Online Appendix for

Population Aging, Economic Growth, and the Importance of Capital

A Proof of Proposition 1

Moving to time $t+1$ and combining equations (6) and (9), gives per-capita output in $t+1$ as

$$y_{t+1} = \left(\frac{\beta}{1+\beta}(1-\alpha)k_{w,t}^{\alpha}\frac{N_{w,t}}{N_{w,t+1}}\right)^{\alpha}\frac{N_{w,t+1}}{N_{c,t+1}+N_{w,t+1}+N_{o,t+1}}.$$

Using this expression, the partial derivative of y_{t+1} with respect to $N_{w,t+1}$ yields

$$\begin{aligned}\frac{\partial y_{t+1}}{\partial N_{w,t+1}} &= (1-\alpha)\left(\frac{\beta}{1+\beta}(1-\alpha)k_{w,t}^{\alpha}N_{w,t}\right)^{\alpha}\frac{N_{w,t+1}^{-\alpha}}{N_{c,t+1}+N_{w,t+1}+N_{o,t+1}} \\ &- \left(\frac{\beta}{1+\beta}(1-\alpha)k_{w,t}^{\alpha}N_{w,t}\right)^{\alpha}\frac{N_{w,t+1}^{1-\alpha}}{(N_{c,t+1}+N_{w,t+1}+N_{o,t+1})^2}.\end{aligned}$$

Algebra shows that this is negative if

$$1-\alpha < \frac{N_{w,t+1}}{N_{c,t+1}+N_{w,t+1}+N_{o,t+1}} = \theta_{t+1}.$$

By taking the partial derivative, we implicitly have held the number of children and the number of retirees constant between time t and $t+1$. Clearly, the number of dependents also affects y_{t+1} directly. Many papers have argued that longer average life-spans have increased the number of retirees and impacted growth rates (see Zhang and Zhang (2001) for one related paper out of many), or even that demographic change itself modifies saving behavior (Curtis et al., 2015) and possibly also the capital intensity of production (Glover and Short, 2017). We have abstracted from these important considerations. We have also assumed (by using log preferences) that capital is unchanged. We revisit this last assumption below.

B Proposition 1: A More General Case

The main paper and the proof above make use of the standard Cobb-Douglas aggregate production function. However, Proposition 1 holds for a more general class of production functions. In a working version (Curtis and Lugauer (2019)) of this paper, we provide further details. Here we give the highlights, and the following section generalizes the model further.

B.1 Model with a Generic Production Function

Demographics As before, the population demographics evolve exogenously. The total number of people alive at time t is N_t, with the working age subset, $N_{w,t}$, inelastically supplying labor. The working fraction of the population θ is

$$\theta_t = \frac{N_{w,t}}{N_t}. \tag{1}$$

Production A representative firm combines capital K and labor N_w to create output using a constant returns to scale technology, represented by production function, $F(K, N_w)$. For simplicity, we abstract from a total factor productivity term (i.e. technology growth). Output per person at time t equals

$$y_t = \frac{F(K_t, N_{w,t})}{N_t}. \tag{2}$$

Allowing for perfectly competitive markets, capital and labor are paid their marginal products. So, the share of output due to capital is $\frac{F_K(K_t,N_{w,t})K_t}{F(K_t,N_{w,t})}$, where F_K stands for the partial derivative of output with respect to a change in the capital input. Therefore, the labor share is

$$ls_t = 1 - \frac{F_K(K_t, N_{w,t})K_t}{F(K_t, N_{w,t})}. \tag{3}$$

The factor shares are critical for determining future economic growth in the face of demographic change; they govern the relative importance of capital versus labor in production.

Proposition 1 holds as before.

Proposition 1 *A decrease in $N_{w,t+1}$ increases y_{t+1}, as long as $\theta_{t+1} > ls_{t+1}$.*

B.2 Proof of Proposition 1

Assume capital at time $t+1$ is not a function of time $t+1$ variables. Output per capita at time $t+1$ is

$$y_{t+1} = \frac{F(K_{t+1}, N_{w,t+1})}{N_{t+1}}.$$

The partial derivative of y_{t+1} with respect to $N_{w,t+1}$ is

$$\frac{\partial y_{t+1}}{\partial N_{w,t+1}} = \frac{F_N(K_{t+1}, N_{w,t+1})}{N_{t+1}} - \frac{F(K_{t+1}, N_{w,t+1})}{N_{t+1}^2}.$$

Note, this implicitly holds the number of non-workers constant. The partial derivative is negative if

$$\frac{F_N(K_{t+1}, N_{w,t+1})}{N_{t+1}} < \frac{F(K_{t+1}, N_{w,t+1})}{N_{t+1}^2}.$$

Algebra shows this is negative if

$$\underbrace{\frac{F_N(K_{t+1}, N_{w,t+1})N_{w,t+1}}{F(K_{t+1}, N_{w,t+1})}}_{ls_{t+1}} < \underbrace{\frac{N_{w,t+1}}{N_{t+1}}}_{\theta_{t+1}}.$$

C Proposition 1: A More General Case including Capital Accumulation

With log utility, the income and substitution effects cancel out leaving K_{t+1} a function of time t variables only. If the elasticity of substitution is not 1, then this is not necessarily the case. Here, we consider the general functional form of (additively separable) preferences. Agents within the working cohort at time t maximize lifetime utility over current consumption and during retirement, based on utility function $\mathbb{U} = u(c_{w,t}) + u(c_{o,t+1})$ and subject to the intertemporal budget constraints.[1]

The Euler Equation with the budget constraints results in

$$R_{t+1}u'(R_{t+1}s_t) = u'(w_t - s_t), \tag{4}$$

which implicitly gives the saving function $s_t = s(w_t, R_{t+1})$, where R is the interest rate. We first show the sign of the response to saving from a change in interest rates, which we will use in the derivation of Proposition 1 under general preferences. Let s_R be the derivative of saving with respect to interest rates. Using the Euler Equation (4) we find through implicit differentiation

$$s_R = -\frac{u'(c_{o,t+1})(1 - \sigma(c_{o,t+1}))}{R_{t+1}^2 u''(c_{o,t+1}) + u''(c_{w,t})}$$

where $\sigma(c) = -u''(c)c/u'(c)$ and $1/\sigma$ is the elasticity of substitution. The equation is a well known result for this type of model. If preferences are, for example, represented with the commonly used Constant Relative Risk Aversion (CRRA) form, most empirical estimates place $\sigma > 1$. Under this restriction, $s_R < 0$. If $\sigma < 1$ then $s_R > 0$.

Next, recall the capital stock at time $t+1$ equals the saving by the previous working generation, $K_{t+1} = N_{w,t}s(w_t, R_{t+1})$. Thus, per capita GDP at time $t+1$ is

$$y_{t+1} = \frac{F(K_{t+1}, N_{w,t+1})}{N_{t+1}} = \frac{F(N_{w,t}s(w_t, R_{t+1}), N_{w,t+1})}{N_{t+1}} = \frac{F(N_{w,t}s(w_t, F_K(K_{t+1}, N_{w,t+1})), N_{w,t+1})}{N_{t+1}}.$$

The partial derivative of y_{t+1} with respect to $N_{w,t+1}$ is

$$\frac{\partial y_{t+1}}{\partial N_{w,t+1}} = \frac{F_N(K_{t+1}, N_{w,t+1})}{N_{t+1}} - \frac{F(K_{t+1}, N_{w,t+1})}{N_{t+1}^2} + \frac{1}{N_{t+1}}F_K(K_{t+1}, N_{w,t+1})\frac{\partial K_{t+1}}{\partial N_{w,t+1}}.$$

This is negative if

$$\theta_{t+1} > ls_{t+1} + \frac{N_{w,t+1}R_{t+1}}{F(K_{t+1}, N_{w,t+1})}\frac{\partial K_{t+1}}{\partial N_{w,t+1}}. \tag{5}$$

[1]For simplicity letting the subjective discount factor $\beta = 1$.

If the elasticity of substitution $\sigma < 1$, the sign of the right-most term is positive $\left(\frac{\partial K_{t+1}}{\partial N_{w,t+1}} > 0\right)$. If the elasticity of substitution $\sigma > 1$, the sign of the last (right-most) term is ambiguous. To see this, let F_{KK} and F_{KN} be the second partial derivatives and consider

$$\frac{\partial K_{t+1}}{\partial N_{w,t+1}} = \frac{s_R N_{w,t} F_{KN}(K_{t+1}, N_{w,t+1})}{1 - s_R N_{w,t} F_{KK}(K_{t+1}, N_{w,t+1})}.$$

The sign is ambiguous since, with $\sigma > 1$, $s_R < 0$ and $F_{KK} < 0$. Thus, the sign of the last term in equation (5) depends on the relative curvature of the utility and the production functions. If the last term is negative, the range in which a decline in the working age cohort leads to higher output increases. As the sign and magnitude depend on the functional forms of the utility and production functions, as well as an empirical determination of the parameter values in these functions, finding an exact solution lies beyond the scope of this paper.

References

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