Modal Logic deals with the modal notions of possibility and necessity. We will have two new logical operators: the box (□) and the diamond (♦), which will mean necessity and possibility respectively. That is, we will have the following:

- □φ “Necessarily, φ” or “It is necessary that φ”
- ♦φ “Possibly, φ” or “It is possible that φ”
We will add one element to our grammar: the box (□). And we will add to definitions of wffs: \(\Box \phi \neg\).

Just as all the logical connectives could be reduced to formulas having \(\sim\) and \(\rightarrow\), so formulas having \(\Diamond\) can be expressed with \(\Box\), if one keeps the following in mind: \(\Diamond \phi \neg\) is short for \(\sim \Box \sim \phi \neg\).

Also, the symbol for strict implication, the fishhook (\(\triangleright\)), can be translated as \(\Box (\phi \rightarrow \psi) \neg\).
Modal Propositional Logic

Symbolizations in MPL

Necessarily, if snow is white, then snow is white or grass is green.
\(\Box (S \rightarrow (S \land G))\)
I’ll go if I must.
\(\Box G \rightarrow G\)
It is possible that Bush will lose the election.
\(\lozenge L\).
It is impossible for snow to be both white and not white.
\(\sim \lozenge (W \land \sim W)\)
Snow is a necessary condition for skiing.
\(\sim W \rightarrow \sim K\)
Thunder implies lightning.
\(T \rightarrow L\)
Problems with the modal operators and our usual translations:

Take Sider’s example, “if Jones is a bachelor, then he must be unmarried.” The surface grammar suggests the following symbolization:

\[ B \rightarrow \Box U \]
But if Jones is in fact a bachelor, what we have written is that he is \textit{necessarily} unmarried. But, in some important sense, Jones could have been unmarried. What we really mean is the following: Necessarily, if Jones is a bachelor, then Jones is unmarried. That is,

\[ \Box(B \rightarrow U) \]

It is the relationship between bachelorhood and being unmarried that is necessary; we are not talking about the essence, as it were, of Jones.
One of the curious aspects of modal logic is that there are a number of different systems of varying degrees of strength with different axioms. The reason for this is that it is unclear which formulas represent logical *truths*.

Moreover, since □ and ♦ are not truth-functional (i.e., they don’t directly affect the truth-value of the proposition to which they are appended), we can’t simply appeal to truth-tables for a representation of the semantics.
Relations
One of the ways to understand necessity and possibility is to say that \( \Box P \) means that \( P \) is true in every possible world and that \( \Diamond P \) means that \( P \) is true in some possible world. Now, there is the metaphysical question of what a possible world is — whether a possible world really exists just like this, the actual, world (modal realism) or whether a (merely) possible (but non actual) world is a like a complete set of propositions (a world-book).
For our purposes, we don’t have to worry about that. Rather, we need to worry about what it means to say that a world is possible for our semantics. Here the important notion is that of accessibility. That is, relative to a given world, \( w_1 \), some state-of-affairs is possible in some other world, \( w_2 \). This means that we need to talk about the ways in which things can stand in relations to one another.
Binary Relations: Let $R$ be any binary relation over some set $A$.

- $R$ is serial (in $A$) iff for every $u \in A$, there is some $v \in A$ such that $Ruv$.
- $R$ is reflexive (in $A$) iff for every $u \in A$, $Ruu$.
- $R$ is symmetric iff for all $u, v$, if $Ruv$ then $Rvu$.
- $R$ is transitive iff for all $u, v, w$, if $Ruv$ and $Rvw$, then $Ruw$.
- $R$ is an equivalence relation (in $A$) iff $R$ is symmetric, transitive, and reflexive (in $A$).
- $R$ is total (in $A$) iff for every $u, v \in A$, $Ruv$. 
Possible-worlds semantics
This follows up on the notion expressed above that \( \Box \phi \) means \( \phi \) is true iff \( \phi \) is true in all possible worlds and \( \Diamond \phi \) is true iff \( \phi \) is true in some possible world.
More precisely. We will talk about \( \phi \)'s truth “at” a possible world. And we will say that \( \Box \phi \) counts as true at a world, \( w_1 \), iff \( \phi \) is true at every possible world that is “accessible” from \( w_1 \).
**Definition of a Model:** An MPL-model is an ordered triple, \( \langle W, R, I \rangle \), where:

- \( W \) is a non-empty set of objects ("possible worlds")
- \( R \) is a binary relation over \( W \) ("accessibility relation")
- \( I \) is a two-place function that assigns a 0 or 1 to each sentence letter, relative to ("at" or "in") each world — that is, for any sentence letter \( \alpha \), and any \( w \in W \), \( I(\alpha, w) \) is either 0 or 1.

\( \langle W, R \rangle \) is often called the model’s *frame*
Definition of Valuation: The important thing to be added is the notion of a valuation at a world. And our valuation of the necessity operator will be this:

\[ V_M(\Box \phi, w) = 1 \text{ iff for each } v \in W, \text{ if } Rwv, \text{ then } V_M(\phi, v) = 1. \]

The idea is (again) \( \Box \phi \) is true iff it is true in every world to which it is accessible. (When we write “\( Rwv \)” think “\( w \) sees \( v \)”.)
Definition of Model for Systems of MPL: An “S-model”, for any of our systems S, is defined as an MPL-model $\langle W, R, I \rangle$ whose accessibility relation $R$ has the formal feature given for system S in the following chart:

- **System** accessibility relation must be
- **K** no requirement
- **D** serial (in $W$)
- **T** reflexive (in $W$)
- **B** reflexive (in $W$) and symmetric
- **S4** reflexive (in $W$) and transitive
- **S5** reflexive (in $W$), symmetric, and transitive
Definition of Validity in an MPL Model: A formula $\phi$ is valid in model $\mathcal{M} (= \langle W, R, I \rangle)$ iff for every $w \in W$, $V_{\mathcal{M}}(\phi, w) = 1$

Definition of Validity and Semantic Consequence for Modal Systems:

- A formula is valid in system $S$ iff it is valid in every $S$-model
- Formula $\phi$ is a semantic consequence in system $S$ of set of wffs $\Gamma$ iff for every $S$-model $\langle W, R, I \rangle$ and each $w \in W$, if $V_{\mathcal{M}}(\gamma, w) = 1$ for each $\gamma \in \Gamma$, then $V_{\mathcal{M}}(\phi, w) = 1$