of rigour versus ontology in the philosophy of mathematics. The theorem itself has already succeeded in bringing to a head the perennial conflict between those who bear down on the side of rigour, on the one hand, and those who bear down on the side of insight, on the other. Speaking for rigour, Skolem himself interprets the theorem as proving that ‘the absolutely non-denumerable is only a fiction.’ Speaking for insight and insisting that the real numbers are indeed ‘absolutely non-denumerable’, Myhill interprets the theorem as positively ‘proclaiming’ our need never to forget completely our intuitions. Itself a meta-theorem concerning the interpretation of formal systems, there is an appropriate irony, and even poetic justice, in the fact that the Löwenheim-Skolem theorem should lend itself to such widely divergent interpretations of its final import. Laying aside all such interpretations, what precisely does the Löwenheim-Skolem theorem entitle us to assert? Given any formal logico-axiomatic system of the predicate calculus which is capable of being assigned a model in a non-denumerable domain, that same system is also capable of being assigned a model in a denumerable domain. In other words, any formal system which has been designed to express the concept of the non-denumerable, may be radically re-interpreted so that the non-denumerable simply drops out. There are two obvious moves that one might make at this point. With Myhill, one may argue as follows. Seeing that the concept of the non-denumerable is incapable of being unambiguously formalized, it is evident that there is more to mathematics that what can be formalized. Alternatively, one may choose, with Skolem, to argue as follows. Seeing that the concept of the non-denumerable is incapable of being unambiguously formalized, it is evident that the ‘absolutely non-denumerable is only a fiction’.

So long as the dispute remains on this level, a contest between the partisans of the formal system and the devotees of the

1 T. Skolem, Sur la portée du théorème de Löwenheim-Skolem (Les Entretiens de Zurich, 1941), pp. 39-49.
informal insight, I doubt if there is much hope for an accommodation. What is required is an ontological grasp of the issue, specifically an ontological grasp of Cantor’s diagonal theorem. What is the ontological cash-value of that theorem? Let us assume, first, that all of the natural numbers actually exist written out somewhere in an infinite sequence. The physical origin of this sequence need not concern us here: we may suppose the sequence to be some lusus naturae. But it is important that the denumerably infinite be understood as an actual, not merely a potential, infinite. Granting this actual infinite sequence of all the natural numbers, is it possible that all of the rational numbers might likewise exist in an infinite sequence, in one-to-one correspondence with all of the natural numbers? Yes. What about all of the algebraic numbers? Again, yes. What about all of the real numbers, specifically all of the irrational numbers? The answer here is: no. Absolutely no. The Löwenheim-Skolem theorem fails in any way to shake this ontological interpretation of Cantor’s theorem. Merely because a denumerable domain will serve as a model for any formal system which is satisfied in a non-denumerable domain, it does not follow that ‘the absolutely non-denumerable is only a fiction’. What does follow from the Löwenheim-Skolem theorem is that the ontological import of the real numbers cannot be adequately expressed in any formal system.

More precisely, the theorem entails that the absolute non-denumerability of the real numbers cannot be expressed in any formal system which is finitist in character. Finitism and formalism are not to be confused. It is quite possible to be a finitist without being a formalist).1 Although the Löwenheim-Skolem theorem is to be understood less a blow against formalism than as a blow against finitism, the formalist account of mathematics has been decisively refuted—oddly enough—by Skolem himself, in his paper of 1933 entitled ‘On the Impossibility of Completely [i.e. uniquely] Characterizing the Number Series by Means of a Finite System of Axioms’. Skolem proves that, just as the five Peano postulates fail to provide a unique characterization of the natural numbers, so, too, any finite list of axioms whatever which are satisfied in the domain of the natural numbers can be shown also to be satisfied in some other domain. A triumph of formalist analysis, Skolem’s paper spells the downfall of formalism.

For almost a hundred years modern mathematics has prided itself on the abstract approach. Any kind of specific content, any kind of ontological import, has been felt to be sub-mathematical or extra-mathematical in nature. But any purely abstract approach, as Skolem has shown, not only fails to define the concept of the non-denumerable (infinite theorems being waived) but equally fails to define the elementary

1 But if one of the formulas asserts that the class of all real numbers is non-denumerable, how can that formula be satisfied in a denumerable domain? This is known as Skolem’s paradox. Skolem answers that the formula must be assigned a sub-standard interpretation. It is to be understood as asserting merely that the class of all real numbers constructible within the system is non-denumerable relative to the system, i.e. there can be no procedure within the system for denumerating them on pain of the Richard paradox. Non-denumerability for Skolem is always non-denumerability relative to the given resources of a particular system.
substratum of all mathematics—the natural numbers. What is the source of this failure? Mathematics is not merely a game, it is not merely a system of rules like chess or bridge: 'it is the use outside mathematics, and so the meaning of the signs, that makes the sign-game into mathematics ... it is essential to mathematics that its signs are also employed in multi.' Any effort to characterize the natural numbers exclusively by their formal or structural properties, quite apart from their teleological import, their use in the world at large both for the vulgar purposes of life and for the theoretical purposes of natural science, any such antiseptic effort simply misses the point of these numbers. A child ignorant of even the simplest procedures of counting might be taught how to 'prove' all the theorems in arithmetic (e.g. 2 + 2 = 4) without any awareness that he was proving anything. He might simply be playing a game, deriving one set of (to him) meaningless marks from another in accordance with certain assigned rules. Although the child's game may be interpreted from the outside as the science of number, it is to be expected that other interpretations might be equally eligible. Skolem's paper merely confirms, on the technical level, a suspicion that we had every right to entertain in advance, namely that no purely formal or abstract account of the natural numbers could succeed in characterizing them uniquely, seeing that their whole point lies in their use outside of pure mathematics.

Less obvious, perhaps, is the fact that the concept of the non-denumerable, even of the absolutely non-denumerable, also has a use outside of pure mathematics, but much of this book is precisely devoted to an elaboration of the wide diversity of ontological uses, in the large as in the small, which the concept of the non-denumerable subserves. In insisting that mathematics, pure as well as applied, is incurably oriented toward ontology, I am not demanding that the working mathematician be committed to some specific dogmatism. Quite the contrary. There are many mansions: there is Euclidean geometry, there is non-Euclidean geometry, there is even the non-Euclidean geometry of the minim, there is constructivist mathematics, there is non-constructivist mathematics, there is hyper-

Contrary to received opinion, snapping the connexion between ontology and mathematics is not the way to enlarge the scope of mathematical research. Far from liberating mathematics from the shackles of reality, formalism succeeds only in impoverishing its substance and contracting its horizons. If mathematics be understood as a game which is parasitic on the ontological problematic of the infinite, then we are invited to range over the most diversified fields of inquiry. Had the mathematician of antiquity undertaken, if only as a sideline, to elaborate the non-Euclidean geometry of the minim (and we have seen that ontologically the issue of the minim versus the continuum was very much an open question), the modern prejudice against any kind of ontological orientation in pure mathematics might never have arisen. The source of the rift between ontology and mathematics (it has been even more disastrous for the former than for the latter) thus lies in anti-quity itself when, in shunning the minim, the classical mathematician shunned the very ontological problematic which underlies his science. Later, oddly enough, it came to be felt that the classical mathematician was too much under the spell of a doctrinaire ontology (grounded in 'self-evident truths' &c.) when actually the reverse was the case, for if Greek mathematics was unduly limited by modern standards, it was limited precisely because of its studied dread of the ontological problematic.

In undertaking to explore the non-Euclidean geometry of the minim, the Greek mathematician could scarcely have resisted the further temptation to cultivate, again perhaps only as a sideline, the mathematics of the actual infinite, for here also the ontological issue of the actual versus the potential infinite was known to be fiercely controversial. I have no doubt that the Greek mathematician, in pinning himself down to the potential infinite, felt himself to be steering a wise middle course between the rock of the minim, on the one hand, and the whirlpool of the actual infinite, on the other. However much one may applaud the prudence of that decision, in the infancy of a new science, one cannot but deplore the fateful consequences, taking the long view. We now know, and I
should suppose that it was also known in antiquity, that Aristotle was guilty of exaggeration when he insisted that 'for the purposes of proof it will make no difference to [the mathematicians] to have ... [the actual] infinite'. There is an element of truth here. Much, perhaps most, of mathematics can dispense with the actual infinite, but many of the more splendid mansions of the subject required it as their foundation.

Ah, but what of the fusillade of paradoxes that threaten to undermine it? Have we any guarantee that the fecundity of the actual infinite is not to be attributed to a latent contradiction? No wonder, then, that the concept should be so fertile: from a contradiction anything whatever, even the creation of the world, can be logically deduced, if not ex nihilo, then *ex contradictione*. I know of no such guarantee in the strict sense, but we have what is almost as good, a quasi-guarantee. If a contradiction could be detected in the bare idea of the actual infinite as such, then it would constitute an *a priori* proof that there are somewhere two bodies A and B such that there is nowhere a body larger than A nor smaller than B. However wildly metaphysical my inquiries may seem to be, I have never dared suppose that an *a priori* proof of such power might be available. A contradiction being detected in the idea of the actual infinite, we should also enjoy an *a priori* proof that the world must have had a beginning in time. For though an infinite past may not count as an *actual* infinite in the strict sense, granting an infinite past, it is conceivable that on each day in the past a different star might have come into being, in which case (assuming that none, or only a finite number, of the stars were to perish in the interim) there would now exist at the present time an actual infinite number of stars—no one of them, incidentally, having been in existence for more than a finite length of time. Furthermore, an actual infinite number of stars would also have been in existence on each day in the past. I wish to be entirely free of dogmatism in my insistence that almost certainly there is no demonstrable contradiction latent in the bare idea of the actual infinite as such. For in any synoptic view of the ontological problematic one cannot but be open to that possibility among others. The fecundity of the actual infinite, in the small as in the large, is seen not least of all in this, namely that if the concept were ever found to be self-contradictory, in the very act of its destruction, out of its own ashes, so to speak, an *a priori* metaphysics would issue forth.

Quite apart from the general warrant that I have offered as grounds for assuming, if only as a reasonable hypothesis, that the actual infinite is internally consistent, it is imperative that each of the paradoxes—are they inexhaustible?—that rise up to bedevil it, be dealt with on its own terms. What, for example, are we to say in regard to the paradox of the peals? Is it not gratuitous to suppose that, although we have not heard a single peal, we have nevertheless been struck deaf? Gratuitous? No. How can our conclusion be gratuitous if its denial leads to a contradiction? This is the short way of dispatching the paradox and, within its limits, it is altogether cogent. But the paradox is worthy of being more generously treated. Properly understood, it discloses an important fact, not that our conclusion is gratuitous, but that it fails to be coercive. That is to say, there are other possibilities that are equally eligible. To mention only one, although each peal may be deafening in isolation, have we any right to assume, from that fact alone, that in these anomalous circumstances (each peal being preceded by an infinite number of peals within a finite duration of time), it retains its deafening character? Why not say that, given the unprecedented circumstances, all of the peals are subdued in loudness or, if not all, then that some infinite sub-set of successive peals crowding the open end are partly muted? In which case there would be a *first* deafening peal. There is, of course, no intrinsic merit to this suggestion. It is quite as arbitrary and quite as factitious as our earlier conclusion, only somewhat more unlovely. But though both suggestions may be equally factitious, if we were to succeed in compiling a comprehensive schedule of *all* the alternatives, then that comprehensive schedule would not be arbitrary in the least: it would be necessary.¹ The point is this. Given merely the pure geometry (as we may style it) of the serrated continuum, it is not possible

¹ Here is another option. Although no one may be allowed to hear more than one of the deafening peals, all of the peals may actually be heard—by an infinite number of different persons, there being for each person (let us say) a different auditory threshold requiring a minimum thickness to a sound before it can be heard by that person.
logically to deduce any unique physics required to flesh it out \textit{in concreto}. The geometry is, in part at least, physically indeterminate. But it is not so indeterminate as not to entail that the physical consequences (whatever they might be) of an actual encounter with the open end, must be desperately bizarre.

How does this discussion of squalid alternatives apply specifically to the sequence of boards? What is the counterpart here of the paradox of the peals? This: if none of the boards is actually seen, then how can it actually block from view what lies behind it? But if none of the boards serves as an actual block, then all of the boards must be seen. But, each board being opaque, at most one board can be seen. Thus we are caught in a contradiction. Here, again, there are many options available to us. Perhaps we actually see all the boards! What would such a monstrous field of vision be like? Think of the cosmological octopus with a tentacle extended between each board and its successor. Having eyes along each tentacle, the octopus is actually engaged in seeing all the boards at once. Absurd? Yes. But it only confirms what we have always known, namely that the actual infinite obeys a peculiar \( \lambda \omega \theta \sigma \) that is radically and uncompromisingly different from the familiar \( \lambda \omega \theta \sigma \) of the finite. I am suggesting, then, that even without the tentacles, standing ten feet from the open end of the sequence, we may be suddenly endowed with the omnibus field of vision of the octopus. It may be added here that if one holds a behaviouristic theory of perception, he will find it even easier to come to terms with the open end of the sequence.

\textit{Part 5: Open and Closed Continua}

Our account of the serrated continuum is in large measure applicable to the continuous continuum as well, at any rate to the open-ended continuous continuum. For there are two types of continua, the open and the closed. No closed continuum can be exhaustively understood in terms of the Z-series of rational values. What distinguishes any familiar solid from the open-ended pile of slabs is, not so much that the latter is serrated (for we may suppose all the schisms to be smoothed out), but rather that the former possesses, whereas the latter lacks, a top surface. Consider any one-inch stick of wood on the right-hand butt-end of which a Z has been inscribed (see Fig. 2). Whether or not the stick has been serrated, in either case the Z cannot be said to belong to any one of the following pieces into which the stick either actually is or potentially may be divided, namely (starting from the far left) \( 1/2 \) inch, \( 1/4 \) inch, \( 1/8 \) inch, &c. \textit{ad infinitum}. Yet the Z certainly belongs to part of the stick: it belongs to the infinitesimal butt-end. It is only the open-ended continuum which may be exhaustively defined by the Z-series of rational values. The closed continuum contains, in addition to the Z-series of rational values, an infinitesimal tail-end.

Hitherto the infinitesimal has come to our attention in two different ways.\(^1\) First, we saw that if a stick be successively divided in halves, quarters, eighths, &c. \textit{ad infinitum} by our Zeno procedure, at the end of the minute we should have decomposed it into an infinite number of infinitesimal chips. Second, we found that in travelling in our metaphorical rocket an infinite distance from the earth in Space § 1 to the meadow in Space § 2, all of our passage in Space § 2 must occur during the infinitesimal tail-end of the temporal Z-series. Now, in contrasting the open and the closed continuum, we find that the infinitesimal comes to sight in a third way. Quite apart from any procedure to be executed, it should be evident that the Z lies on a planum

\(^1\) Cf. A. Fraenkel, \textit{Abstract Set Theory} (Amsterdam, 1953), pp. 163–5. "Philosophers sometimes content themselves with just "posing" a concept, and this act in itself will but rarely entail contradiction—certainly not in the case of infinitesimals. The difficulty begins at the moment when one starts doing something with the concept, in our case operating with infinitesimals and applying them to scientific problems. \textit{Now in this respect the infinitesimal has on the whole proved a failure}. The types of infinitely small magnitudes conceived hitherto have not produced a really interesting arithmetic, nor yielded useful applications. \ldots \text{So far infinitely small magnitude has essentially continued to be conceived in the sense of potential and not of actual infinity, in accordance with Gauss's statement}. \ldots \text{To be sure, we cannot exclude the possibility that a suitable arithmetical foundation of infinitesimals in a way not attempted so far, might open a new access to calculus or to other fields of analysis. In the present state of mathematical science, however, this prospect seems extremely unlikely." (Italics Fraenkel's.)
of the stick which does not belong to the 1/2 inch piece on the far left, nor to the 1/4 inch piece following it, nor to the 1/8 inch piece following that, &c. ad infinitum. We cannot fail to raise the following question. Is it at all possible to sever the infinitesimal tail from the rest of the stick? The answer is, yes. Let us first cut the stick in half and inscribe the number 1 on the right-hand butt-end of the left-hand piece (see Fig. 3). Now let us throw the

![Fig. 3](image)

left-hand piece into a box, leaving the right-hand piece on the table. Next, we shall cut this latter piece again in half, inscribing the number 2 on the right-hand butt-end of the new left-hand piece (see Fig. 4). Again, we shall throw this number 2 piece into the box, to join the first. Once more we shall cut the Z piece in half, &c. ad infinitum. At the end of the minute we shall have an infinite number of pieces in the box, each being of some rational length, and lying on the table there will be a single infinitesimal chip with the Z plainly marked on one side. It is interesting to note that, though each of the rational pieces resulted from a specifiable cut, there was no single cut which severed the infinitesimal chip from the rest of the stick. We may indeed speak of an imaginary cut here, though this can only be a tropological way of saying that there was no proper cut at all. The infinite sequence of rational cuts, in and of itself, logically entails the imaginary cut.

Let us now undertake (as far as possible) to reconstitute the original stick, omitting the infinitesimal chip. First, we shall glue the number 2 piece to the number 1 piece. These two pieces on being glued together will extend \(1/2 + 1/4\) inches plus the glue. We shall see to it that the glue adds no more than 1/2 inch to the length of the whole. At the maximum, then, the two pieces plus the glue will extend \(1/2 + 1/2 + 1/4\) inches. Next, we shall glue the number 3 piece to the number 2 piece. The three pieces joined together (neglecting the glue) will extend \(1/2 + 1/4 + 1/8\) inches. In this latter case we shall see to it that the new glue adds no more than 1/4 inch to the whole. At the maximum, then, these three pieces plus the glue will extend \((1/2 + 1/2 + 1/4) + 1/4 + 1/8\) inches. At the end of the minute, having prosecuted our procedure ad infinitum, we shall have, if not quite the old stick, at any rate a new stick extending not more than two inches in length. This new stick will not only be a serrated continuum, it will also lack a right-hand butt-end. For even before undertaking to glue the rational pieces together, we took special pains to lock the infinitesimal chip in an iron vault (throwing away the key) lest we yield to temptation, during the infinitesimal tail-end of the temporal Z-series, to clap the chip on to the open end.

Viewed in abstracto, the open-ended stick would seem perfectly intelligible; but we have seen that it proves to be highly perplexing when we attempt to come to grips with it in concreto. Grasping the stick by its closed end, let us undertake to drive the open end on to some hard surface, say the top of the table. What will happen? Who can say? All kinds of things might happen. Thus the stick might get stuck fast at its open end, as if by suction, to the top of the table. That would be unfortunate. For we have other plans for the open end: we wish to perform another perplexing experiment, namely to undertake to restore the infinitesimal chip to the open end of the stick, thereby to close it. Ah, but how is that to be done? Having broken open the vault and recovered the chip, what is there on the stick for us to glue the chip to? Glue will not do the job. Although it has been easy enough (metaphysically speaking) to sever the infinitesimal butt-end from the stick, it is by no means so easy to reinstate it in situ. Perhaps there is actually no way at all of capping it on to the open end. That, certainly is one physical possibility. There are others, however, Thus there is always the possibility of the suction effect. Perhaps if we undertake to drive the open end of the stick on to the infinitesimal chip, the stick will simply pick it up, as it were, off the table. In this connexion the suction effect may be mentioned as a peculiarly eligible possibility for what happens when we undertake to walk on into the open end of the sequence of boards. May we not simply be
sucked into the open end? Not that our chest need actually touch any one of the boards. We may simply find ourselves stuck to the open end in such a way that an infinite number of boards are interposed between our chest and any one of the boards in the sequence. After all, something like this actually obtains in the case of the original untampered-with stick. Each of the rational pieces (let them all be understood as only potentially articulated) is separated from the infinitesimal butt-end by an infinite number of pieces. Hence it is only right (we might say) that the severed chip should get readily stuck to the open end of the reconstituted stick at the moment of ‘contact’. ‘Contact’ may be defined here as that maximum proximity to the open end which is characterized by the absence of any small rational distance to the left of the chip that is not occupied by an infinite number of sub-sections of the stick. (What would happen if the open ends of two open-ended sticks were to come in ‘contact’ with each other?)

We are now in a position to correct an error into which we lapsed earlier. We argued, not implausibly, that since there is no actual point of separation (be it rational or irrational) between the infinitesimal tail of any closed continuum and the main body, therefore it must be impossible to lop off the tail. We now know that, though our premise was sound (at best there is only an imaginary point dividing the tail from the rest of the closed continuum), our conclusion was false: the tail is eliminable. The imaginary cut (not to be confused with the altogether different imaginary point of standard mathematics) is thus seen to be much more than a mere façon de parler: it signifies the ontological possibility of disengaging the infinitesimal tail from the main body of the closed continuum. This detachability of the infinitesimal tail is of some importance for our metaphysical rocket. Undertaking to travel an infinite distance from the earth, what if we were to sustain our flight only during the Z-series of rational temporal intervals? What then? We should succeed, certainly, in travelling an infinite distance but all of it would lie in our own infinite Space. Might we not simply vanish, rocket and all, at the imaginary point separating Space §1 from Space §2? What is the ontological cash-value of this suggestion? This: at any small rational interval of time prior to the end of the minute, we would be found at some finite distance from the earth, and at any infinitesimal interval of time prior to the end of the minute (as also at any interval of time whatever after the minute), we should simply not be in existence at all. Perishing at an imaginary point of time and at an imaginary point of space, we should nonetheless have travelled an actual infinite distance, perhaps even visiting en route an infinite number of different people, each on a star of his own.

The detachability of the infinitesimal tail has an even more striking consequence. The paradox of the alternating divergent series returns in full force. Here on earth let us celebrate the paradox of the apple but only during the Z-series of rational temporal intervals. At any small rational interval of time prior to the end of the minute we are to be found engaged either in restoring the apple to the table or in removing it; and at any infinitesimal interval of time prior to the end of the minute (as also at any interval of time whatever after the end of the minute) we are to be found continuously with our hands locked together behind our back. What, then, will be the state of affairs that obtains on the table? Where will the apple be, on the table or off? What is the result of the whole infinite sequence of alternately removing and restoring the apple? Result: There is a radical indeterminacy here, and almost anything may actually prove to be the case. Perhaps the world itself, confounded by this ontological crisis, will simply come to an end, not indeed at any rational or irrational instant of time, but at that imaginary instant that separates the body of the temporal continuum from its infinitesimal tail. Relieved of all embarrassment, we shall not then be found with our hands behind our back.

Zeno’s misgivings are now seen to be vindicated in an unexpected way. Achilles undertakes to travel in one minute from the earth to the moon. During the first half minute he travels half the distance, during the next 1/4 minute he travels the next quarter of the distance, &c. ad infinitum. Assuming that he actually exhausts the entire Z-series of rational intervals, must he then be found on the moon at the end of the minute? Not necessarily. It is quite possible that at the precise end of the minute (or at any infinitesimal interval of time prior to the end of the minute) Achilles will be found on the earth or on Mars or
Perhaps nowhere at all. He will indeed have satisfied Cauchy’s criterion by penetrating into every rational neighbourhood of the moon, but he may well have failed actually to touch the surface of the moon at any time. Merely because one succeeds in actually approaching arbitrarily close to the moon, does not in and of itself logically entail that he will ever reach it.

The net result (so to speak) of a convergent series is found to be quite as indeterminate as that of a divergent series. The divergent series is no longer to be viewed as an ontological anomaly. It may be subsumed under a general theory of infinite series which accommodates the two types with equal ease. Of the two it is the divergent series which proves to be the more illuminating. For its very lack of any strict limit has ontological reverberations even in the case of the convergent series. Returning to the piece of string that puzzled us early in our researches, we are now to understand it as follows. In the first half minute we employ the string to form an equilateral triangle, in the next $\frac{1}{4}$ minute we employ it to form a square, &c. ad infinitum. What figure will the string be found to have assumed at the end of the minute? A circle? Perhaps, but we should not be surprised to find any other figure in its place. If we are to guarantee the circle here or the success of Achilles in actually touching the moon, we must presuppose some kind of postulate of continuity.

The fact that every closed continuum requires an infinitesimal tail, enables us to understand an otherwise perplexing construction that is suggested by an ingenious point set of Cantor. Take a stick of wood. Inscribe the number 1 on the left-hand butt-end and the number 2 on the right-hand butt-end. Divide the stick into three equal parts, and throw the middle third into a box. Inscribe the number 3 on the right-hand butt-end of the $\frac{1}{3}$ piece and the number 4 on the left-hand butt-end of the $\frac{2}{3}$ piece. Divide each of these two pieces again into three equal parts, and throw the middle third of each into the box. There are now four pieces on the table. Inscribe the numbers 5, 6, 7 and 8 on the vacant butt-ends. Divide each of the four pieces into three equal parts, &c. ad infinitum. At the end of the minute the box will contain an infinite number of pieces of wood each of some rational length. None of the inscribed numerals will be found on any of the pieces in the box. Where then are they to be found? On the infinitesimal chips lying on the table that correspond to the points $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{9}$, $\frac{2}{9}$, $\frac{7}{9}$, $\frac{8}{9}$ ... by which the successive segments were trisected. As for the pieces in the box, their total length is

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \frac{1}{1 - \frac{1}{2}} = 2.$$ 

Inside the brackets is a geometric series whose sum, by the method of convergence, is $\frac{1}{1 - \frac{1}{2}} = 2$; which makes the total length of the pieces in the box equal to 1. But if the total length of the original stick is exhausted by the pieces in the box, the infinitesimal chips must then be metrically superfluous. Actually the pieces in the box constitute a convergent series open at one end. Their total length is 1 minus an actual infinitesimal, in this case a very big infinitesimal seeing that its magnitude is equal to the combined magnitude of all the infinitesimal chips.

The intelligibility of the continuum has been found—many times over—to require that the domain of real numbers be enlarged to include infinitesimals. This enlarged domain may be styled the domain of continuum numbers. It will now be evident that ‘9999 ... does not equal 1 but falls infinitesimally short of it. I think that ‘9999 ... should indeed be admitted as a number (it makes sense to speak of an open-ended stick as being ‘9999 ... inches long), though not as a real number. In general, the decimal expansion of any real number $r$, be it rational or irrational, is equal to $r$ minus an actual infinitesimal. Although the decimal expansion itself fails to express a real number, it does succeed in expressing a continuum number.

In the present discussion we have been addressing ourselves to the small infinite. If it is richness of complexity that is sought, one may be invited to orchestrate together, in one coherent system of hyper-geometry, both the large infinite and the small. Our earlier account of a well-ordered pluriverse will be seen to be fairly naive, for nothing less than a continuum of Spaces is rather to be expected. Launching from point A two rockets, one ascending straight up at an angle of 90°, the other at some other angle, say 89°, we are to project them no more than a short infinite distance, the one rocket landing at points B, the other at point C. (Any infinite line AB will be said to extend but a short infinite distance if each point on the line is a finite
What now is the distance between point B and point C? Seeing that AB and AC are to be understood as Euclidean straight lines, though very much extended Euclidean straight lines, the distance between point B and point C must be infinite. Hence we shall say that points B and C lie in different Spaces, in accordance with our convention that a Space is the locus of all points which are a finite distance from one another. This inter-Spatial triangle ABC is highly curious. The distance BA+AC, being equal to two short infinite distances, is found to be infinitely shorter than the base BC of the triangle. For the straight line BC cuts across a whole continuum of distinct Spaces. Consider the 'mid-point', D, of this enormously long infinite line BC. This 'mid-point' D may be defined as the point on BC at which a rocket launched from point A at the precise angle of 89°5 will intersect the line BC. This mid-point D, being removed an infinite distance from both B and C, is seen to lie in a Space of its own. In general, given any angle of incidence greater than 89° and less than 90°, there exists on line BC a unique Space which corresponds to that angle of incidence. There being infinitely many, indeed non-denumerably many angles of incidence (counting the irrational as well as the rational angles of incidence) between 89° and 90°, there must then be non-denumerably many Spaces which are pierced by line BC.

If point A be said to lie within Space X, then it is evident that Space X is immediately adjacent, so to speak, to non-denumerably many Spaces which surround it along its 'circumference', so to say, in a continuous ring. Each angle of incidence at point A (and we actually have the whole range of 360° to exploit) proves to be a unique route into a unique Space. On the assumption, now, that between any two points a straight line can be drawn, what kind of straight line will connect point A with some point E, lying on line BC, which is precisely five miles from point B? Point E is seen to lie within the same Space as point B. At what angle of incidence should the rocket be launched from point A so as to intersect line BC at point E? Clearly, the angle of incidence must be 90° minus some infinitesimal quantity. This type of angle is not unimportant. Seeing that there exist non-denumerably many points on line BC which are a finite distance from point B, each of these points is connected to point A by a straight line whose angle of incidence at A is 90° minus some infinitesimal. Each of these straight lines being unique, each of the infinitesimals will be definably greater or less than any of the others. There must, then, exist a non-denumerable set of distinguishable infinitesimal quantities which are themselves arranged in a continuum. For if point E is five miles, and point F six miles, removed from point B on line BC, then the infinitesimal which defines the angle of separation between line AB and line AF will be greater than the infinitesimal which defines the angle of separation between line AB and line AE, and this difference between the two angles, itself an infinitesimal, is seen to be infinitely divisible.

As a result of our explorations here, we are led almost inevitably to doubt one of our premises, namely that lines AB and AC extend but a short infinite distance. How can there be any short infinite distances anywhere? Seeing that it is impossible to travel a short infinite distance from point B along line BC—there simply is no point G on line BC which is an infinite distance from B, such that every point on BG is a finite distance from either B or G—it is unreasonable to suppose that BA is a short infinite distance. Assuming that our multiverse is isotropic, why should there exist a short infinite distance only in the vertical direction BA and not in the horizontal direction BC? If there is a continuum of Spaces lying between point B and any point P on line BC which is an infinite distance from point B, then we should equally suppose that there is a continuum of Spaces lying between point B and any point P on line BA which is an infinite distance from point B. Such being the case, there must then be a continuum of Spaces lying between point A and point B; and line AB must be a very
long infinite line indeed. We are thus liberated from the paradoxical assertion that the straight line \( BC \) is not the shortest distance between points \( B \) and \( C \), the sum-total of \( BA + AC \) having held been to be infinitely shorter.

What happens, now, if we undertake to launch a rocket from point \( A \) with the design of penetrating the whole continuum of Spaces between \( A \) and \( B \)? Following our standard procedure, we shall find that at any rational interval of time prior to the end of the allotted minute our rocket will remain within a finite distance of point \( A \), and at any infinitesimal interval of time prior to the end of the minute (but after the imaginary cut) our rocket will be separated from point \( A \) by a long infinite distance. That is to say, at any infinitesimal interval of time prior to the end of the minute a whole continuum of Spaces will lie between point \( A \) and the rocket.

At the end of the minute we shall have succeeded in traversing non-denumerably many Spaces (all but one being traversed during the infinitesimal tail of the minute), and if each of the real numbers be found recorded in some one of those Spaces, we shall have succeeded in scanning all of the real numbers, in a linear order, during an infinitesimal span of time. We may even be so fortunate as to scan them in order of magnitude. In which case we shall have a guarantee that none has been omitted from our inspection. Non-denumerably many books, each of infinite length, may also be read by us in linear succession during that infinitesimal interval. If all of these books should prove to be merely chapters in one great book, we shall be reading an infinite book of non-denumerable extent. May we not hope to find therein a proof of the non-existence of random numbers or a proof of the continuum hypothesis? The very least, a proof of a hyper-Gödelian sentence?

Although this particular cosmography is logically consistent, one may yet wish to retain the short infinite distance. Is it possible, after all, that there may exist a point \( G \) on line \( BC \) which is removed from point \( B \) by a short infinite distance? Yes, but only if we are willing to extend the scope of the infinitesimal. For such a point \( G \) must be connected to point \( A \) by a straight line whose angle of incidence at \( A \) is \( 90^\circ \) minus an infinitesimal, albeit a very large infinitesimal indeed. I think that this possibility may be credited. Merely because any rational or irrational angle of incidence less than \( 90^\circ \) must inevitably project us an infinite distance from point \( B \), it does not follow that any angle of incidence which is infinitesimally less than \( 90^\circ \) must inevitably project us only a finite distance from point \( B \). The smaller infinitesimal angles may project us only a finite distance from point \( B \) along line \( BC \), but there may also exist larger infinitesimal angles that are capable of projecting us an infinite, i.e. a short infinite, distance from point \( B \) along line \( BC \) (meaning here by a short infinite distance any finite number of short infinite distances). Granting that the prospect of retaining line \( AB \) as a short infinite distance cannot but be agreeable to us, there is an attendant liability that gives us pause. The sum of \( BA + AC \) being equal \textit{ex hypothesi} to two short infinite distances, once again the very long infinite line \( BC \) cannot be the shortest distance between points \( B \) and \( C \).

We seem to have a choice. On the one hand, we may retain the short infinite distance but only at the price of surrendering the Euclidean axiom that the straight line is always the shortest distance between any two points. On the other hand, we may retain the Euclidean axiom but only at the price of surrendering the short infinite distance.

In our efforts to draught a cosmography, metaphysics has become virtually transformed into mathematics, albeit a highly speculative mathematics. Here, then, is perhaps as good a place as any to call a halt to what are, essentially, endless inquiries. If there is to be a last word, it is only appropriate, I think, that the privilege should be accorded to the old high priest of the Swahilis. I can see him shaking his head sadly. 'This is what comes', I can hear him say, 'of estranging oneself from the security of one's mother-tongue. It is not as if we do not accept the Peano postulates: they are mother's milk to us, and though you may feel that we assign to them a sub-standard interpretation, they are no less adequate for all that.

'So you hanker after the actual infinite? Why, then, have you not been content with the actual infinite that is your proper birthright, instead of aspiring to ape the gods? God having created man in his own image, man is indeed the ape of God, but that only means that one should acquiesce in the reduced form of the actual infinite that is suitable for man. I do not mean the potential infinite, for having abandoned the actual infinite
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of your mother-tongue, you did not dare—at first—to arrogate to yourself the exalted actual infinite of God Almighty, but still somewhat diffident you devised this intermediate concept of the potential infinite lest your total estrangement from your mother-tongue be too abrupt. Alas! This intermediate concept, though it satisfied you for centuries, was yet to prove but the stepping-stone to even higher aspirations, as one might have seen in advance. You would have your actual infinite in the end, even as you had it in the beginning, only now in such an inflated form that it signifies sheer megalomania. Folly! Folly! Folly!

EPILOGUE

If metaphysics is a form of Ἀφίλα, then we have been justly punished for our presumption.

Happily, there is a comic as well as a tragic form of Ἀφίλα, and there is not only low, there is also high, even divine, comedy.

I freely admit that the metaphysical adventure very much smacks of the quixotic. In a spirit of metaphysical hyperbole, one finds oneself engaged in an investigation of utopian concepts that transcend the empirical. There have always been, there always will be, philosophers of a severe and sober cast of mind who set their faces against all such utopian transports. However much I may inveigh against them, these philosophers are my friends. For I am persuaded that we are engaged in a common enterprise and that it belongs to the very rhythm of philosophy itself that the downward dialectic should alternate with the upward. Nor should I wish to deny that it is the downward dialectic which possesses a power and an authority to which the upward dialectic can only fitfully aspire.

At the same time I should be reluctant to believe that our metaphysical ascent has been altogether a failure. In hearkening to the lure of the infinite, we have pursued it both in the small and in the large, and if we have looked to mathematics for light, we have no less undertaken to fetch light of our own into mathematics. Almost in a Spinozistic mood I am tempted to say that apart from the infinite nothing can either be or be conceived. For the finite and the infinite are polar opposites, and though only one of the two may be empirically accessible, neither is intelligible apart from the other. Both logically and cosmologically, it is the peculiar destiny of mind to open up onto the prospect of an infinite horizon.

In the end, our conclusions (such as they are) can only be viewed as problematic. I submit this sheaf of studies as scarcely more than an interim report in an ongoing metaphysical inquiry. If I have succeeded in extending the limits within which such inquiry has heretofore been conducted, I shall have accomplished perhaps as much as any philosopher can reasonably hope to achieve.
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