10

The Cobb-Douglas Production Function

This chapter describes in detail the most famous of all production functions used to represent production processes both in and out of agriculture. First used in 1928 in an empirical study dealing with the productivity of capital and labor in the United States, the function has been widely used in agricultural studies because of its simplicity. However, the function is not an adequate numerical representation of the neoclassical three stage production function. One of the key characteristics of a Cobb Douglas type of production function is that the specific corresponding dual cost function can be derived by making use of the first order optimization conditions along the expansion path. Examples of constrained output or revenue maximization problems using a Cobb Douglas type of function are included.

Key terms and definitions:

Cobb Douglas Production Function
True Cobb Douglas
Base 10 Logarithm
Base *e* Logarithm
Cobb Douglas Type of Function
Technology and the Parameter *A*
Homogeneity
Partial Elasticities of Production
Function Coefficient
Total Elasticity of Production
Asymptotic Isoquants
Three-Dimensional Surface
Duality of Cost and Production
Cost Elasticity
Finite Solution

10.1 Introduction

The paper describing the Cobb Douglas production function was published in the journal *American Economic Review* in 1928. The original article dealt with an early empirical effort to estimate the comparative productivity of capital versus labor within the United States.

Since the publication of the article in 1928, the term *Cobb Douglas* production function has been used to refer to nearly any simple multiplicative production function. The original production function contained only two inputs, capital (*K*) and labor (*L*). Moreover, the function was assumed to be homogeneous of degree 1 in capital and labor, or constant returns to scale.

Economists of this period, while recognizing that the law of diminishing returns (or the law of variable proportions) applied when units of a variable factor were added to units of a fixed factor, were fascinated with the possibility of constant returns to scale, when all factors of production were increased or decreased proportionately. They probably believed that as the scale of the operation changed, it was no longer possible to divide inputs into the categories fixed and variable. In the long run, the marginal product of the bundle of inputs that comprise the resources or factors of production for the society should be proportionate to the change in the size of the bundle, or the amount of resources available to the society.

There were other constraints in 1928. Econometrics, the science of estimating economic relationships using statistics, was only in its infancy. The function had to be very simple to estimate. The lack of computers and even pocket calculators meant that at most, statistical work had to be done on a mechanical calculator. Estimates of parameters of the function derived from the data had to be possible within the constraints imposed by the calculation tools of the 1920s.

10.2 The Original Cobb Douglas Function

The function proposed in the 1928 article was

†10.1 $y = Ax_1^{\alpha} x_2^{1-\alpha}$

where x_1 = labor

x_2 = capital

The function had three characteristics viewed at that time as desirable

1. *It was homogeneous of degree 1 with respect to the input bundle, which was consistent with the economics of the day that stressed that production functions for a society should have constant returns to scale*.

2. *The function exhibited diminishing marginal returns to either capital or labor, when the other was treated as the fixed input, so the law of variable proportions held*. The parameter A was thought to represent the technology of the society that generated the observations upon which the parameters of the function were to be estimated.

3. *The function was easily estimated with the tools of the day*. Both sides of the function could be transformed to logarithms in base 10 or natural logarithms in base *e* (2.71828...)

†10.2 $\log y = \log A + \alpha \log x_1 + (1 - \alpha) \log x_2$

The resulting equation is referred to as *linear in the parameters* or *linear in the coefficients*. In other words, $\log y$ is a linear function of $\log x_1$ and $\log x_2$. The transformed function is the equation for a simple two variable regression line in which all observations in the data set used for estimating the regression line have been transformed into base 10 or natural (base e = 2.71828..) logarithms:

†10.3 $$\log y = b_o + b_1 \log x_1 + b_2 \log x_2 + \,,$$

where

$A = e^{b_o}$ if the transformation is to the natural logarithms, or

10^{b_o} if the transformation is to base 10 logarithm

$b_1 = "$

$b_2 = 1\ !\ "$

, = regression error term

There was no point in empirically estimating b_2 if the assumption was made that the parameters on capital and labor summed to 1. The function could be estimated with only one input or independent variable. Cobb ! and Douglas estimated the parameter on labor using regression analysis and saved their statistical clerks a lot of work on the mechanical calculator.

It is important to recognize that the Cobb Douglas production function, when originally proposed, was not intended to be a perfect representation for the United States of the technical relationships governing the transformation of labor and capital into output. Rather, it was chosen because it retained the two key economic assumptions of the day (diminishing returns to each input and constant returns to scale) and because its parameters were easy to obtain from actual data.

The Cobb Douglas function had economic properties clearly superior to what was the probable alternative of the day, a simple linear function with constant marginal products for both inputs

†10.4 $$y = ax_1 + bx_2$$

As will be seen shortly, the Cobb Douglas function lacked many features characteristic of the three stage production function proposed by the neoclassical economists, which was graphically developed earlier. Had Cobb and Douglas perceived the massive impact of their early work on both economists and agricultural economists, they perhaps would have come up with something more complicated and sophisticated. Part of the appeal of the function rested with its utter simplicity in estimation. Agricultural economists today use only slightly modified versions of the Cobb Douglas production function for much the same reasons that the function was originally developed-it is simple to estimate but allows for diminishing marginal returns to each input.

10.3 Early Generalizations

The first generalization of the Cobb Douglas production function was to allow the parameters on the inputs to sum to a number other than 1, allowing for returns to scale of something other than 1. The function

†10.5 $$y = Ax_1^{\beta_1}x_2^{\beta_2}$$

where $\beta_1 + \beta_2$ sum to any number,

is sometimes referred to as a Cobb Douglas type of production function, but it is not the true Cobb Douglas function. This function was also readily transformed to logs. Parameters could still be estimated by least squares regression with two inputs or explanatory variables, and with the advent of the computer, this could be done very easily.

As the use of the function moved from the problem of estimating the relationships between capital, labor, and output at the society level to problems of representing production processes at the individual farm level, the interpretation of some of the parameters changed. Cobb and Douglas assumed that output could be produced with only capital and labor. At the farm firm level, x_1 and x_2 more likely represent two variable inputs that are under the control of the manager. The remaining inputs are treated as fixed. The parameter A might be thought of as the combined impact of these fixed factors on the production function. In this context

†10.6 $$A = \Pi\, x_i^{\beta_i}$$

For all $i = 3, ..., n$

In equation †10.6 , there are n inputs, with all but $n - 2$ being treated as fixed. Technology could have an impact on the magnitude of the β_i themselves. The parameters β_1 and β_2 might be expected to sum to a number substantially less than 1, particularly if there are a large number of fixed inputs contained in the parameter A. Thus, a restriction that forced the coefficients on the variable inputs to sum to 1 would be silly.

The second generalization was to expand the function in terms of the number of inputs. The four input expansion is

†10.7 $$y = Ax_1^{\beta_1}x_2^{\beta_2}x_3^{\beta_3}x_4^{\beta_4}$$

A function of the general form of equation †10.7 with any number of inputs was readily transformed to logs, and the parameters were empirically estimated from appropriate data using ordinary least squares regression techniques. As the number of inputs treated as variable expanded, the sum of the parameters on the variable inputs should also increase, assuming that each variable input has a positive marginal product.

In this text, the term Cobb Douglas function or true Cobb Douglas function is used only in reference to the two-input multiplicative function in which the sum of the individual production elasticities is equal to 1. The term *Cobb Douglas type of function* is used in reference to a multiplicative function where the elasticities of production sum to a number other than 1, or in a case where there are more than two inputs or factors of production.

10.4 Some Characteristics of the Cobb Douglas Type of Function

The Cobb Douglas type of function is homogeneous of degree $\Sigma\beta_i$. The returns to scale parameter or function coefficient is equal to the sum of the β values on the individual inputs, assuming that all inputs are treated explicitly as variable. The β values represents the elasticity of production with respect to the corresponding input and are constants.

The partial elasticities of production for each input are simply the β *parameters for the input.* This can easily be shown. The partial elasticity of production for input x_i is the ratio of *MPP* to *APP* for that input. The *MPP* for input x_i is

†10.8 $$MPP_{x_i} = \beta_i A x_i^{\beta_i - 1} \Pi x_j^{\beta_j}$$

for all $j \neq i = 1, ..., n$ where n is the number of inputs. The *APP* for input x_i is

†10.9 $$APP_{x_i} = A x_i^{\beta_i - 1} \Pi x_j^{\beta_j}$$

The ratio of *MPP* to *APP* for the ith input is β_i. Hence, the elasticities of production for the Cobb Douglas type of production function are constant irrespective of the amounts of each input that are used. The ratio of *MPP* to *APP* is constant, which is very unlike the neoclassical three-stage production function.

Moreover, MPP *and* APP *for each input never intersect, but stay at the fixed ratio relative to each other as determined by the partial elasticity of production.* The only exception is an instance where the partial production elasticity is exactly equal to 1 for one of the inputs. If this were the case, the *MPP* and the *APP* for that input would be the same everywhere irrespective of how much of that input were used.

All inputs must be used for output to be produced. Since the Cobb Douglas function is multiplicative, the absence of any one input will result in no total output, even if other inputs are readily available. This characteristic may not be extremely important when there are but a few categories of highly aggregated inputs, but if there are a large number of input categories, this characteristic may be of some concern, since it is unlikely that every input would be used in the production of each commodity.

There is no finite output maximum at a finite level of input use. The function increases up the expansion path at a rate that corresponds to the value of the function coefficient. If the function coefficient is 1, the function increases at a constant rate up the expansion path. If the function coefficient is greater than 1, the function increases at an increasing rate. If the function coefficient is less than 1, the function increases at a decreasing rate. Agricultural production functions of the Cobb Douglas type when estimated usually have function coefficients of less than 1.

For a given set of parameters, the function can represent only one stage of production for each input, and ridge lines do not exist. If the elasticities of production are for each input less than 1, the function will depict stage II everywhere.

If the function coefficient is less than 1, there will normally be a point of global profit maximization at a finite level of input use. Pseudo scale lines exist and will intersect on the expansion path at this finite level.

10.5 Isoquants for the Cobb Douglas Type of Function

The Cobb Douglas type of production function, as given by

†10.10 $$y = A x_1^{\beta_1} x_2^{\beta_2}$$

has the corresponding marginal products

†10.11 $$MPP_{x_1} = \partial y/\partial x_1 = \beta_1 x_1^{\beta_1 - 1} x_2^{\beta_2}$$

†10.12 $$MPP_{x_2} = \partial y/\partial x_2 = \beta_2 x_1^{\beta_1} x_2^{\beta_2 - 1}$$

The $MRS_{x_1x_2}$ is obtained by finding the negative ratio of MPP_{x_1}/MPP_{x_2}.

†10.13 $$MRS_{x_1x_2} = -(\beta_1 x_2)/(\beta_2 x_1)$$

The *MRS* is a linear function of the input ratio x_2/x_1.

The equation for an isoquant is obtained by fixing the output of y at some constant level y° and solving for x_2 in terms of x_1

†10.14 $$y^\circ = Ax_1^{\beta_1}x_2^{\beta_2}$$

†10.15 $$x_2^{\beta_2} = y^\circ/(Ax_1^{\beta_1})$$

†10.16 $$x_2 = [y^\circ/(Ax_1^{\beta_1})]^{1/\beta_2}$$

†10.17 $$x_2 = y^{\circ(1/\beta_2)} A^{-1/\beta_2} x_1^{-\beta_1/\beta_2}$$

†10.18 $$dx_2/dx_1 = -(\beta_1/\beta_2)y^{\circ(1/\beta_2)}A^{-1/\beta_2} x_1^{(-\beta_1/\beta_2)-1} < 0$$

The isoquants for a Cobb Douglas type of production function have a downward slope as long as the individual production elasticities are positive. This is true irrespective of the values of β_1 and β_2.

Moreover,

†10.19 $$d^2x_2/dx_1^2 = [(-\beta_1/\beta_2) - 1][-\beta_1/\beta_2]y^{\circ(1/\beta_2)}A^{-1/\beta_2}x_1^{(-\beta_1/\beta_2)-2} > 0$$

if the individual production elasticities are positive.

The sign on equation †10.19 indicates that the isoquants for the Cobb−Douglas type of production function are asymptotic to the x_1 and x_2 axes irrespective of the values of the partial production elasticities, as long as the partial production elasticities are positive. Isoquants for a Cobb Douglas type of function are illustrated in Figure 10.1, with Y1 - Y7 indicating various specific output levels represented by each isoquant. Although these isoquants appear to be rectangular hyperbolas, their position relative to the x_1 and x_2 axes will depend upon the relative magnitudes of β_1 and β_2. The isoquant will be positioned closer to the axis of the input with the larger elasticity of production.

To reemphasize, the general shape of the isoquants for a Cobb Douglas type of function are not conditional on the values of the individual production elasticities. As long as the individual production elasticities are greater than zero, the isoquants will always be downward sloping, convex to the origin of the graph, and asymptotic to the axes. The convexity of the isoquants for the function occurs because of the diminishing marginal rate of substitution and because the function is multiplicative, not additive, resulting in a synergistic influence on output when inputs are used in combination with each other. That is, output is the product of that attributed to each input, not the sum of that attributed to each input.

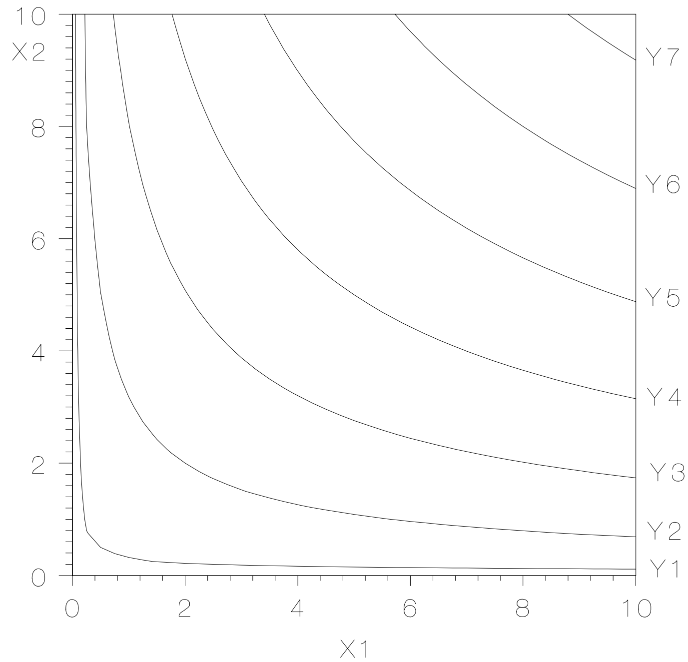


Figure 10.1 Isoquants for the Cobb-Douglas Production Function

The expansion path generated by a Cobb Douglas function in the x_1 and x_2 plane has a constant slope equal to $(v_2/v_1)(\$_1/\$_2)$. The expansion path is obtained by setting the $MRS_{x_1x_2}$ equal to the inverse price ratio

†10.20 $$MRS_{x_1x_2} = (\$_1 x_2)/(\$_2 x_1) = v_1/v_2$$

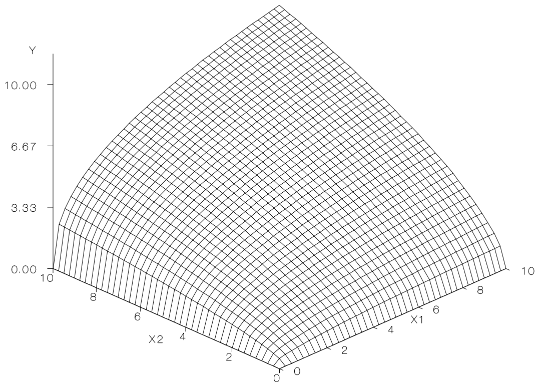
†10.21 $$\$_2 x_1 v_1 = \$_1 x_2 v_2$$

†10.22 $$\$_2 x_1 v_1 \; ! \; \$_1 x_2 v_2 = 0$$

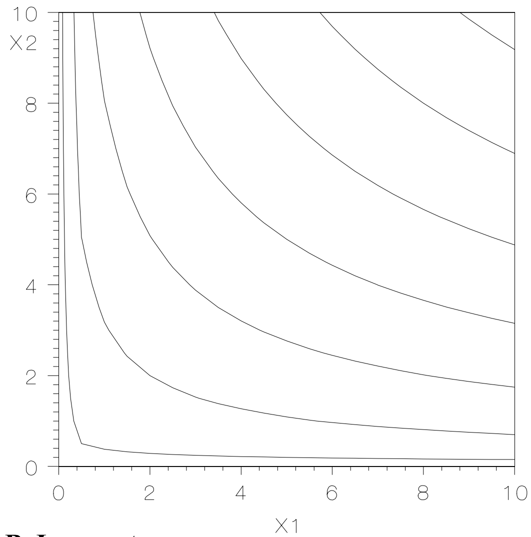
†10.23 $$x_2 = (v_1/v_2)(\$_2/\$_1)x_1$$

10.6 The Production Surface of the Cobb Douglas Production Function

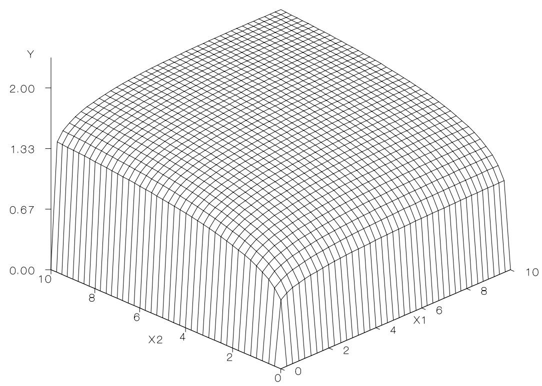
Figure 10.2 illustrates the three dimensional surface of the Cobb Douglas type of production function with two inputs, under varying assumptions with respect to the values of the $\$$ coefficients. Depending on the specific coefficients, the production surface for the Cobb Douglas type of function can vary rather dramatically. Diagrams A and B illustrate the surface and isoquants for the case in which the parameters on the two inputs sum to 1. In this illustration, $\$_1$ is 0.4 and $\$_2$ is 0.6. Each line on the diagram represents a production function for one of the inputs holding the other input constant. Production functions for x_2 begin at the x_1 axis. Production functions for x_1 begin at the x_2 axis. Since x_2 is the more productive input, production functions for x_2 have a steeper slope than do the production functions for x_1. Now move along an imaginary diagonal line midway between the x_1 and x_2 axes. The production surface directly above this imaginary diagonal line has a constant slope. The slope of the surface above this line represents the function coefficient or returns-to-scale parameter of 1 for this two input Cobb Douglas production function which has constant returns to scale.



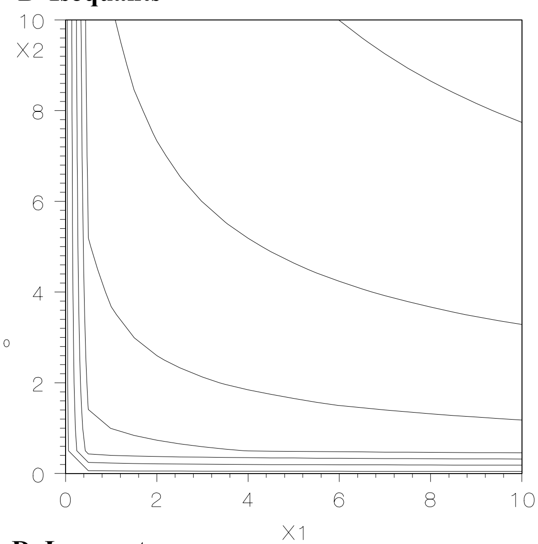
A Surface $y = x_1^{0.4} x_2^{0.6}$



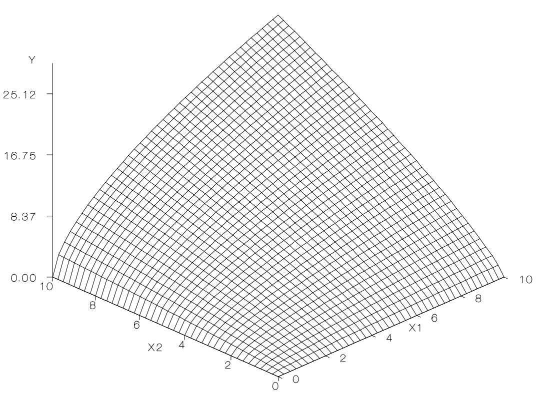
B Isoquants

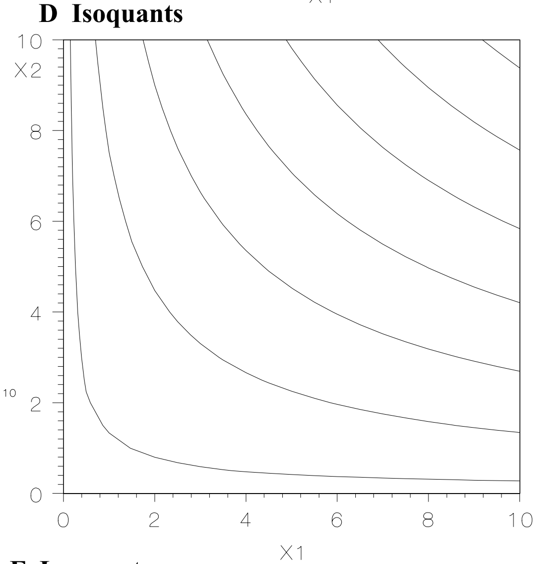


C Surface $y = x_1^{0.1} x_2^{0.2}$



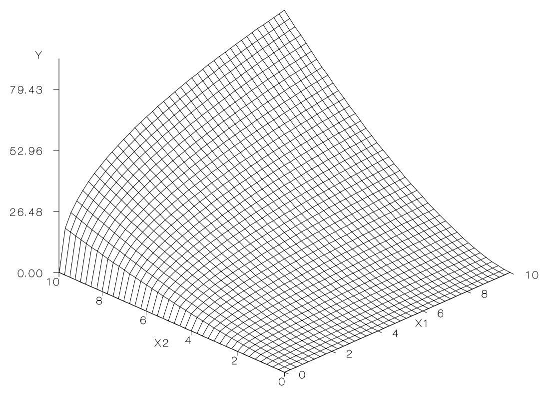
D Isoquants



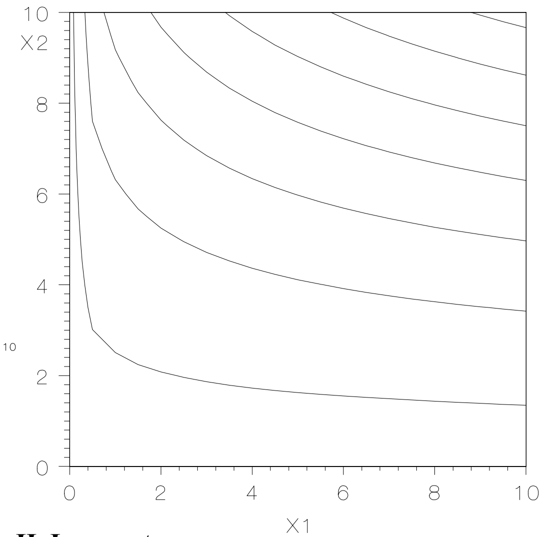


F Isoquants

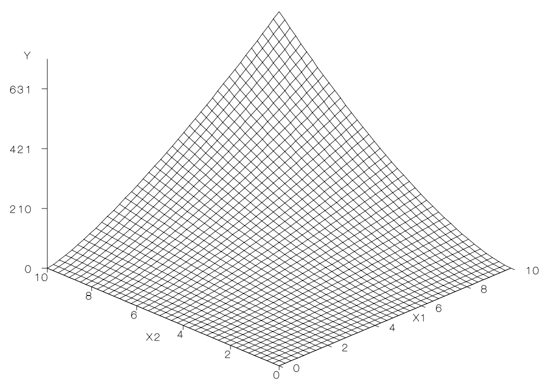
E Surface $y = x_1^{0.6} x_2^{0.8}$



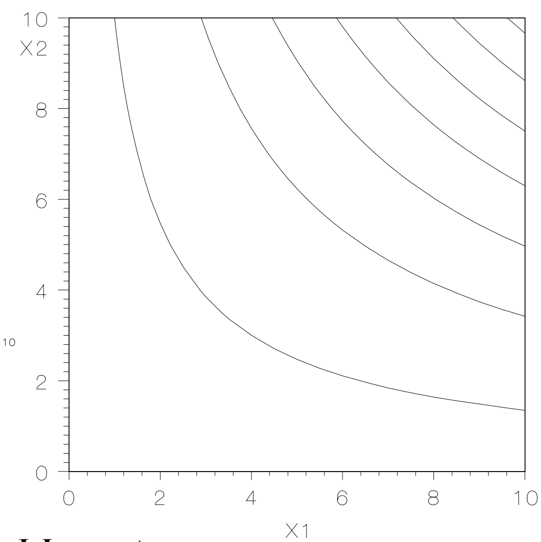
G Surface $y = x_1^{0.4} x_2^{1.5}$



H Isoquants



I Surface $y = x_1^{1.3} x_2^{1.5}$



J Isoquants

Figure 10.2. Surfaces and Isoquants for a Cobb-Douglas Type Production Function

Diagrams C and D illustrate the surface and isoquants of a Cobb Douglas type of production function in which the elasticities of production sum to a number smaller than 1. In this example, $\$_1$ is 0.1, and $\$_2$ is 0.2. The function is homogeneous of degree 0.3 and has a function coefficient of 0.3. Again, x_2 is the more productive input, as verified by its larger elasticity of production.

Production functions for x_2 are found by starting upward at the x_1 axis, and production functions for x_1 are found by following a line upward from the x_2 axis. Input x_2 has the production functions with the steepest slope. The marginal product of each input appears to be very great at small values for x_1 and x_2, but drops off rapidly as input use is increased. The output (y) produced by the function for a specific quantity of x_1 and x_2 is much smaller than for the function illustrated in diagram A. The production surface above the imaginary diagonal line is concave from below. The function coefficient of .3 indicates that the marginal product of incremental units of a bundle of x_1 and x_2 is declining.

Diagrams E and F illustrate a Cobb Douglas type of production function in which the individual elasticities of production for each input are less than 1 but the elasticities of production sum to a number greater than 1. In this example, $\$_1$ is 0.6 and $\$_2$ is 0.8, yielding a function coefficient of 1.4. The marginal product for individual production functions is declining, but the marginal product for the bundle along the imaginary diagonal line is increasing. This implies that the production surface above this imaginary scale line is convex, not concave from below.

Diagrams G and H illustrate a Cobb Douglas type of production function in which one input has an elasticity of production greater than 1, but the elasticity of production for the other input is less than 1. In this example, $\$_1$ is 0.4 and $\$_2$ is 1.5. Starting at the x_1 axis, follow a line representing the production function for x_2. Note that this production function is curving upward, or increasing at an increasing rate, and the marginal product of input x_2 is increasing as the level of x_2 is increased. But the production functions for x_1, which start at the x_2 axis, have a declining marginal product, as evidenced by the fact that they increase at a decreasing rate. The production surface above the imaginary diagonal line is convex from below. The marginal product of the input bundle defined by the diagonal line increases as the size of the bundle increases, and the function coefficient is 1.9.

Diagrams I and J illustrate a Cobb Douglas type of production function in which both inputs have elasticities of production of greater than 1. In this example, $\$_1$ is 1.3 and $\$_2$ is 1.5 yielding a total elasticity of production or function coefficient of 2.8. That the marginal product of both x_1 and x_2 is increasing is clearly evident from a careful examination of individual production functions in diagram E. The production surface above the imaginary diagonal line representing the input bundle is clearly convex from below.

10.7 Profit Maximization with the Cobb Douglas Function

Regardless of the values for the elasticities of production, a multiplicative production function of the Cobb Douglas type never achieves an output maximum for a finite level of x_1 and x_2. Upon learning that the first order conditions of a maximum are achieved by setting the first partial derivatives of a function equal to zero, students are sometimes tempted to try this with a Cobb Douglas type of function. Unless the elasticities of production for each input are zero, (in which case, increases in each input produce no additional output, since any number raised to the zero power is 1) the only way for these first-order conditions to hold is for no input to be used, and if that were the case, there would also be no output. A Cobb Douglas type of function has no finite maximum where the ridge lines would intersect. This is not surprising, since the ridge lines do not exist, and any point on any isoquant for a Cobb Douglas type of function, irrespective of the parameter values, will have a negative slope.

Profit maximization is possible only if the function coefficient is less than 1, assuming that the purely competitive model holds, with constant input and output prices. In the purely competitive model, the price of the output is a constant p, total revenue is *py* and the total value of the product (*TVP*) is

†10.24 $$TVP = px_1^{\$_1} x_2^{\$_2}$$

The function coefficient or total elasticity of production for the production function indicates the responsiveness of output to changes in the size of the input bundle. It is the percentage change in output divided by the percentage change in the size of the input bundle. Assuming a constant output price, the function coefficient also represents the responsiveness of total value of the product to changes in the size of the input bundle. It is the percentage change in the total value of the product or output divided by the percentage change in the input bundle.

With constant input prices, the marginal cost of acquiring an additional unit of the input bundle along the expansion path is also constant, not decreasing or increasing. If the function coefficient is greater than 1, each additional unit of the bundle produces more and additional output and total value of the product. Incremental units of the bundle can be obtained at a fixed constant price per unit. The manager would be better off in terms of increased revenue by acquiring more and more additional units of the bundle. This process could occur indefinitely, and both input use and output would be expanded up to the point where the purely competitive assumptions with regard to both input and output prices are no longer met. Either the manager is producing so much output that it can no longer be sold at the going market price, or so much input is being purchased that more cannot be bought without causing the price of the input bundle to increase.

Now consider a case in which each input has a production elasticity of less than 1 but the total elasticity of production or function coefficient is exactly 1. If this case, a 1 percent increase in the size of the bundle is accompanied by a 1 percent rise in revenue. The marginal cost of an additional unit of the bundle is a constant V. The revenue from the use of an additional unit of the bundle is the revenue generated from an additional unit of output, or p. The manager would attempt to equate the marginal cost of the bundle (V) with the marginal revenue from the additional unit of output (p). Both numbers are constants. If $p > V$, the manager could make additional profits by expanding use of the input bundle indefinitely, which is the same solution as the first case. Total profit is (p ! V)y where y is the number of units of output produced. If p were less than V, then the manager should shut down since each and all incremental units of output cost more that they generated in additional revenue. The total loss is (p ! V)y, where y is the number of units of output produced. Only if p were equal to V would the manager be indifferent to producing or shutting down. Each incremental unit of output cost exactly what it returned, so there would be zero profit everywhere.

Finally, consider a case in which the sum of the individual elasticities of production is less than 1. Again the price of the input bundle is treated as a constant, but in this case the value of the marginal product is declining. Profits can be maximized at some finite level of use of the input bundle, assuming that for certain levels of input use, *TVP* does exceed the cost of the bundle.

Another way of looking at the profit maximization conditions for the Cobb Douglas type of function is with the aid of the pseudo scale lines developed earlier. Assume constant input and output prices. If the function coefficient is less than 1, the pseudo scale lines exist and converge at some finite level of input use along the expansion path. The convergence of the pseudo scale lines represents the global point of profit maximization. If the function coefficient is equal to 1, the pseudo scale lines exist but diverge from each other, so that they do not intersect for any finite level of use of the input bundle along the expansion path. If the function coefficient is greater than 1 but the individual elasticities of production are less than 1, the pseudo scale line exist but diverge, going farther and farther apart as the use of the bundle is expanded. A pseudo scale line for an individual input does not exist if the elasticity of production for that input is greater than or equal to 1.

10.8 Duality and the Cobb Douglas Function

The Cobb Douglas type of function is homogeneous, and its corresponding dual cost function exists. It is possible to derive the specific cost function in terms of output for a Cobb ! Douglas type of production function. Assume the production function

†10.25 $$y = Ax_1^{\$_1} x_2^{\$_2}$$

The input cost function is

†10.26 $C = v_1 x_1 + v_2 x_2$

The dual cost function for a Cobb Douglas type of production function is found using the following procedure. First, the equation for the expansion path is found by partially differentiating the production function with respect to x_1 and x_2, to find the marginal products. The negative ratio of the marginal products is the $MRS_{x_1 x_2}$. This is equated to the inverse input price ratio. The result can be written as

†10.27 $\$_2 v_1 x_1 = \$_1 v_2 x_2$

Equation †10.27 defines the points of least cost combination along the expansion path.

Equation †10.27 is solved for x_1 to yield

†10.28 $x_1 = \$_1 v_2 x_2 \$_2^{!1} v_1^{!1}$

Equation †10.28 is inserted into equation †10.26 and x_2 is factored out

†10.29 $C = x_2(\$_1 v_2 \$_2^{!1} + v_2)$

Equation †10.29 defines the quantity of x_2 that is used in terms of cost (C) and the parameters of the production function

†10.30 $x_2 = C/(\$_1 v_2 \$_2^{!1} + v_2)$

Similarly, for input x_1,

†10.31 $x_1 = C/(\$_2 v_1 \$_1^{!1} + v_1)$

Inputs x_1 and x_2 are now defined totally in terms of cost C, the input prices (v_1 and v_2) and the parameters of the production function. Inserting equations †10.30 and †10.31 into the original production function [equation †10.25] and rearranging, results in

†10.32 $y = C^{\$_1+\$_2} A(\$_2 v_1 \$_1^{!1} + v_1)^{!\$_1} (\$_1 v_2 \$_2^{!1} + v_2)^{!\$_2}$

Solving equation †10.32 for C in terms of y, the production function parameters and the input prices yields

†10.33 $C = y^{1/(\$_1+\$_2)} A^{!1/(\$_1+\$_2)} (\$_1^{!1}\$_2 v_1 + v_1)^{\$_1/(\$_1+\$_2)} (\$_2^{!1}\$_1 v_2 + v_2)^{\$_2/(\$_1+\$_2)}$

or $C = y^{1/(\$_1+\$_2)} Z$

where

$$Z = A^{!1/(\$_1+\$_2)} (\$_1^{!1}\$_2 v_1 + v_1)^{\$_1/(\$_1+\$_2)} (\$_2^{!1}\$_1 v_2 + v_2)^{\$_2/(\$_1+\$_2)}$$

Notice that y is raised to the power 1 over the degree of homogeneity of the original production function. The value of Z is a constant, since it is dependent only on the assumed constant prices of the inputs and the assumed constant parameters of the production function. Notice that prices for inputs are available, all of the information needed to obtain the corresponding dual cost function can be obtained from the production function. The coefficients or parameters of a Cobb Douglas type of production function uniquely define a corresponding dual cost function. C is cost in terms of output.

Marginal cost is

†10.34 $$MC = dC/dy = [1/(\beta_1+\beta_2)]y^{1/(\beta_1+\beta_2)-1}Z$$

The slope of *MC* is positive if the sum of the individual partial production elasticities or function coefficient is less than 1. If the individual production elasticities sum to a number greater than 1, then *MC* is declining. *MC* has a zero slope when the production elasticities sum exactly to 1. The supply function for a firm with a Cobb Douglas type of production function can be found by equating marginal cost (equation †10.34) with marginal revenue or the price of the product and solving the resultant equation for y.

Average cost is

†10.35 $$AC = TC/y = y^{1/(\beta_1+\beta_2)-1}Z$$

Since Z is positive, average cost decreases when the partial production elasticities sum to a number greater than 1. Average cost increases if the partial production elasticities sum to a number less than 1. If the production function is a true Cobb-Douglas, total cost is given by

†10.36 $$TC = yZ$$

Both marginal and average cost are given by the constant Z, and therefore both *MC* and *AC* have a zero slope. For a Cobb-Douglas type of production function, *MC* and *AC* never intersect, except in the instance where the function coefficient (or the cost elasticity) is 1, in which case *MC* and *AC* are the same everywhere.

The ratio of marginal to average cost or the cost elasticity (ρ) is

†10.37 $$(\rho) = 1/(\beta_1 + \beta_2)$$

$$= 1/E,$$

where E is the returns-to-scale parameter, or function coefficient.

If total product along the expansion path is increasing at a decreasing rate, then costs are increasing at an increasing rate. If total product along the expansion path is increasing at an increasing rate, than costs are increasing at a decreasing rate. If total product along the expansion path is increasing at a constant rate (the true Cobb Douglas function), then costs are also increasing at a constant rate. If the product sells for a fixed price, that price is a constant marginal revenue (*MR*). Marginal revenue (*MR*) can be equated to marginal cost (*MC*) only if *MC* is increasing. With fixed input prices and elasticities of production, this can happen only if the cost elasticity is greater than 1, which means that the function coefficient for the underlying production function is strictly less than 1.

The profit function can be written as

†10.38 $$\pi = TR - TC$$

†10.39 $$\pi = py - Zy^{1/E}$$

where E is the function coefficient.

Maximum profits occur if

†10.40 $$d\Pi/dy = p - Z\,(1/E)y^{(1/E)-1} = 0$$

$$MR - MC = 0$$

and

†10.41 $$d^2\Pi/dy^2 = -Z\,(1/E)[(1/E)-1]y^{(1/E)-2} < 0$$

E is positive. The only way the second derivative can be negative is for E to be smaller than 1. This implies that MC is increasing. If E is equal to 1, the second derivative of the profit function is zero, and that MC is constant. If E is greater than 1, the second derivative of the profit function is positive, and MC is decreasing.

10.9 Constrained Output or Revenue Maximization

A finite solution to the problem of globally maximizing profits could be found only in those instances where the production function had a function coefficient of less than 1. The same conditions do not hold for the problem of finding the least cost combination of inputs required to produce a particular level of output or revenue. The isoquants generated by a Cobb−Douglas type of production function are convex to the origin if the partial elasticities of production are positive, and as a result, points of tangency that meet second order conditions are easy to find. For example, suppose that the production function is

†10.42 $$y = x_1x_2$$

The individual partial elasticities of production for each input is 1, and the function coefficient is 2. Despite its strange appearance, this is a production function of the Cobb Douglas type.

Suppose that the price of both x_1 and x_2 is \$1 per unit. The Lagrangean would be

†10.43 $$L = x_1x_2 + \lambda(C^\circ - 1x_1 - 1x_2)$$

With the corresponding first order conditions

†10.44 $$\partial L/\partial x_1 = x_2 - 1\lambda = 0$$

†10.45 $$\partial L/\partial x_2 = x_1 - 1\lambda = 0$$

†10.46 $$\partial/\partial\lambda = C^\circ - 1x_1 - 1x_2 = 0$$

The second order conditions require that the determinant of the following matrix be positive

†10.47 $$\begin{matrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{matrix}$$

The determinant of †10.47 is 2, which is clearly positive, thus meeting the second order conditions for a constrained output maximization. Despite the fact that the production function in equation †10.42 meets both first- and second-order conditions for a constrained revenue maximization, there is no assurance that revenue less costs will be positive when the point of least-cost combination is found.

10.10 Concluding Comments

The Cobb Douglas type of production function has been estimated by agricultural economists for virtually any production process involving the transformation of inputs into outputs in an agricultural setting. Economists have used a Cobb Douglas type of specification for virtually every conceivable type of production process. To review specific applications of the Cobb Douglas type of function would be to review a large share of the literature in which empirical attempts have been made to estimate production functions. Some of this literature is cited in the reading list.

The appeal of the Cobb Douglas type of function rests largely with its simplicity. Even when the Cobb Douglas form is not used as the final form of the function, it is often used as a benchmark specification for comparison with other functional forms. The null research hypothesis might be that the production function is of the Cobb Douglas type. The alternative hypothesis is that another specification provides a better fit to the data.

Cobb and Douglas never intended that the Cobb Douglas production function represent the subtle details of the three-stage production function of the neoclassical economists. However, the elegant simplicity of the algebra surrounding the Cobb Douglas type of production function seems to appeal to both economists and agricultural economists alike. Never mind that the relationships were not always as the neoclassical economists had proposed.

The neoclassical three-stage production function was a marvelous invention. However, as subsequent chapters will show, the three stage production function as originally conceived is not always the easiest thing to represent with mathematics. The problem becomes especially difficult as extensions are made to multiple input categories. Agricultural economists use the Cobb ! Douglas specification for no better reason than that the algebra is simplified.

Problems and Exercises

1. For a Cobb Douglas type of function

$$y = Ax_1^{"}x_2^{\$}$$

For each case, does there exist the following?

a. A global point of output maximization.

b. A global point of profit maximization (assume constant input and output prices).

c. A series of points of constrained output maximization.

Case	Value for *A*	Value for "	Value for $
(1)	1.0	0.2	0.3
(2)	1.0	0.4	0.6
(3)	1.0	0.6	0.8
(4)	1.0	1.0	1.0
(5)	1.0	2.0	2.0
(6)	1.0	! 0.3	0.5
(7)	! 1.0	0.4	0.6

2. For each case outlined above, find *MPP* and *APP* for each input, holding the other input constant at some predetermined level. What is the relationship between *MPP* and *APP* in each case?

3. Suppose that the production function is

$$y = x_1x_2$$

The input x_1 sells for \$1 per unit and input x_2 sells for \$2 per unit. The farmer has \$200 to spend on x_1 and x_2. How much of each input will the farmer purchase in order to be at a point of constrained output maximization?

4. Making certain that the scale on both the x_1 and the x_2 axes is the same, draw a graph for an isoquant generated by the function

$$y = x_1^{0.5}x_2^{0.33}$$

Assume that the length of each axis represents 10 units of input use. Is the isoquant closer to the x_1 axis or the x_2 axis? Why?

5. Assume that the production function is

$$y = x_1^{0.5}x_2^{0.33}$$

x_1 costs \$1 per unit; x_2 costs \$2 per unit. Find the corresponding total cost function with total cost expressed as a function of output (y), the input prices, and the production function parameters.

Reference

Cobb, Charles W. and Paul H. Douglas. "A Theory of Production." *American Economic Review* 18: Suppl.(1928) pp 139 ! 156.