Part I: Axiomatic Systems

In addition to the axiomatic system from Chapter 2 of Sider there is one from Frege and Łukasiewicz. It also consists of three axioms, which we shall designate “CL1-CL3” (for classical propositional logic axiomatic system). CL1 and CL2 are identical to PL1 and PL2, but CL3 differs from PL3. The axioms are the following:

\begin{align*}
\text{CL1} & \quad \Gamma \phi \rightarrow (\psi \rightarrow \phi)^	op \\
\text{CL2} & \quad \Gamma (\phi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \chi))^	op \\
\text{CL3} & \quad \Gamma (\sim \phi \rightarrow \sim \psi) \rightarrow (\psi \rightarrow \phi)^	op
\end{align*}

And we also have our rule of inference MP (Modus Ponens): \( \Gamma \phi \rightarrow \psi, \phi \vdash \psi \). 

Assignment: go step-by-step through the following. Some of this will be familiar, some not. Do each derivation using only the axioms and the rules that you have already derived or been given. That is, don’t use the Deduction Theorem in the somewhat slippery we did before (e.g., in (4)) and forget about Sider’s toolkit. If you can’t get one, go to the next and assume the previous axiom or rule. (40 pts.)

1. Derive first CLD1: \( \Gamma \phi \rightarrow \phi \). 

2. Now derive CLD2: \( \Gamma (\psi \rightarrow \chi) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \chi)) \). 

3. Show that with CLD2 and MP, our rule of HS (Hypothetical Syllogism) is justified, that is, \( \Gamma \phi \rightarrow \psi, \psi \rightarrow \chi \vdash \phi \rightarrow \chi \). \(^1\)

\(^1\)Here and below ((4) and (8)), you should have a derivation that looks something like this:

\begin{align*}
1. & \quad \phi \rightarrow \psi \quad \text{Premise} \\
2. & \quad \psi \rightarrow \chi \quad \text{Premise} \\
\vdots & \quad \text{blah} \quad \text{Some axiom or rule} \\
\vdots & \quad \text{blah} \quad \text{Some axiom or rule} \\
n. & \quad \phi \rightarrow \chi \quad \text{Some axiom or rule}
\end{align*}
(4) Next, derive the rule Trans (Transposition): \( \Gamma \phi \rightarrow (\psi \rightarrow \chi) \vdash \psi \rightarrow (\phi \rightarrow \chi)^\updownarrow \).

(5) Now, derive the axiom schema CLD3: \( \Gamma \sim \sim \phi \rightarrow \phi^\updownarrow \).

(6) And CLD4: \( \Gamma \phi \rightarrow \sim \sim \phi^\updownarrow \).

(7) Now, using only our axioms and derived rules, derive CLD5: \( \Gamma (\phi \rightarrow \psi) \rightarrow (\sim \psi \rightarrow \sim \phi)^\updownarrow \) which is, of course, the converse of CL3.

(8) Finally, use CLD5 to show that we are justified in our standard rule of MT (Modus Tollens): \( \Gamma \phi \rightarrow \psi, \sim \psi \vdash \sim \phi \).

Part II: Soundness and Completeness

In no more than 250 words, explain the results and the importance of the soundness and completeness proofs. You should not use any logical or metalogical symbols, but you should feel free to employ metalogical concepts, such as, for example, “semantic consequence” and “syntactic consequence”. (20 pts.)

Part III: Many-Valued Logic

(1) Using truth tables, show that the following pairs of formulas are equivalent in both \( L_3 \) (Łukasiewicz’s trivalent system) and \( K^S_3 \) (Kleene’s strong trivalent system), so that the second formula can be used as a definition of the first in either system. (10 pts.)

(a) \( P \land Q \sim (\sim P \lor \sim Q) \)

(b) \( P \leftrightarrow Q \sim (P \rightarrow Q) \land (Q \rightarrow P) \)

(2) Using truth tables, show that the following formulas are not equivalent in \( L_3 \), so that the second cannot be used as a definition of the first in that system. (5 pts.)

\( P \rightarrow Q \sim (P \land \sim Q) \)

(3) Are the above formulas (from (2)) equivalent in \( K^S_3 \). Show this. (5 pts.)
(4) Determine whether the following hold in the systems $L_3$, $K_3^S$, and $LP$. Demonstrate your answers using either truth tables or brief argument.

(20 pts.)

(a) $P \rightarrow R \not\vdash (P \land Q) \rightarrow R$

(b) $P \rightarrow Q, Q \rightarrow R \vdash P \rightarrow R$

(c) $\vdash P \rightarrow (Q \lor \sim Q)$

(d) $\vdash (P \land \sim P) \rightarrow Q$