Predicate Logic

In what we’ve discussed thus far, we haven’t addressed other kinds of valid inferences: those involving quantification and predication. For example:

All philosophers are wise
Socrates is a philosopher
∴ Socrates is wise

All psychiatrists are doctors
All doctors are college graduates
∴ All psychiatrists are college graduates
All Epicureans are charming bon vivants
Some philosophers are Epicureans
∴ Some philosophers are charming bon vivants
These arguments, of course, can be symbolized in the following way:

\[
\forall x (Px \rightarrow Wx) \\
P_s \\
\therefore W_s
\]

\[
\forall x (Px \rightarrow Dx) \\
\forall x (Dx \rightarrow Cx) \\
\therefore \forall x (Px \rightarrow Cx)
\]
∀x(Ex → Cx)
∃x(Px ∧ Ex)

∴ ∃x(Ex ∧ Cx)
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Grammar of Predicate Logic

**Primitive Vocabulary**

- Connectives: $\rightarrow$, $\sim$, and $\forall$
- Variables $x, y \ldots$, with or without subscripts
- For each $n > 0$, $n$-place predicates $F, G \ldots$, with or without subscripts
- Individual constants (names) $a, b \ldots$, with or without parentheses

Just as all the truth functional connectives could be expressed by $\sim$ and $\rightarrow$, so $\exists$ can be expressed by what we have in our vocabulary, for $\exists x Fx$ is equivalent to $\sim \forall x \sim Fx$. 
Definition of wff

(i) If $\Pi$ is an $n$-placed predicate and $\alpha_1 \ldots \alpha_n$ are terms, then $\Pi\alpha_1 \ldots \alpha_n$ is a wff

(ii) If $\phi$, $\psi$ are wffs, and $\alpha$ is a variable, then $\sim \alpha$, $(\phi \rightarrow \psi)$, and $\forall \alpha \phi$ are wffs

(iii) Only strings that can be shown to be wffs using (i) and (ii) are wffs

Quine: “To be assumed as an entity is, purely and simply, to be reckoned as the value of a variable.” (“On What There Is”)
Variables are either free or bound. A variable is free iff it does not belong to a quantifier; a variable is bound iff it belongs to a quantifier. That is, a variable is bound if it is within the scope of a quantifier.

For example, consider the following wffs:

1. $Fx$
2. $\forall xFx$

The occurrence of “$x$” in (1) is free; the second occurrence of “$x$” in (2) is bound.

When a formula has no free occurrences of variables, it is “closed”; otherwise it is an open formula.
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Semantics of Predicate Logic

**Definition of a Model** A PC-model is an order pair $\langle D, I \rangle$ such that:

- $D$ is a non-empty set ("the domain")
- $I$ is a function ("the interpretation function") obeying the following constraints:
  - if $\alpha$ is a constant, then $I(\alpha) \in D$
  - if $\Pi$ is an $n$-place predicate, then $I(\Pi) = \text{some set of } n\text{-tuples of members of } D$
Models represent configurations of the world.

\(N\)-place predicates are assigned to sets of \(n\)-tuples drawn from \(\mathcal{D}\). This set is the \textit{extension} of the predicate of the model.

Consider (again) Tarski’s World.
Our notion of truth will now be “truth in a model” — or truth relative to a model $\langle D, I \rangle$. And we will want a valuation function that assigns truth values to our propositions. But it is a little more complicated than that in predicate logic.

Think first of the statement $Fa$, i.e., “$a$ is $F$”. What we are saying is that there is some object of our domain that is among the set of $F$ things; that is, that $a$ is some object, call it “$u$”, and there is a subset, call it “$S$”, of the domain to which we assign the predicate $F$. So, if $Fa$ is true, $u \in S$. Or, in saying $Fa$ is true, we mean that $I(a) \in I(F)$. 
Definition of a Variable Assignment: $g$ is a variable assignment for model $\langle D, I \rangle$ iff $g$ is a function that assigns to each variable some object in $D$.

And *this* in turn means that $g$ is a *function* that assigns variables to objects in some domain. Per Sider, we will define “$g^\alpha_u$” as the variable assignment that assigns $u$ to $\alpha$. Hence, $g^\alpha_u(\alpha) = u$. 
The *denotation* of a variable \( \alpha \) relative to a model \( M(= \langle D, I \rangle) \) is defined as follows:

\[
[\alpha]_{M,g} = \begin{cases} 
I(\alpha) & \text{if } \alpha \text{ is a constant} \\
g(\alpha) & \text{if } \alpha \text{ is a variable}
\end{cases}
\]
Definition of Valuation: The PC valuation function, $V_{\mathcal{M},g}$, for PC-model $\mathcal{M} = \langle \mathcal{D}, \mathcal{I} \rangle$ and variable assignment $g$, is defined as the function that assigns to each wff either 0 to 1 subject to the following constraints:

1. for any $n$-place predicate $\Pi$ and any terms $\alpha_1 \ldots \alpha_n$,
   $$V_{\mathcal{M},g}(\Pi \alpha_1 \ldots \alpha_n) = 1 \text{ iff } \left\langle [\alpha_1]_{\mathcal{M},g} \ldots [\alpha_n]_{\mathcal{M},g} \right\rangle \in \mathcal{I}(\Pi)$$

2. for any wffs $\phi, \psi$, and any variable $\alpha$:
   2.1 $V_{\mathcal{M},g}(\sim \phi) = 1 \text{ iff } V_{\mathcal{M},g}(\phi) = 0$
   2.2 $V_{\mathcal{M},g}(\phi \rightarrow \psi) = 1 \text{ iff } V_{\mathcal{M},g}(\phi) = 0 \text{ or } V_{\mathcal{M},g}(\psi) = 1$
   2.3 $V_{\mathcal{M},g}(\forall \alpha \phi) = 1 \text{ iff for every } u \in \mathcal{D}, V_{\mathcal{M},g_u}^\alpha(\phi) = 1$
Definition of Truth in a Model: \( \phi \) is true in PC model \( M \) iff \( V_M^g(\phi) = 1 \), for each variable assignment \( g \) for \( M \).

Definition of Validity: \( \phi \) is PC-valid (\( \models_{PC} \phi \)) iff \( \phi \) is true in all PC-models.
Definition of Semantic Consequence: \( \phi \) is a PC-semantic consequence of a set of wffs \( \Gamma \) ("\( \Gamma \models_{PC} \phi \") iff for every PC-model \( \mathcal{M} \) and every variable assignment \( g \) for \( \mathcal{M} \), if \( V_{\mathcal{M},g}(\gamma) = 1 \) for each \( \gamma \in \Gamma \), then \( V_{\mathcal{M},g}(\phi) = 1 \).
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Establishing Validity and Invalidity

Let’s do one of Sider’s exercises:

We are asked to show that $\vdash \forall x (Fx \rightarrow (Fx \lor Gx))$.

1. So, suppose that $V_g(\forall x (Fx \rightarrow (Fx \lor Gx))) = 0$ (for some assignment in some model).

2. Given (1), for some $v \in \mathcal{D}$, $V_{g^v}(Fx \rightarrow (Fx \lor Gx)) = 0$.

3. Call one such $v$ “$u$”. So, $V_{g^u}(Fx \rightarrow (Fx \lor Gx)) = 0$.

4. Given (3), $V_{g^u}(Fx \rightarrow (Fx \lor Gx)) = 0$ iff $V_{g^u}(Fx) = 1$ and $V_{g^u}(Fx \lor Gx) = 0$.

5. $V_{g^u}(Fx) = 1$ iff $\langle [x]g^u \rangle \in \mathcal{I}(F)$.

6. But $V_{g^u}(Fx \lor Gx)) = 0$ iff $V_{g^u}(Fx) = 0$ and $V_{g^u}(Gx) = 0$.

7. And $V_{g^u}(Fx) = 0$ iff $\langle [x]g^u \rangle \notin \mathcal{I}(F)$, which contradicts (5).

$\blacksquare$
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Axiomatic Proofs in PC

Axiomatic System for PC:

- **Rules:** MP plus *Universal Generalization* (UG):
  \[
  \frac{\phi}{\forall \alpha \phi} \quad \text{UG}
  \]

- **Axioms:** PL1-PL3, plus:
  
  - **PC1** \( \forall \alpha \phi \rightarrow \phi(\beta/\alpha) \)
  - **PC2** \( \forall \alpha (\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \forall \alpha \psi) \)

Conditions:

- \( \phi, \psi, \) and \( \chi \) are any wffs of predicate logic, \( \alpha \) is any variable, and \( \beta \) is any term

- \( \phi(\beta/\alpha) \) results from \( \phi \) by “correct substitution” of \( \beta \) for \( \alpha \)

- in PC2, no occurrences of variable \( \alpha \) may be free in \( \phi \).
So, let’s do some problems. Prove
\[ \forall x (F_x \rightarrow G_x), \forall x (G_x \rightarrow H_x) \vdash_{PC} \forall x (F_x \rightarrow H_x) \]

1. \[ \forall x (F_x \rightarrow G_x) \]
   Premise
2. \[ \forall x (G_x \rightarrow H_x) \]
   Premise
3. \[ \forall x (F_x \rightarrow G_x) \rightarrow (F_a \rightarrow G_a) \]
   PC1
4. \[ \forall x (G_x \rightarrow H_x) \rightarrow (G_a \rightarrow H_a) \]
   PC1
5. \[ F_a \rightarrow G_a \]
   1,3 MP
6. \[ G_a \rightarrow H_a \]
   2,4 MP
7. \[ F_a \rightarrow H_a \]
   5,6 HS
8. \[ \forall x (F_x \rightarrow H_x) \]
   7, UG

The move from 7-8 via UG is justified because \(a\) does not occur in the premises, nor does it occur in \[ \forall x (F_x \rightarrow H_x) \].
Or the relatively easy proof of $\vdash_{PC} Fa \rightarrow \exists xFx$

1. $\forall x \sim Fx \rightarrow \sim Fa$ \hspace{10pt} PC1
2. $Fa \rightarrow \sim \forall x \sim Fx$ \hspace{10pt} PL
3. $Fa \rightarrow \exists xFx$ \hspace{10pt} Def. of $\exists$
Here is another justification for what Sider calls Distribution:

\[ \vdash_{PC} \forall x(\phi \to \psi) \to (\forall x\phi \to \forall x\psi) \].

1. \( \forall x(\phi \to \psi) \)  
   \hspace{1cm} \text{Premise}
2. \( \forall x\phi \)  
   \hspace{1cm} \text{Premise}
3. \( \forall x(\phi \to \psi) \to (\phi \to \psi) \)  
   \hspace{1cm} \text{PC1}
4. \( \phi \to \psi \)  
   \hspace{1cm} 1,3 \text{ MP}
5. \( \forall x\phi \to \phi \)  
   \hspace{1cm} \text{PC1}
6. \( \phi \)  
   \hspace{1cm} 2,5 \text{ MP}
7. \( \psi \)  
   \hspace{1cm} 4,6 \text{ MP}
8. \( \forall x\psi \)  
   \hspace{1cm} 7, \text{ UG}
9. \( \forall x(\phi \to \psi), \forall x\phi \vdash \forall x\psi \)  
   \hspace{1cm} 1-8
10. \( \forall x(\phi \to \psi) \vdash \forall x\phi \to \forall x\psi \)  
    \hspace{1cm} 9, \text{ DT}
11. \( \forall x(\phi \to \psi) \to (\forall x\phi \to \forall x\psi) \)  
    \hspace{1cm} 10, \text{ DT}
Soundness and Completeness:
As in the system of Propositional Logic, it can be shown that
\( \Gamma \vdash_{PC} \phi \) iff \( \Gamma \models_{PC} \phi \).

Compactness Theorem:
Let \( \Gamma \) be a set of wffs of language \( L \). If for each finite subset of \( \Gamma \) there is a first-order structure making this subset of \( \Gamma \) true, then there is a first-order structure \( \mathcal{M} \) that makes all the sentences true. (As Sider mentions, this is intuitively surprising, because it means that there is a true model even when \( \Gamma \) is infinite.)