Propositional Logic

Grammar
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Following Sider, we will just have the following **Primitive Vocabulary**:

- **Connectives**: $\rightarrow$, $\sim$
- **Sentence letters**: $P, Q, R, \ldots$ with or without numerical subscripts
- **Parentheses**: (, )
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- **Parentheses:** (,)
We will have the following *Definition of a wff*:

(i) Every sentence letter is a wff. (That is, each sentence letter is an atomic sentence.)

(ii) If $\phi$ and $\psi$ are wffs, then $\neg(\phi \to \psi)$ and $\neg\neg\phi$ are also wffs.

(iii) Only strings that can be shown to be wffs using (i) and (ii) are wffs.
We will have the following *Definition of a wff*: 

(i) Every sentence letter is a wff. (That is, each sentence letter is an atomic sentence.)

(ii) If \( \phi \) and \( \psi \) are wffs, then \( \neg (\phi \rightarrow \psi) \) and \( \neg \sim \phi \) are also wffs.

(iii) Only strings that can be shown to be wffs using (i) and (ii) are wffs.

Note: in our language, PL, we will no longer have the logical connectives \( \lor, \land, \) and \( \leftrightarrow \). Why not? Because we can *define* those connectives in terms of \( \sim \) and \( \rightarrow \).
Redefining the other logical connectives in terms of \( \sim \) and \( \rightarrow \):

\[
\begin{array}{ccc}
\phi & \psi & \neg(\phi \land \psi) \quad \neg(\phi \rightarrow \neg \psi) \\
T & T & T \\
T & F & F \\
F & T & F \\
F & F & F \\
\end{array}
\]

\[
\begin{array}{ccc}
\phi & \psi & \neg(\phi \lor \psi) \quad \neg(\neg \phi \rightarrow \psi) \\
T & T & T \\
T & F & T \\
F & T & T \\
F & F & F \\
\end{array}
\]
\[ \phi \quad \psi \quad \neg (\phi \leftrightarrow \psi) \quad \neg (\neg ((\phi \rightarrow \psi) \rightarrow (\psi \rightarrow \phi))) \]

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<th>(\phi)</th>
<th>(\psi)</th>
<th>(\neg (\phi \leftrightarrow \psi))</th>
<th>(\neg (\neg ((\phi \rightarrow \psi) \rightarrow (\psi \rightarrow \phi))))</th>
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Note: we will only use the logical connectives \( \sim \) and \( \rightarrow \) because it makes metalogic easier. But we could also have used just \( \sim \) and \( \lor \) or \( \sim \) and \( \land \).

In fact, as we will see, there are two cases of logical connectives that are in themselves sufficient to ‘translate’ our standard set of logical connectives.
Propositional Logic
The semantic approach to logic

Let’s look at the way Sider introduces this in 2.2: “A semantics for a language is a way of assigning meanings to words and sentences of that language. The central concept to be used in characterizing meaning will be that of truth . . . Philosophers disagree over how to understand the notion of meaning in general. But the idea that the meaning of a sentence has something to do with truth-conditions is hard to deny, and at any rate has currency within logic. On this approach, one explains the meaning of a sentence by showing how that sentence depends for its truth or falsity on the way the world is.” (pp.22-23)
Some of the philosophy behind this. Consider the following:

- “Schnee ist weiß” means “Snow is white”.
- “Snow is white” is true iff snow is white.
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What’s going on here? A German-speaker means by his sentence what an English-speaker means by his: namely, that snow is white. That is, there is a fact of the matter about the way the world is, and this — *mirabile dictu!* — has something to do with truth and meaning.
Now, consider this:

➤ The slithy toves did gyre and gimble in the wabe.

This looks meaningful but isn’t. Why not? One answer: ‘toves’ and ‘wabe’ (at the very least) do not refer to anything in the world; nor is ‘gimble’ any kind of action.
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Suppose, however, we agree to rename nostril hairs ’toves’; ’gimbling’ is just swirling around; and a ’wabe’ is a rush of mucous. So, when I blow my nose . . . You get the idea.
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The point is, we have now a certain interpretation that allows us to assign a truth-value to some event or fact of the matter.
This isn’t exactly what interpretation is going to mean for us, however. Rather, for us, an interpretation will consist in the following:

1. A **domain** — that is, some collection of objects. (The domain always has at least one object)

2. A **name** for each object in the domain. (Objects could have more than one name.)

3. A list of **predicates**. (Though we’ll talk about predicates more next chapter.)

4. A specification of the objects of which each predicate is true and of the objects of which each predicate is false. (E.g. the predicate ‘is a student’ is true of each of you; ‘blond’ is false of me.)

5. An interpretation may include an atomic sentence letter, in which case the interpretation specifies a truth value of that sentence letter.
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Semantics of propositional logic

In introductory logic, you were exposed to truth tables as a way of understanding the truth values associated with the various logical connectives. And you were told how to use truth tables in your evaluations of logical equivalence, validity, and so on.

Truth tables are *pictorial* representations of combinations of truth values.

Now, we will give a metalinguistic account of the truth values to be assigned for different possible combinations of truth values for our atomic sentences.
**Definition of Interpretation:** A PL-interpretation is a function $I$, that assigns either 1 or 0 to every sentence letter.

Nothing mysterious here: $1 = \text{True}; 0 = \text{False}$.

And, for $\neg(\phi \rightarrow \psi)$, we can represent the truth values in the following way:

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<td>0</td>
<td>1</td>
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Definition of Valuation: For any PL-interpretation, \( \mathcal{I} \), the PL-valuation for \( \mathcal{I} \), \( V_{\mathcal{I}} \), is defined as the function that assigns to each wff either 1 or 0, and which is such that, for any sentence letter \( \alpha \) and any wffs \( \phi \) and \( \psi \):

\[
V_{\mathcal{I}}(\alpha) = I(\alpha)
\]
\[
V_{\mathcal{I}}(\phi \rightarrow \psi) = 1 \text{ iff either } V_{\mathcal{I}}(\phi) = 0 \text{ or } V_{\mathcal{I}}(\psi) = 1
\]
\[
V_{\mathcal{I}}(\sim \phi) = 1 \text{ iff } V_{\mathcal{I}}(\phi) = 0
\]
Some more definitions from Sider:

**Definition of Validity:** A formula $\phi$ is PL-valid iff every PL-interpretation, $\mathcal{I}$, $V_\mathcal{I}(\phi) = 1$.

We write "$|=_{PL} \phi$" for "$\phi$ is PL-valid".

**Definition of Semantic Consequence:** Formula $\phi$ is a PL-semantic consequence of a set of formulas $\Gamma$ iff for every PL-interpretation, $\mathcal{I}$, if $V_\mathcal{I}(\gamma) = 1$ for each $\gamma$ in $\Gamma$, then $V_\mathcal{I}(\phi) = 1$.

Look closely: what is being said is that $\phi$ is a consequence just in case if all the premises are true, then it is true.
Sequent proofs in propositional logic

In introductory logic, you learned how to prove logical formulas are true or consequences of other formulas by means of natural deduction or the tree method. Now, we are going to use a different approach: *sequent proofs.*
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A sequent looks like this:

\[ \Gamma \Rightarrow \phi \]

This simply means that \( \phi \) is a conclusion or consequence of the premise \( \Gamma \), where \( \Gamma \) is to be thought of as a set of premises, \( \{ \gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_n \} \).
As with natural deduction, there are *rules* that allow us to make certain inferences or that allow us to go from one sequent or set of sequents to another.
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These sequent rules are supposed to represent natural language sequent rules that preserve logical correctness. (That is, if the ‘from’ logical sequents are true, then the ‘to’ logical sequents ought to be true, too.)
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**Definition of a Sequent Proof:** A sequent proof is a series of sequents, each of which is either of the form $\phi \Rightarrow \phi$, or follows from earlier sequents in the series by some sequent rule.
Some additional examples:

Show that $Q, (P \land Q) \rightarrow R \Rightarrow P \rightarrow R$

1. $(P \land Q) \rightarrow R \Rightarrow (P \land Q) \rightarrow R$  RA
2. $P \Rightarrow P$  RA
3. $Q \Rightarrow Q$  RA
4. $P, Q \Rightarrow P \land Q$  $2,3 \land I$
5. $P, Q, (P \land Q) \rightarrow R \Rightarrow R$  $1,4 \rightarrow E$
6. $Q, (P \land Q) \rightarrow R \Rightarrow P \rightarrow R$  $2,5 \rightarrow I$
Show that $\sim (P \land Q), P \rightarrow Q \Rightarrow \sim P$

1. $\sim (P \land Q) \Rightarrow \sim (P \land Q)$  
   RA
2. $P \rightarrow Q \Rightarrow P \rightarrow Q$  
   RA
3. $P \Rightarrow P$  
   RA
4. $P, P \rightarrow Q \Rightarrow Q$  
   2,3 $\rightarrow$ E
5. $P, P \rightarrow Q \Rightarrow P \land Q$  
   3,4 $\land$ I
6. $P, \sim (P \land Q), P \rightarrow Q \Rightarrow (P \land Q) \land \sim (P \land Q)$  
   1,5 $\land$ I
7. $\sim (P \land Q), P \rightarrow Q \Rightarrow \sim P$  
   6 RAA