Online Appendix for Precautionary Saving of Chinese and US Households

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Abstract

This online appendix presents the analytical derivations and estimation details referenced in the main body of the paper.

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Online Appendix

A. Estimation of the Labor Income Process

Carroll (1992) contains a more complete explanation and justification for the estimation procedure. Here we sketch the steps involved.

The procedure for estimating the probability of zero (transitory) income is as follows:

1. For each year, divide actual household income by the cross-sectional mean of income. Call the result detrended household income. Normalization by the mean is intended to remove cycle and trend components.

2. Regress detrended income on age, occupation, education, the interactions of these terms, age squared, and gender. Use this regression to predict life-cycle (age-specific) movements in income for each household.

3. Divide detrended income by predicted income. Call this $YL_{i,t}$.

4. Take the average income over all observations for household i. Call this average permanent income.

5. Take $YL_{i,t}$ and divide by average permanent income. This creates up to 8 observations per household for a total of 4,550 observations on urban households. The entire procedure was repeated separately for 12,163 rural households, since their income stream could be different. Categorize a zero-income event as occurring when YL divided by average permanent income is less than 0.1. A substantial portion of the observations are concentrated near zero income. Following Carroll (1992), negative observations are counted as zero. A total of 69, about 1.5%, of the observations of urban households occur at or below 0.10 (i.e. 90% below trend income). The percentage for rural households is 2.5%. A weighted average across urban and rural households gives $p = 0.0224$.

Table 1: Frequency of Zero Non-Capital Income Events, China

Head of Household	Observations	Near-Zero Events	% Near-Zero Events
Urban Chinese	4,550	69	1.52
Rural Chinese	12,163	307	2.52

The entire process was repeated separately for the US PSID data, resulting in $p = 0.0010$.

To determine the relative magnitudes of the transitory and permanent shocks (σ_n, σ_u), we further restrict the sample to heads of households whose marital status never changed, who never ran a business as their primary occupation, and who never experienced a near-zero income event. Note, determining who owns a business in the Chinese data is less straightforward than in the US data. These restrictions should all reduce variability. The variance of the shocks are then estimated by regressing the sample variance of $\ln YL_{it-m} - \ln YL_{it}$ on m and a constant for all values of m that can be calculated.

B. Analytical Characteristics of the Model

First, we discuss the analytical characteristics of the model relevant to the key results in the paper. Then, we give the details on making the model economy stationary. The derivations necessary for the analytical analysis are collected together in the third sub-section (B.3).

B.1. How the Parameters affect the Saving Rate

Closed-form solutions to the model are not available, but we can deduce some of its properties from an analysis of the household's Euler equation. Throughout, we assume the (log) normality and conditional homoskedasticity of the stochastic variables. As we show below, the analytical results with the log-normality assumption are similar to those using the second order approximation. We begin with general characteristics of the model, and then study how saving behavior responds to changes in income growth.

To ease notation, we drop the i subscript and let

$$Z_{t+1} \equiv \frac{V_{t+1}}{\left[E_t\left(V_{t+1}^{1-\gamma}\right)\right]^{\frac{1}{1-\gamma}}}. \tag{1}$$

By analogy to the development in Parker and Preston (2005), we refer to Z_{t+1} as a preference shifter. With this notation, we can write the household's Euler equation as

$$1 = e^{r-\delta} E_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\frac{1}{\sigma}} Z_{t+1}^{\frac{1-\gamma\sigma}{\sigma}}\right]. \tag{2}$$

Assuming that consumption and utility are log-normally distributed (which is equivalent to taking a second-order approximation of (2) around the deterministic steady state) gives the log-linearized version of the Euler equation

$$E\left[\Delta \ln C_{t+1} - (1-\gamma\sigma)\ln Z_{t+1}\right] = \sigma(r-\delta) + \frac{1}{2\sigma}\text{Var}\left[\Delta \ln C_{t+1} - (1-\gamma\sigma)\ln Z_{t+1}\right]. \tag{3}$$

Before discussing (3), some context under constant relative risk aversion (CRRA) utility can be provided by setting $\gamma = 1/\sigma$

$$E\left(\Delta \ln C_{t+1}\right) = \frac{r-\delta}{\gamma} + \frac{\gamma}{2}\text{Var}\left(\Delta \ln C_{t+1}\right). \tag{4}$$

Under CRRA, an effect that raises the expected future consumption growth rate is inferred to raise the saving rate, since high consumption growth means low current consumption and hence high saving. Under CRRA utility, saving behavior is summarized by a mean-variance relationship for consumption growth. A high consumption variance household has higher demand for precautionary saving. This depresses current consumption and raises expected consumption growth. The first term in (4) is typically identified with the effect of the IES on consumption growth, because the coefficient on $(r-\delta)$ is the IES $1/\gamma$. This is made clear in (3), where the IES σ shows up explicitly. The second term in (4) is the effect of precautionary saving on mean consumption growth. An increase in the volatility of consumption growth raises expected consumption growth.

Now, comparing (4) to (3), under recursive preferences the mean-variance relationship of consumption growth generalizes to a mean-variance relationship for $\Delta \ln C_{t+1} - (1-\gamma\sigma) \ln Z_{t+1}$, which includes the preference shifter term.

As we do not have explicit expressions for the saving rate, we infer the effect of parameter values on the saving rate through their effect on the expected consumption growth rate. To assess how variations in the parameter values affect the saving rate, express (3) as

$$E\left(\Delta \ln C_{t+1}\right) = \iota + \phi + \psi. \tag{5}$$

where

$$\iota \equiv \sigma\left(r-\delta\right), \tag{6}$$

$$\phi \equiv \left(1-\gamma\sigma\right) E\left(\ln Z_{t+1}\right), \tag{7}$$

$$\psi \equiv \frac{1}{2\sigma} \mathrm{Var}\left[\Delta \ln C_{t+1} - \left(1-\gamma\sigma\right) \ln Z_{t+1}\right]. \tag{8}$$

We refer to ι as the intertemporal substitution effect, ϕ as the 'preference shifter,' and ψ as the precautionary effect on expected consumption growth. The preference shifter embodies the preference for the timing of resolution of uncertainty. $E\left(\ln Z_{t+1}\right)$ can be interpreted as the cost of carrying uncertainty to the future and $\gamma\sigma$ as the risk adjusted elasticity of substitution for uncertainty resolution. If ϕ is positive (negative), individuals prefer later (earlier) resolution of uncertainty and raise (lower) consumption growth by consuming less (more) today. Under log-normality, the cost of carrying uncertainty to the future is $E\left(\ln Z_{t+1}\right) = (\gamma-1)\mathrm{Var}(\ln Z_{t+1})/2$. Substituting into Equation (7) gives

$$\phi = \frac{\left(1-\gamma\sigma\right)\left(\gamma-1\right)\mathrm{Var}\left(\ln Z_{t+1}\right)}{2}. \tag{9}$$

We use equations (6),(8) and (9) to study how changing the underlying parameters affects household saving decisions.

Income Growth g. We begin with the income growth rate because the large difference in g between China and the US is quantitatively the most important factor for explaining the difference in saving rates within the model simulations. While an exact analytical solution does not exist, we still can use the equations to show the rich relationship between income growth and saving.

First, we note that consumption volatility (hence precautionary saving) increases with the income growth rate, g. This can be inferred by looking at the steady state version of (3), where average consumption grows at the same rate as income $\ln E\left(e^{\Delta \ln C_{t+1}}\right) = \ln E\left(e^{\Delta \ln Y_{t+1}}\right) = g + \mu_n$. As the growth rate increases on the left hand side of (3), the only thing that can increase on the right hand side are variances of utility or consumption growth.

Second, let $M = A + Y$ be 'cash-on-hand,' and s be the ratio of saving to total income (labor income plus interest on assets). Then the direct relationship between the saving rate and the growth rate implied by the budget constraint is (we ignore the stochastic nature of the model for illustration)

$$\frac{s}{1-s} = \left(\frac{M}{C} - 1\right)\left(1 - e^{-(g+\mu_n)}\right). \tag{10}$$

Holding M/C constant, an increase in the growth rate has a positive effect on the saving rate s. This formula is a bit cumbersome, however. We can get the same intuition by looking at saving as a fraction of labor income, S/Y. For a given growth rate g, in the steady state, wealth will be proportional to income $W = \omega Y$. Hence, in the steady state,

$$\frac{S}{Y} = \frac{\Delta W}{Y} = \frac{\Delta W}{\Delta Y}\frac{\Delta Y}{Y} = e^{g+\mu_n}\frac{W}{Y}. \tag{11}$$

An increase in the growth rate,

$$\frac{\partial(S/Y)}{\partial g} = e^{g+\mu_n}\left[\frac{W}{Y} + \frac{\partial(W/Y)}{\partial g}\right], \tag{12}$$

has a positive direct effect (W/Y) and an ambiguous indirect effect $\partial(W/Y)/\partial g$. Hence, for a given target wealth-to-income ratio, the saving rate increases with the growth rate because a higher g causes the denominator Y to grow faster, and households need to save more aggressively to get the numerator W to grow at the new higher rate. This relationship captures the main mechanism driving our quantitative simulation results.

However, the target wealth-to-income ratio (equivalently M/C) need not be invariant to g, which gives rise to the indirect effect. Higher future income from higher income growth makes households less vulnerable to income risk. Households may reduce their target wealth-to-income ratio (and M/C), which can depress the saving rate. Assessing the relative strength of these two effects must be done numerically. Clearly, the saving rate is zero when growth is zero and positive for some positive growth rates. Also, the direct effect is diminishing in g as shown in (10). Hence, the saving rate either increases with g or, if the indirect effect dominates when income growth is high, exhibits a hump-shaped pattern with respect to g. The simulations in the paper further highlight the potentially non-monotonic relationship between income growth and saving.

The remainder of this sub-section studies the analytical properties of the model with respect to the preference parameters. Some readers may find these analytical results interesting because they highlight the use of Epstein-Zin (1989)-Weil (1989) preferences within the precautionary framework. The analytical derivations also contain insights relevant to the simulation results.

Intertemporal elasticity of substitution. The relationship between the saving rate, the IES and the RRA is non monotonic. For clarity of exposition, we concentrate our discussion by assuming that risk aversion is not too low $(\gamma > 1)$ and that people prefer early resolution of uncertainty $(\gamma\sigma > 1)$.

If people are impatient $(\delta > r)$, then increasing σ lowers expected consumption growth (and hence saving) directly by depressing the intertemporal substitution effect, $\partial\iota/\partial\sigma < 0$. When shifting consumption across time periods is easy for impatient people, they will shift consumption towards the present.

The effect of increasing the IES on the preference shifter is

$$\frac{\partial\phi}{\partial\sigma} = \underbrace{-\frac{\gamma(\gamma-1)\text{Var}\left(\ln Z_{t+1}\right)}{2}}_{\text{Direct}} + \underbrace{\frac{(1-\gamma\sigma)\left(\gamma-1\right)}{2}\frac{\partial\text{Var}\left(\ln Z_{t+1}\right)}{\partial\sigma}}_{\text{Indirect}}. \tag{13}$$

The first term in (13) is the direct effect which is negative when $\gamma > 1$. Raising σ strengthens the preference for early resolution of uncertainty and lowers the preference shifter $\partial\phi/\partial\sigma < 0$. Thus, the consumption growth (and hence saving) fall with σ. The second term is an indirect effect that works through the variance of utility. When it becomes easier for people to move consumption across time periods, higher σ increases the volatility of consumption and utility (and Z_{t+1}), which increases precautionary saving. The indirect effect on the preference shifter ϕ is negative when the risk aversion and intertemporal substitution are high ($\gamma > 1$ and $\gamma\sigma > 1$).

The effect of increasing the IES on the precautionary component is

$$\frac{\partial\psi}{\partial\sigma} = \underbrace{\frac{1}{2\sigma^2}\left[(\gamma\sigma)^2 \operatorname{Var}(\ln Z_{t+1}) - \operatorname{Var}(\ln Z_{t+1} - \Delta \ln C_{t+1})\right]}_{\text{Direct}} \tag{14}$$
$$+ \underbrace{\frac{1}{2\sigma}\frac{\partial \operatorname{Var}(\Delta \ln C_{t+1})}{\partial\sigma} + (\gamma\sigma - 1)^2 \frac{\partial \operatorname{Var}(\ln Z_{t+1})}{\partial\sigma} + 2(\gamma\sigma - 1)\frac{\partial \operatorname{Cov}(\Delta \ln C_{t+1}, \ln Z_{t+1})}{\partial\sigma}}_{\text{Indirect}}.$$

The direct effect of increasing σ on the precautionary component is ambiguous. When people prefer early resolution of uncertainty, on the one hand, raising σ reduces precautionary saving. When people can more easily substitute consumption across time, they will move consumption to the present. On the other hand, since variation of the preference shifter contributes positively to the overall volatility, precautionary saving increases with σ through $\operatorname{Var}(\ln Z_{t+1} - \Delta \ln C_{t+1})$.

The indirect effect works through the comovement with consumption growth. The stochastic part of $\ln Z_{t+1}$ is the (log) utility forecast error $\ln V_{t+1} - E_t(\ln V_{t+1})$. See below for the derivation. A surprise improvement in utility is positively correlated with consumption growth, making the covariance term positive. The last term in (14) is the indirect effect working through the changes in the variations of consumption growth and utility. As the variability of consumption and utility rise with the substitutability, the indirect effect is positive when $\gamma\sigma > 1$.

Although the overall relationship between the intertemporal elasticity of substitution and the saving rate cannot be unambiguously signed, we conjecture that increasing σ lowers the saving rate when risk aversion is low and raises the saving rate when risk aversion is high. Combining all effects, we have

$$\frac{\partial E(\Delta \ln C_{t+1})}{\partial\sigma} = \underbrace{(r - \delta) + \frac{\gamma}{2}\operatorname{Var}(\ln Z_{t+1}) - \frac{1}{2\sigma^2}\operatorname{Var}(\ln Z_{t+1} - \Delta \ln C_{t+1})}_{\text{Direct}} \tag{15}$$
$$+ \frac{1}{2\sigma}\left\{\frac{\partial \operatorname{Var}(\Delta \ln C_{t+1})}{\partial\sigma} + (\gamma\sigma - 1)(\sigma - 1)\frac{\partial \operatorname{Var}(\ln Z_{t+1})}{\partial\sigma}\right.$$
$$\left. + 2(\gamma\sigma - 1)\frac{\partial \operatorname{Cov}(\Delta \ln C_{t+1}, \ln Z_{t+1})}{\partial\sigma}\right\}.$$

The first three terms are direct effects and the term in curly brackets is the indirect effect. The overall indirect effect, the last term, is positive when $\gamma\sigma > 1$ and $\sigma > 1$. The overall direct effect is increasing in γ and σ given volatility. For the direct effect, increasing σ lowers (raises) the saving rate when γ is relatively low (high).

For $\gamma > 1$, the saving rate profile has a U shape with respect to σ. The desire to accumulate a buffer-stock of assets to hedge against adverse income shocks intensifies with greater risk aversion. Raising σ makes moving consumption around across time easier and leads to higher saving if γ is high enough (for there to be buffer stock asset accumulation). On the other hand, if risk-aversion is low, people do not build up a large buffer stock. There is less desire to sacrifice current consumption, and when it is easy for people to move consumption across time periods, they will, due to their impatience, move it to the present.

Risk Aversion. The overall effect of increasing risk-aversion on the saving rate is also ambiguous. Risk aversion has no effect on the intertemporal substitution effect, ι. Increasing risk aversion has the following effect on the preference shifter,

$$\frac{\partial \phi}{\partial \gamma} = \underbrace{\frac{(1-\gamma\sigma)\,\mathrm{Var}\,(\ln Z_{t+1})}{2} - \frac{\sigma\,(\gamma-1)\,\mathrm{Var}\,(\ln Z_{t+1})}{2}}_{\text{Direct}} + \underbrace{\frac{(1-\gamma\sigma)\,(\gamma-1)}{2}\frac{\partial \mathrm{Var}\,(\ln Z_{t+1})}{\partial \gamma}}_{\text{Indirect}}. \tag{16}$$

The first term is the effect of change in the uncertainty cost, $E\,(\ln Z_{t+1})$ which is increasing in risk aversion. When $\gamma\sigma > 1$, an increase in the uncertainty cost lowers ϕ and hence saving. The second term is the direct effect of change in the risk adjusted elasticity of substitution. When $\gamma > 1$, the uncertainty cost is positive and raising risk adjusted substitutability strengthens the desire for early resolution of uncertainty. This channel lowers ϕ and hence saving. Combining these two effects, increasing risk aversion has a negative impact (lowering the saving rate) when risk aversion is relatively high $\gamma > \frac{\sigma+1}{2\sigma}$. When risk aversion is low (high), increasing γ raises (lowers) the preference shifter and contributes towards higher (lower) saving. Thus, for the direct effects, the preference shifter profile has a hump shape with respect to the risk-aversion coefficient. The third term is the indirect effect. We conjecture that consumption and utility volatility declines with higher risk aversion, which leads the last term to exert a positive impact on ϕ when $\gamma\sigma > 1$ and $\gamma > 1$.

The effect of increasing risk aversion on the precautionary component is

$$\frac{\partial \psi}{\partial \gamma} = \underbrace{\left[(\gamma\sigma-1)\mathrm{Var}\,(\ln Z_{t+1}) + \mathrm{Cov}\,(\Delta C_{t+1}, \ln Z_{t+1})\right]}_{\text{Direct}} \tag{17}$$

$$+ \frac{1}{2\sigma}\left\{\frac{\partial \mathrm{Var}\,(\Delta \ln C_{t+1})}{\partial \gamma} + (\gamma\sigma-1)^2\frac{\partial \mathrm{Var}\,(\ln Z_{t+1})}{\partial \gamma} + 2\,(\gamma\sigma-1)\frac{\partial \mathrm{Cov}\,(\Delta \ln C_{t+1}, \ln Z_{t+1})}{\partial \gamma}\right\}. \tag{18}$$

The sign of the direct effect is positive under preference for early resolution of uncertainty $\gamma\sigma > 1$. The predicted profile of the saving rate with respect to the direct effect of risk aversion is either the saving rate rises with γ or that it displays a U shape. The indirect effect (in curly brackets) is negative when $\gamma\sigma > 1$ since the variability of consumption and utility declines with γ.

Combining all effects, we have

$$\frac{\partial E\left(\Delta \ln C_{t+1}\right)}{\partial \gamma} = \underbrace{\left(\frac{\sigma-1}{2}\right) \operatorname{Var}\left(\ln Z_{t+1}\right) + \operatorname{Cov}\left(\Delta C_{t+1}, \ln Z_{t+1}\right)}_{\text{Direct}} \tag{19}$$
$$+ \frac{1}{2\sigma}\left\{\frac{\partial \operatorname{Var}\left(\Delta \ln C_{t+1}\right)}{\partial \gamma} + (\gamma\sigma - 1)(\sigma - 1)\frac{\partial \operatorname{Var}\left(\ln Z_{t+1}\right)}{\partial \gamma} + 2(\gamma\sigma - 1)\frac{\partial \operatorname{Cov}\left(\Delta \ln C_{t+1}, \ln Z_{t+1}\right)}{\partial \gamma}\right\}.$$

The overall direct effect is positive for $\sigma > 1$. The overall indirect effect (in curly brackets) is negative when $\gamma\sigma > 1$ and $\sigma > 1$, but is ambiguous for other values. When $\sigma > 1$, increasing γ raises (lowers) the saving rate when γ is relatively low (high) so that the indirect effect is relatively small or positive (high or negative). Thus the saving rate profile should exhibit a 'hump shape' with respect to γ, and the peak occurs earlier with lower σ. Note, if the volatility of consumption (hence utility) is relatively high so that the positive direct effect always dominates, the saving rate will monotonically rise with γ.

CRRA utility. The separation of intertemporal substitution and risk aversion gives additional richness to saving behavior beyond what is present under CRRA utility where γ regulates both risk aversion and intertemporal substitution. As a comparison, under CRRA utility, differentiating (4) with respect to γ gives

$$\frac{\partial E\left(\Delta \ln C_{t+1}\right)}{\partial \gamma} = \frac{\delta - r}{\gamma^2} + \frac{1}{2}\operatorname{Var}\left(\Delta \ln C_{t+1}\right) + \frac{\gamma}{2}\frac{\partial \operatorname{Var}\left(\Delta \ln C_{t+1}\right)}{\partial \gamma}. \tag{20}$$

The direct effect, given in the first two terms are positive with $\delta > r$. The indirect effect, however has a negative effect. Higher risk aversion (or lower substitutability $\sigma = 1/\gamma$) leads to lower consumption volatility. Under CRRA utility, the saving rate profile should be increasing in γ as long as the indirect effect is not too high.

Rate of time preference. The saving rate decreases with the rate of time preference, δ. Less patient individuals (high δ) place relatively more importance on the present over the future and consume relatively more in the present. The rate of time preference does not directly impact the precautionary component of saving.

Income Shocks σ_n, σ_u, and p. Higher volatility of income shocks increase saving. An increase in the volatility of the income shocks raises the volatility of consumption and utility and hence the precautionary saving component. Volatility also affects the preference shifter. The preference shifter effect is dominated by the precautionary saving effect in all our simulations because volatility is strongly related to precautionary saving.

B.2. Stationarity and Convergence

The exogenous income process has a random walk component, so the model must be transformed to induce stationarity before conducting the simulations. We do this by normalizing variables by permanent income. Let lower case letters denote the normalized variables $c_t = C_t/P_t$, $a_t = A_t/P_t$, and $v_t = V_t/P_t$. Normalizing the budget constraint in this way yields

$$a_{t+1} = (a_t + y_t - c_t)\, e^{(r-g-n_t)}. \tag{21}$$

Similarly, the stationary form of utility is

$$v_t = \left\{ c_t^{\frac{\sigma-1}{\sigma}} + e^{-\delta+\left(\frac{\sigma-1}{\sigma}\right)g} \left[E_t \left(v_{t+1}^{1-\gamma} e^{(1-\gamma)(g+n_{t+1})} \right) \right]^{\frac{\sigma-1}{\sigma(1-\gamma)}} \right\}^{\frac{\sigma}{\sigma-1}}, \tag{22}$$

and the normalized form of the Euler equation is

$$1 = E_t \left\{ e^{r-\delta} \left(\frac{c_{t+1}}{c_t} e^{g+n_{t+1}} \right)^{-\frac{1}{\sigma}} \left[\frac{v_{t+1}}{\left(E_t \left(v_{t+1}^{1-\gamma} e^{(1-\gamma)(g+n_{t+1})} \right) \right)^{\frac{1}{1-\gamma}}} \right]^{\frac{1-\gamma\sigma}{\sigma}} \right\}. \tag{23}$$

Convergence requires that two conditions be jointly satisfied. First, as in Deaton (1991) and Carroll (1997), convergence of the model, from the Euler equation (23), requires

$$\sigma(r-\delta) + \frac{[\gamma(\sigma+1)-1]\,\sigma_n^2}{2} \leq g - \frac{\sigma_n^2}{2}. \tag{24}$$

Clearly, impatience $\delta > r$ helps to achieve stationarity. The left hand side of (24) is increasing in γ. When $\delta > r + \gamma\sigma_n^2/2$, the left hand side is decreasing in σ and the first term helps to achieve stationarity. This condition reduces to Carroll's and Deaton's condition for CRRA utility upon setting $\gamma\sigma = 1$. The second condition comes from the maximized utility function (22), which is given by

$$\sigma(r-\delta) + \frac{[\gamma(\sigma+1)-1]\,\sigma_n^2}{2} + \sigma\left(g - r - \gamma\sigma_n^2\right) < g - \frac{\sigma_n^2}{2}. \tag{25}$$

When income growth is relatively high, $g > r + \gamma\sigma_n^2$, the stationarity condition is governed by the utility function (25). In this case, once the first stationarity condition is satisfied, higher risk aversion helps to achieve stationarity as the left hand side is decreasing in γ. Impatience δ also helps to achieve stationarity. When $g > \delta + \gamma\sigma_n^2/2$, higher intertemporal substitutability raises the left hand side of (25). In this case, the first term may prevent convergence when σ is sufficiently large.

B.3. Derivations

This sub-section contains the derivations of the equations discussed above.

Derivation of the Euler equation (2). Begin with the utility function. If the household is given extra consumption today (dC_t), it lowers tomorrow's assets by $dA_{t+1} = -e^r dC_t$. Exploiting the envelope theorem, if the household is on the optimal path, this infinitesimal reallocation results in no change in welfare. That is,

$$\frac{\partial V_t}{\partial C_t} dC_t = E_t \left(\frac{\partial V_t}{\partial V_{t+1}} \frac{\partial V_{t+1}}{\partial A_{t+1}} \right) (e^r dC_t) \tag{26}$$

where

$$\frac{\partial V_t}{\partial C_t} = \left(1 - e^{-\delta}\right) V_t^{\frac{1}{\sigma}} C_t^{-\frac{1}{\sigma}} \tag{27}$$

$$\frac{\partial V_t}{\partial V_{t+1}} = V_t^{\frac{1}{\sigma}} e^{-\delta} \left[E_t \left(V_{t+1}^{1-\gamma} \right) \right]^{\frac{\gamma\sigma-1}{\sigma(1-\gamma)}} V_{t+1}^{-\gamma}. \tag{28}$$

Since the intertemporal marginal rate of substitution is

$$\text{IMRS}_{t+1} = \frac{E_t \left(\frac{\partial V_t}{\partial V_{t+1}} \frac{\partial V_{t+1}}{\partial A_{t+1}} \right)}{\frac{\partial V_t}{\partial C_t}}, \tag{29}$$

using (26) in (29) gives

$$E \left(\text{IRMRS}_{t+1} e^r \right) = 1.$$

Now the budget constraint gives

$$\frac{\partial A_{t+1}}{\partial A_t} = \left(1 - \frac{\partial C_t}{\partial A_t} \right) e^r. \tag{30}$$

Furthermore,

$$\begin{aligned}
\frac{\partial V_t}{\partial A_t} &= \frac{\partial V_t}{\partial C_t} \frac{\partial C_t}{\partial A_t} + E_t \left[\frac{\partial V_t}{\partial V_{t+1}} \frac{\partial V_{t+1}}{\partial A_{t+1}} \underbrace{\left(\frac{\partial A_{t+1}}{\partial A_t} \right)}_{(30)} \right] \\
&= \frac{\partial V_t}{\partial C_t} \frac{\partial C_t}{\partial A_t} + E_t \left[\frac{\partial V_t}{\partial V_{t+1}} \frac{\partial V_{t+1}}{\partial A_{t+1}} \left(1 - \frac{\partial C_t}{\partial A_t} \right) e^r \right] \\
&= \left\{ \frac{\partial V_t}{\partial C_t} - E_t \left[\frac{\partial V_t}{\partial V_{t+1}} \frac{\partial V_{t+1}}{\partial A_{t+1}} e^r \right] \right\} \frac{\partial C_t}{\partial A_t} + E_t \left[\frac{\partial V_t}{\partial V_{t+1}} \frac{\partial V_{t+1}}{\partial A_{t+1}} e^r \right] \\
&= \frac{\partial V_t}{\partial C_t}
\end{aligned}$$

where the last equality comes from the envelope condition. Notice that the term in braces (by equation (26)) is zero. This means,

$$\frac{\partial V_t}{\partial A_t} = \frac{\partial V_t}{\partial C_t}. \tag{31}$$

Now substitute (31), (27), and (28) into (29) to get

$$\begin{aligned}
\text{IMRS}_{t+1} &= E_t \left(\frac{\frac{\partial V_t}{\partial V_{t+1}} \frac{\partial V_{t+1}}{\partial C_{t+1}}}{\frac{\partial V_t}{\partial C_t}} \right) \\
&= E_t \left\{ \frac{V_t^{\frac{1}{\sigma}} e^{-\delta} \left[E_t \left(V_{t+1}^{1-\gamma} \right) \right]^{\frac{\gamma\sigma-1}{\sigma(1-\gamma)}} V_{t+1}^{-\gamma} V_{t+1}^{\frac{1}{\sigma}} C_{t+1}^{-\frac{1}{\sigma}}}{V_t^{\frac{1}{\sigma}} C_t^{-\frac{1}{\sigma}}} \right\} \\
&= \left\{ e^{-\delta} \left[\frac{V_{t+1}}{\left(E_t \left(V_{t+1}^{1-\gamma} \right) \right)^{\frac{1}{1-\gamma}}} \right]^{\frac{1-\gamma\sigma}{\sigma}} \left(\frac{C_{t+1}}{C_t} \right)^{-\frac{1}{\sigma}} \right\}.
\end{aligned}$$

Using this IMRS to price the asset available to households, gives the Euler equation in the text.

Derivation of log-linearized Euler equation (3) by second-order approximation. That (3) follows from (2) under log-normality of consumption growth and utility is obvious. Here, we show that a second-order approximation around a deterministic steady state gives the same result. To see this, we use the deterministic steady state as the evaluation point:

$$1 = \exp \left\{ r - \delta - \left(\frac{1}{\sigma} \right) g_c + \left(\frac{(1-\gamma\sigma)}{\sigma} \right) \ln \overline{Z} \right\},$$

where $g_c = \overline{\Delta \ln C}$, steady state consumption growth. Note, we do not take normalization of variables here. Hence, the steady state value C grows over time, $g_c > 0$. The second order approximation gives

$$\begin{aligned}
0 = E_t & \left\{ - \left(\frac{1}{\sigma} \right) \left[\Delta \ln \left(C_{t+1} \right) - g_c \right] + \left(\frac{(1-\gamma\sigma)}{\sigma} \right) \ln \left(Z_{t+1} / \bar{Z} \right) \right\} \\
& + \frac{1}{2\sigma^2} E_t \left\{ \left[\Delta \ln \left(C_{t+1} \right) - g_c - (1-\gamma\sigma) \ln \left(Z_{t+1} / \bar{Z} \right) \right]^2 \right\}.
\end{aligned}$$

Using the deterministic steady state condition, $r - \delta - \left(\frac{1}{\sigma} \right) g_c + \left(\frac{1-\gamma\sigma}{\sigma} \right) \ln \overline{Z} = 0$, we have

$$\begin{aligned}
E_t \Delta \ln C_{t+1} &= \sigma (r - \delta) + (1-\gamma\sigma) E_t \ln Z_{t+1} \\
& \quad + \frac{1}{2\sigma} E_t \left\{ \left[\Delta \ln C_{t+1} - (1-\gamma\sigma) \ln \left(Z_{t+1} \right) - \left[g_c - (1-\gamma\sigma) \ln \left(\overline{Z} \right) \right] \right]^2 \right\}.
\end{aligned}$$

Taking unconditional expectations on both sides gives the result.

Property of $\ln Z_{t+1}$. Before proceeding, we state and prove a pair of useful results.

Result 1 *Under log-normality of consumption and utility,*

$$\ln Z_{t+1} = \epsilon_{v,t+1} + \left(\frac{\gamma - 1}{2} \right) \text{Var} \left(\ln v_{t+1} + n_{t+1} \right), \quad (32)$$

$$E \left(\ln Z_{t+1} \right) = \left(\frac{\gamma - 1}{2} \right) \text{Var} \left(\ln v_{t+1} + n_{t+1} \right), \quad (33)$$

$$\text{Var} \left(\ln Z_{t+1} \right) = \text{Var} \left(\ln v_{t+1} + n_{t+1} \right), \quad (34)$$

where $\epsilon_{v,t+1} = \ln v_{t+1} - E_t \left(\ln v_{t+1} \right) + n_{t+1} - \mu_n$ *and* $v_{t+1} = V_{t+1} / P_{t+1}$.

To obtain (33), let $\ln v_{t+1} = \ln(V_{t+1}/P_{t+1})$ be conditionally (on date t information) normally distributed with conditional mean $\mu_{v,t} = E_t(\ln v_{t+1})$ and variance $\omega_{v,t} = \mathrm{Var}_t(\ln v_{t+1}) = \mathrm{Var}(\ln v_{t+1})$ with the assumption of conditional homoskedasticity.[1] Then,

$$\begin{aligned}\ln Z_{t+1} &= \ln V_{t+1} - \frac{1}{1-\gamma}\ln\left[E_t\left(V_{t+1}^{1-\gamma}\right)\right] \\ &= \ln v_{t+1} + n_{t+1} - \frac{1}{1-\gamma}\ln\left[E_t\left(v_{t+1}e^{n_{t+1}}\right)^{1-\gamma}\right] \\ &= \ln v_{t+1} - E_t(\ln v_{t+1}) + n_{t+1} - \mu_n + \left(\frac{\gamma-1}{2}\right)\mathrm{Var}(\ln v_{t+1} + n_{t+1}),\end{aligned} \tag{35}$$

where the last equation uses the lognormality and conditional homoskedasticity. The last equation is (32). Taking expectations on both sides gives (33), and taking variances gives (34).

Property of $E(\ln Z_{t+1})$. With the log-normality assumption, we have

$$E(\ln Z_{t+1}) = \left(\frac{\gamma-1}{2}\right)\mathrm{Var}(\ln Z_{t+1}).$$

Thus, $E(\ln Z_{t+1})$ is positive (negative) for $\gamma > 1$ (< 1), and is increasing in γ.

Without log-normality we have the same qualitative properties. From Jensen's inequality,

$$\begin{aligned}\exp\{E[(1-\gamma)\ln v_{t+1}]\} &\leq E\left(e^{(1-\gamma)\ln v_{t+1}}\right), \\ (1-\gamma)E(\ln v_{t+1}) &\leq \ln E\left(e^{(1-\gamma)\ln v_{t+1}}\right).\end{aligned}$$

This gives

$$(1-\gamma)\left[E(\ln v_{t+1}) - \frac{1}{1-\gamma}\ln E\left(v_{t+1}^{1-\gamma}\right)\right] = (1-\gamma)E_t(\ln Z_{t+1}) \leq 0.$$

Thus, $sgn(E\ln Z_{t+1}) = sgn(\gamma - 1)$. We also have $\frac{\partial E(\ln Z_{t+1})}{\partial\gamma} > 0$, since $\frac{1}{1-\gamma}\ln\left[E_t\left(v_{t+1}^{1-\gamma}\right)\right]$ is decreasing in γ from the property of the generalized mean.

Derivation of (13). Substituting (33) into (7) and differentiating with respect to σ gives (13).

Derivation of (14). The derivation of equation (14) uses the following result.

Result 2 *Let $y(x)$ be a random variable that depends on a parameter x. Then*

$$\frac{\partial \mathrm{Var}[y(x)]}{\partial x} = 2E\left[y(x)\frac{\partial y(x)}{\partial x}\right] - 2E[y(x)]E\left(\frac{\partial y(x)}{\partial(x)}\right). \tag{36}$$

Direct differentiation of $\mathrm{Var}[y(x)] = E\left[y(x)^2\right] - [E(y(x))]^2$ *with respect to x gives the result (36).*

[1] Note that if $\mathrm{Var}_t(\ln v_{t+1}) = \mathrm{Var}(\ln v_{t+1})$ with the assumption of conditional homoskedasticity for $\ln v_{t+1}$, $\mathrm{Var}_t(\ln v_{t+1} + n_{t+1}) = \mathrm{Var}(\ln v_{t+1} + n_{t+1})$.

Using the rule (36) to differentiate ψ with respect to σ gives

$$\frac{\partial \psi}{\partial \sigma} = \underbrace{-\frac{1}{2\sigma^2}\mathrm{Var}\left(\Delta \ln C_{t+1} - (1-\gamma\sigma)\ln Z_{t+1}\right) + \frac{\gamma}{\sigma}\left\{\mathrm{Cov}\left(\Delta \ln C_{t+1}, \ln Z_{t+1}\right) - (1-\gamma\sigma)\mathrm{Var}\left(\ln Z_{t+1}\right)\right\}}_{\text{Direct}}$$

$$+ \underbrace{\frac{1}{2\sigma}\frac{\partial \mathrm{Var}\left(\Delta \ln C_{t+1}\right)}{\partial \sigma} + (\gamma\sigma - 1)^2 \frac{\partial \mathrm{Var}\left(\ln Z_{t+1}\right)}{\partial \sigma} + 2(\gamma\sigma-1)\frac{\partial \mathrm{Cov}\left(\Delta \ln C_{t+1}, \ln Z_{t+1}\right)}{\partial \sigma}}_{\text{Indirect}}.$$

For the direct effects of the first two terms, we have

$$\begin{aligned}
\left.\frac{\partial \psi}{\partial \sigma}\right|_{\text{direct}} &= \frac{1}{2\sigma^2}\left[2\gamma\sigma\mathrm{Cov}\left(\Delta \ln C_{t+1}, \ln v_{t+1}\right) - 2\gamma\sigma(1-\gamma\sigma)\mathrm{Var}\left(\ln v_{t+1}\right)\right. \\
&\quad \left.- \mathrm{Var}\left(\Delta \ln C_{t+1}\right) - (1-\gamma\sigma)^2\mathrm{Var}\left(\ln v_{t+1}\right) + 2(1-\gamma\sigma)\mathrm{Cov}\left(\Delta \ln C_{t+1}, \ln v_{t+1}\right)\right] \\
&= \frac{1}{2\sigma^2}\left[2\mathrm{Cov}\left(\Delta \ln C_{t+1}, \ln v_{t+1}\right) - \mathrm{Var}\left(\Delta \ln C_{t+1}\right) + \left(\gamma^2\sigma^2 - 1\right)\mathrm{Var}\left(\ln v_{t+1}\right)\right] \\
&= \frac{1}{2\sigma^2}\left[(\gamma\sigma)^2\,\mathrm{Var}\left(\ln v_{t+1}\right) - \mathrm{Var}\left(\Delta \ln C_{t+1} - \ln v_{t+1}\right)\right].
\end{aligned}$$

Note also that the direct effect can be rewritten as

$$\frac{1}{2\sigma^2}\left[(\gamma\sigma)^2\,\mathrm{Var}\left(\ln Z_{t+1}\right) - \mathrm{Var}\left(\ln Z_{t+1} - \Delta \ln C_{t+1}\right)\right]$$

which is the form presented in the text.

Derivation of (16). Substitute (33) in (7), and differentiating with respect to γ gives the result.

Derivation of (17). Substitute (33) in (8), and using the rule (36) to differentiate ψ with respect to γ gives

$$\begin{aligned}
\frac{\partial \psi}{\partial \gamma} &= \frac{1}{2\sigma}\left[2\sigma(\gamma\sigma-1)\mathrm{Var}\left(\ln Z_{t+1}\right) + 2\sigma\mathrm{Cov}\left(\Delta C_{t+1}, \ln Z_{t+1}\right)\right] \\
&\quad + \frac{1}{2\sigma}\left[\frac{\partial \mathrm{Var}\left(\Delta \ln C_{t+1}\right)}{\partial \gamma} + (\gamma\sigma-1)^2\frac{\partial \mathrm{Var}\left(\ln Z_{t+1}\right)}{\partial \gamma}\right. \\
&\quad \left.+2(\gamma\sigma-1)\frac{\partial \mathrm{Cov}\left(\Delta \ln C_{t+1}, \ln Z_{t+1}\right)}{\partial \gamma}\right] \\
&= \left[(\gamma\sigma-1)\mathrm{Var}\left(\ln Z_{t+1}\right) + \mathrm{Cov}\left(\Delta C_{t+1}, \ln Z_{t+1}\right)\right] + \frac{1}{2\sigma}\left[\frac{\partial \mathrm{Var}\left(\Delta \ln C_{t+1}\right)}{\partial \gamma}\right. \\
&\quad \left.+ (\gamma\sigma-1)^2\frac{\partial \mathrm{Var}\left(\ln Z_{t+1}\right)}{\partial \gamma} + 2(\gamma\sigma-1)\frac{\partial \mathrm{Cov}\left(\Delta \ln C_{t+1}, \ln Z_{t+1}\right)}{\partial \gamma}\right].
\end{aligned}$$

Derivation of (10). Begin with the budget constraint

$$A_{t+1} = e^r\left(A_t + Y_t - C_t\right),$$

and divide both sides by C_t to get

$$\begin{aligned}
\left(\frac{A_{t+1}}{C_{t+1}}\right)e^{g_{c,t+1}} &= e^r\left(\frac{A_t + Y_t}{C_t} - 1\right) \qquad (37)\\
\left(\frac{A_{t+1}}{C_{t+1}}\right)e^{g_{c,t+1}-r} &= \frac{M_t}{C_t} - 1,
\end{aligned}$$

where $g_{c,t+1} = \ln(C_{t+1}/C_t)$ and $M_t = A_t + Y_t$, cash-at-hand. The saving rate is defined as

$$S_t = 1 - \frac{C_t}{\left(\frac{e^r - 1}{e^r}\right) A_t + Y_t}.$$

Then, we have

$$\frac{1}{1 - S_t} = \frac{A_t + Y_t}{C_t} - e^{-r}\left(\frac{A_t}{C_t}\right).$$

Applying (37), we obtain

$$\begin{aligned}
\frac{S_t}{1 - S_t} &= \left(\frac{A_{t+1}}{C_{t+1}}\right) e^{g_{c,t+1} - r} - e^{-r}\left(\frac{A_t}{C_t}\right) \\
&= \left(\frac{A_{t+1}}{C_{t+1}}\right) e^{g_{c,t+1} - r}\left[1 - \left(\frac{A_t/C_t}{A_{t+1}/C_{t+1}}\right) e^{-g_{c,t+1}}\right] \\
&= \left(\frac{M_t}{C_t} - 1\right)\left[1 - \left(\frac{A_t/C_t}{A_{t+1}/C_{t+1}}\right) e^{-g_{c,t+1}}\right].
\end{aligned}$$

Lastly, taking the steady state and the condition that income growth equals the consumption growth $g_c = g + \mu_n$ gives (10).

Stationary Transformation. Normalize the income process by P_t to get

$$\frac{Y_t}{P_t} = e^{u_t}.$$

Normalizing the budget constraint gives

$$\frac{A_{t+1}}{P_t}\frac{P_{t+1}}{P_t} = e^r\left(\frac{A_t}{P_t} + \frac{Y_t}{P_t} - \frac{C_t}{P_{t-1}}\right).$$

Let lower case denote variables normalized by P_t :

$$a_{t+1} e^{g + n_{t+1}} = e^r\left(a_t + e^{u_t} - c_t\right). \tag{38}$$

Now, normalizing the utility function gives

$$\begin{aligned}
\left(\frac{V_t}{P_t}\right)^{\frac{\sigma-1}{\sigma}} &= \left(\frac{C_t}{P_t}\right)^{\frac{\sigma-1}{\sigma}} + e^{-\delta}\left\{E_t\left[\left(\frac{V_{t+1}}{P_{t+1}}\frac{P_{t+1}}{P_t}\right)^{1-\gamma}\right]\right\}^{\frac{\sigma-1}{(1-\gamma)\sigma}} \\
v_t^{\frac{\sigma-1}{\sigma}} &= c_t^{\frac{\sigma-1}{\sigma}} + e^{-\delta}\left\{E_t\left[\left(v_{t+1} e^{g+n_{t+1}}\right)^{1-\gamma}\right]\right\}^{\frac{\sigma-1}{(1-\gamma)\sigma}} \\
&= c_t^{\frac{\sigma-1}{\sigma}} + e^{-\delta + \left(\frac{\sigma-1}{\sigma}\right)g}\left\{E_t\left[\left(v_{t+1} e^{n_{t+1}}\right)^{1-\gamma}\right]\right\}^{\frac{\sigma-1}{(1-\gamma)\sigma}}.
\end{aligned} \tag{39}$$

Normalizing the Euler equation (2) gives

$$1 = e^{r-\delta} E_t \left\{ \left(\frac{C_{t+1}/P_{t+1}}{C_t/P_t} \left(\frac{P_{t+1}}{P_t} \right) \right)^{-\frac{1}{\sigma}} \left[\frac{V_{t+1}/P_{t+1}}{E\left[\left(\frac{V_{t+1}}{P_{t+1}} \frac{P_{t+1}}{P_t} \right)^{1-\gamma} \right]^{\frac{1}{1-\gamma}}} \left(\frac{P_{t+1}}{P_t} \right) \right]^{\frac{1-\gamma\sigma}{\sigma}} \right\} \tag{40}$$

$$= e^{r-\delta} E_t \left\{ \left(\frac{c_{t+1}}{c_t} \right)^{-\frac{1}{\sigma}} e^{-\frac{1}{\sigma}(g+n_{t+1})} \left[\frac{v_{t+1} e^{g+n_{t+1}}}{E\left[\left(v_{t+1} e^{g+n_{t+1}} \right)^{1-\gamma} \right]^{\frac{1}{1-\gamma}}} \right]^{\frac{1-\gamma\sigma}{\sigma}} \right\}$$

$$= e^{r-\delta-\frac{g}{\sigma}} E_t \left\{ \left(\frac{c_{t+1}}{c_t} \right)^{-\frac{1}{\sigma}} e^{-\gamma n_{t+1}} \left[\frac{v_{t+1}}{E\left[\left(v_{t+1} e^{n_{t+1}} \right)^{1-\gamma} \right]^{\frac{1}{1-\gamma}}} \right]^{\frac{1-\gamma\sigma}{\sigma}} \right\}.$$

Convergence Criterion (25) and (24). The stationarity conditions concern non-exploding a_t and v_t. The borrowing constraint always prevents degeneration of a_t and v_t. If a household builds up enough assets a with stationarity, the household can completely stabilize the consumption by setting $c_t = \overline{c}$, which equals the expected income $E(Y_t/P_t)$ plus the expected annuity $(e^{r-g} - 1) E(a_t)$. Therefore, the stationarity conditions are (i) the normalized utility in (39) does not explode, and (ii) $E(a_t)$ does not explode, which is equivalent to the condition that the right hand side of (40) is less than with $c_t = \overline{c}$. From (39), with $c_t = \overline{c}$, we have constant utility $v_t = \overline{v}$. Applying this to the Euler equation

$$\left(\frac{c}{v} \right)^{\frac{\sigma-1}{\sigma}} = 1 - \exp\left[-\delta + \left(\frac{\sigma-1}{\sigma} \right) g - \frac{\gamma(\sigma-1)\sigma_n^2}{2\sigma} \right].$$

Since $\overline{c} > 0$ and $\overline{v} > 0$, the stationarity condition from the utility function is given by

$$-\sigma\delta + (\sigma - 1) g - \frac{\gamma(\sigma-1)\sigma_n^2}{2} < 0.$$

From (40) with $c_t = \overline{c}$ and $v_t = \overline{v}$ together with the inequality, we have

$$\sigma(r - \delta) - g + \frac{\gamma(1+\sigma)\sigma_n^2}{2} \leq 0.$$

C. Simulated Moments Estimation

Letting $c_{i,t}$ be the logarithm of real consumption expenditures for household i in year t, the four moments we use are

$$h(c_{i,t}) = \begin{pmatrix} h_1(c_{i,t}) \\ h_2(c_{i,t}) \\ h_3(c_{i,t}) \\ h_4(c_{i,t}) \end{pmatrix} = \begin{pmatrix} c_{i,t} \\ \left(\frac{c_{i,t}-\bar{c}}{\bar{c}}\right)^2 \\ \left(\frac{c_{i,t}-\bar{c}}{\bar{s}}\right)^3 \\ \left(\frac{c_{i,t}-\bar{c}}{s}\right)^4 \end{pmatrix},$$

where $\bar{c} = \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T} c_{i,t}$ is the grand sample mean and $s = \sqrt{\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}(c_{i,t}-\bar{c})^2}$ is the sample standard deviation. Recall, that the distribution has been rendered stationary by the transformation described in the previous section. Also, the higher-ordered moments are scaled so that they are all similar in magnitude.

Let $\beta = (\delta, \sigma, \gamma)$ be the parameter vector. For the simulated observations, let

$$h(\beta) = \begin{pmatrix} h_1(\beta) \\ h_2(\beta) \\ h_3(\beta) \\ h_4(\beta) \end{pmatrix} = \begin{pmatrix} c_{i,t}(\beta) \\ \left(\frac{c_{i,t}(\beta)-\bar{c}}{\bar{c}}\right)^2 \\ \left(\frac{c_{i,t}(\beta)-\bar{c}(\beta)}{s}\right)^3 \\ \left(\frac{c_{i,t}(\beta)-\bar{c}(\beta)}{s}\right)^4 \end{pmatrix}.$$

We simulate $N_s = 50000$ individuals over many periods, t. The simulated moments are calculated at $t = 20$.

Note, $\bar{c}$ in $h_2(\beta)$ is the mean computed from the sample, and s in $h_3(\beta)$ and $h_4(\beta)$ is the standard deviation computed from the sample. The idea is to scale the sample and simulated moments with the same scaling factors so that we can treat them like constants.

Now let

$$H(c) = \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T} h(c_{i,t})$$

be the vector of sample moments and

$$H(c(\beta)) = \frac{1}{N_s T_s}\sum_{i=1}^{N_s}\sum_{t=1}^{T_s} h(c_{i,t}(\beta))$$

be the vector of simulated moments and define $u(c_{i,t}) = h(c_{i,t}) - H(c)$. Then, $\Omega_{NT} = \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T} u(c_{i,t})\, u(c_{i,t})'$ is the sample covariance matrix. The simulated method of moments estimator selects the vector β that minimizes

$$g(\beta, c)' W_{NT}^{-1} g(\beta, c),$$

where $g(\beta, c) = H(c) - H(c(\beta))$ and $W_{NT} = \left(1 + \frac{NT}{N_s T_s}\right) \Omega_{NT}$. Asymptotically,

$$\sqrt{NT}\left(\hat{\beta} - \beta_0\right) \xrightarrow{D} N\left(0, V_\beta\right)$$
$$V_\beta = \left(1 + \frac{1}{n}\right)\left(B'\Omega B\right)^{-1},$$

where $n = \lim\left(\frac{NT}{N_s T_s}\right)$ and $B = E\left[\frac{\partial g(\beta, c)}{\partial \beta'}\right]$. Inference is drawn using the sample counterparts.