11

Other Agricultural Production Functions

In addition to the Cobb Douglas, agricultural economists have made use of a diverse array of other functional forms. The earliest efforts to develop production functions from agricultural data predate the Cobb Douglas work, using a production function developed by Spillman. The transcendental production function represented an attempt conducted in the 1950s to develop a specification closely tied to the characteristics of the neoclassical three-stage production function. Production functions with variable rather than constant input elasticities represented a development during the 1960s. In the early 1970s de Janvry showed that the Cobb Douglas function with either fixed or variable input elasticities and the transcendental production functions were all members of a family of production functions called generalized power production functions. All of these production functions have been used as a basis for estimating relationships within agriculture. This chapter will be of primary interest to students interested in doing research in agricultural economics.

Key terms and definitions:

Spillman Production Function
Transcendental Production Function
Cobb Douglas Function with Variable Elasticities
Generalized Power Production Function
Polynomial Forms

11.1 Introduction

Despite the widespread use of the Cobb Douglas production function, it was not the first or the only production function to be used by agricultural economists for representing production relationships. Agricultural economics as a formal discipline is relatively new, having had its start as a separate discipline in the first decade of the twentieth century. The first work in agricultural economics was conducted by biological scientists who were interested in providing farmers with useful information with regard to designing plans for feeding livestock or fertilizing crops. Even these early efforts, conducted by biological scientists with little or no training in economics, had a central focus in obtaining estimates of parameters of agricultural production functions as a basis for the development of recommendations to farmers.

11.2 The Spillman

One of the earliest efforts to estimate a production function in agriculture was conducted by Spillman, and was published in the newly created *Journal of Farm Economics* (later to become the *American Journal of Agricultural Economics*) in two articles in 1923 and 1924. The first article was titled "Application of the Law of Diminishing Returns to Some Fertilizer and Feed Data.' The second was "Law of the Diminishing Increment in the Fattening of Steers and Hogs." It is not surprising that Spillman was interested in determining whether or not the law of diminishing returns had empirical support within some rather basic agricultural production processes.

The empirical efforts by Spillman were published prior to the work by Cobb and Douglas in 1928, and the form of the production functions used by Spillman differed slightly. The Spillman function was

†11.1 $$y = A(1 - R_1^{x_1})(1 - R_2^{x_2})$$

where A, R_1 and R_2 are parameters to be estimated. The parameters R_1 and R_2 would normally be expected to fall between zero and 1. The sum of $R_1 + R_2$ would normally be less than or equal to 1.

An example of the Spillman function is

†11.2 $$y = 1(1 - 0.3^{x_1})(1 - 0.4^{x_2})$$

In equation †11.2 , if one of the inputs is increased, output increases, but at a decreasing rate. The marginal products of x_1 or x_2 are positive but decreasing.

The marginal product of input x_1 (MPP_{x_1}) is

†11.3 $$\partial y/\partial x_1 = -\ln R_1(1 - R_2^{x_2})AR_1^{x_1} > 0$$

since $A, R_1 > 0$,

†11.4 $$(1 - R_2^{x_2})\ln R_1 < 0$$

Like the Cobb Douglas function, the marginal product is positive for any level of input use.

Moreover,

†11.5 $$\partial^2 y/\partial x_1^{\,2} = -\ln^2 R_1(1 - R_2^{x_2})AR_1 x_1 < 0$$

MPP is declining for any level of input use.

The production surface of the Spillman function is somewhat different from the Cobb Douglas. Figure 11.1 illustrates the surface and isoquants under the assumption that $R_1 = 0.4$ and $R_2 = 0.6$ and $A = 10$. Compared with a Cobb Douglas with similar parameters (diagram A, Figure 10.1), the function appears to initially increase at a much more rapid rate, and then increase very slowly.

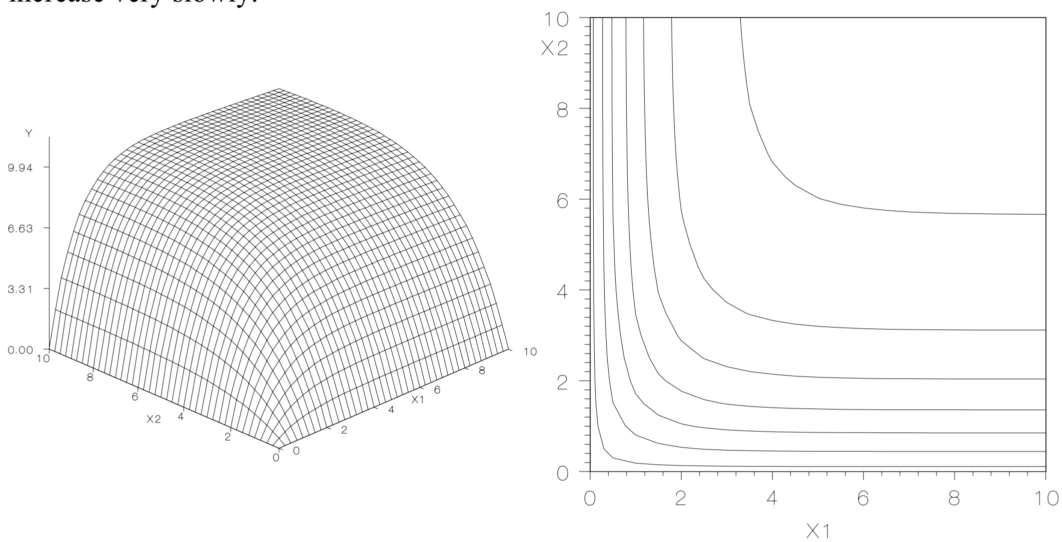


Figure 11.1 The Spillman Production Function

Since the advent of the the Cobb Douglas, the Spillman has seldom been used by agricultural economists. It is primarily of historical interest because the Spillman research represented one of the first efforts to estimate parameters of a production function for some basic agricultural processes.

11.3 The Transcendental Production Function

By the mid-1950s, both economists and agricultural economists were very much aware of many of the limitations of the Cobb Douglas production function. They recognized that although parameters of the function were very easy to estimate from data, the function did not very well represent the neoclassical three stage production function. The problem of greatest concern at that time was the fixed production elasticities, which require that *APP* and *MPP* be at a fixed proportion to each other. This issue was not unrelated to the fact that the Cobb Douglas could represent only one stage of production at a time, very much unlike the neoclassical presentation.

Halter, Carter, and Hocking were concerned with the lack of compatibility between the Cobb Douglas and the neoclassical three-stage production function. The researchers sought to make modifications in the Cobb Douglas to allow for the three stages of production and variable production elasticities, yet at the same time retain a function that was clearly related to the Cobb Douglas and was easy to estimate from agricultural data.

The function that Halter et al. introduced in 1957 looked like a slightly modified version of the Cobb Douglas. The base of the natural logarithm, e was added and raised to a power that was a function of the amount of input that was used.

The two-input function was

†11.6 $$y = Ax_1^{\alpha_1} x_2^{\alpha_2} e^{\gamma_1 x_1 + \gamma_2 x_2}$$

The corresponding single input function was

†11.7 $$y = Ax^{\alpha} e^{\gamma x}$$

The *MPP* for the single input version, using the composite function rule, was

†11.8 $$dy/dx = \alpha A x^{\alpha - 1} e^{\gamma x} + \gamma e^{\gamma x} A x^{\alpha}$$

$$= (\alpha/x + \gamma)y$$

Since *APP* is y/x and the elasticity of production is *MPP*/*APP*, the elasticity of production for the single input transcendental is

†11.9 $$\varepsilon = (\alpha/x + \gamma)y(x/y)$$

$$= \alpha + \gamma x$$

The elasticity of production, and hence the ratio of *MPP* to *APP*, is clearly dependent on the amount of input that is used. The change in the elasticity of production (ε) with respect to a change in the use of x ($d\varepsilon/dx$) is equal to the parameter γ. In other words, the size of γ indicates how rapidly the elasticity of production is declining. In the case of a single input power production function such as $y = Ax^b$, the elasticity of production is a constant b, and hence $d\varepsilon/dx$ is 0. This function is a special case of the single input transcendental with the parameter γ equal to zero. Since illustrations of the neoclassical production function show a declining elasticity of production as the use of the input increases, the transcendental production functions of greatest interest are those in which γ is negative.

Halter et al. worked out the properties of the transcendental production function for the single-input case under varying assumptions with respect to the values of α and γ. Table 11.1 summarizes their findings.

11.4 The Two-Input Transcendental

Halter et al. proposed an extension of the single-input transcendental to two inputs

†11.10 $$y = Ax_1^{\alpha_1} x_2^{\alpha_2} e^{\gamma_1 x_1 + \gamma_2 x_2}$$

The *MPP* of x_1 is

†11.11 $$\partial y/\partial x_1 = (\alpha_1/x_1 + \gamma_1)y$$

The *MPP* of x_2 is

†11.12 $$\partial y/\partial x_2 = (\alpha_2/x_2 + \gamma_2)y$$

Table 11.1 Properties of the Single-Input Transcendental Under Varying Assumptions with Respect to Parameters " and (

Value of "	Value of (	What Happens to y and ,
0 < " ≤ 1	< 0	y increases at a decreasing rate until x = ! "/(, then decreases; as x increases, , is declining.
> 1	< 0	The neoclassical case. y increases at an increasing rate until x = (! " + / ")/(, increases at a decreasing rate until x = ! "/(, then decreases; as x increases, , is declining.
0 < " < 1	0	y increases at a decreasing rate; , is constant equal to ".
1	0	y increases at a constant rate; , is 1; *MPP* and *APP* are the same everywhere.
> 1	0	y increases at an increasing rate; , is constant equal to ".
0 < " < 1	> 0	y increases at a decreasing rate until x = (! " + / ")/(, then increases at increasing rate; , is increasing.
≥ 1	> 0	y increases at an increasing rate; , is increasing

Source: Adapted from Halter et al.

APP_{x_1} is y/x_1 and APP_{x_2} is y/x_2.

Therefore, the partial elasticity of production with respect to x_1 is

†11.13 ,$_1$ = "$_1$ + ($_1x_1$

and with respect to x_2 is

†11.14 ,$_2$ = "$_2$ + ($_2x_2$

Each production elasticity is dependent on the quantity of that input being used but not on the quantity of the other input. If a measurement of returns to scale is the sum of the individual production elasticities, the returns to scale are not constant but are dependent on the amount of x_1 and x_2 that is used. The two input transcendental is not homogeneous of any degree.

†11.15 $\partial \varepsilon_1/\partial x_1 = \gamma_1 \quad \partial \varepsilon_1/\partial x_2 = 0$

†11.16 $\partial \varepsilon_2/\partial x_1 = 0 \quad \partial \varepsilon_2/\partial x_2 = \gamma_2$

The marginal rate of substitution of x_1 for x_2 is equal to the negative ratio of the marginal products

†11.17 $MRS_{x_1x_2} = dx_2/dx_1 = -[(\alpha_1/x_1 + \gamma_1)y]/[(\alpha_2/x_2 + \gamma_2)y]$

$= -(\alpha_1/x_1 + \gamma_1)/(\alpha_2/x_2 + \gamma_2)$

$= -[x_2(\alpha_1 + \gamma_1 x_1)]/[x_1(\alpha_2 + \gamma_2 x_2)]$

The isoquants for the transcendental when α_1 and $\alpha_2 > 0$ and γ_1 and $\gamma_2 < 0$ consist of a series of concentric rings or lopsided ovals centered at the global output maximum for the function (Figure 11.2). The exact shape of the rings is determined by the value of the parameters for the function. The exact center of the rings occurs at $x_1 = -\alpha_1/\gamma_1$, $x_2 = -\alpha_2/\gamma_2$.

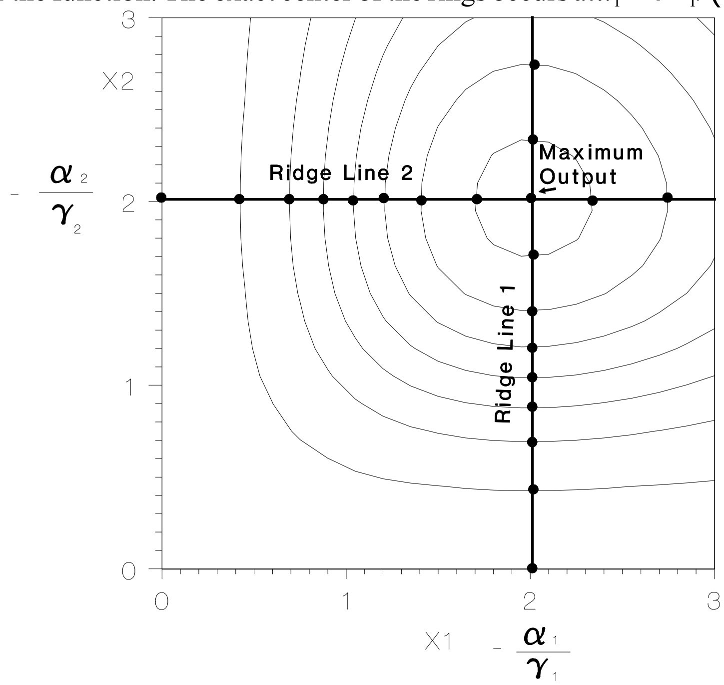


Figure 11.2 Isoquants and Ridge lines for the Transcendental, $\gamma_1 = \gamma_2 = -2;\ \alpha_1 = \alpha_2 = 4;\ \gamma_3 = 0$

The first-order conditions for profit maximization can be derived by setting the marginal rate of substitution equal to the negative ratio of the input prices ($-v_1/v_2$). The resultant equation defines the expansion path along which the farmer would move as output is expanded. The first-order conditions are defined by

†11.18 $(\alpha_1/x_1 + \gamma_1)/(\alpha_2/x_2 + \gamma_2) = v_1/v_2$

The expansion path equation is defined by

†11.19 $$x_2 v_2(\alpha_1 + \gamma_1 x_1) = x_1 v_1(\alpha_2 + \gamma_2 x_2)$$

†11.20 $$x_2 = v_1 x_1 \alpha_2/(v_2 \alpha_1 + v_2 \gamma_1 x_1 - v_1 x_1 \gamma_2)$$

The expansion path for the transcendental production function is clearly nonlinear unless γ_1 and γ_2 are zero.

The ridge lines for the transcendental are present only when γ_1 and γ_2 are negative, and are straight lines that form a right angle at the point of maximum output, where $x_1 = -\alpha_1/\gamma_1$. The position of the ridge line for x_1 in the horizontal axis is determined by the value of α_1 and γ_1. Similarly, the position of the ridge line on the x_2 axis corresponds to the point where $x_2 = -\alpha_2/\gamma_2$. (The slope of this ridge line dx_2/dx_1 is clearly zero.)

The resultant square is very much unlike the football shape defined by the ridge lines for the neoclassical case. This ridge line pattern suggests that the maximum output for the family of production functions for the input x_1 occurs at the same level of use for input x_1, regardless of how much of the second input is used. The same holds for input x_2. This is not consistent with the neoclassical case in which an increase in the use of x_2 pushes the maximum of the production function for x_1 farther and farther to the right.

A modification of the transcendental suggested by this author to make the function more closely correspond to the neoclassical diagram would be to include an interaction term in the power of *e*. The function is

†11.21 $$y = Ax_1^{\alpha_1} x_2^{\alpha_2} e^{\gamma_1 x_1 + \gamma_2 x_2 + \gamma_3 x_1 x_2}$$

the corresponding *MPP* for x_1 is $(\alpha_1/x_1 + \gamma_1 + \gamma_3 x_2)y$. *APP* is y/x_1, so the corresponding partial elasticity of production for input x_1 is

†11.22 $$\epsilon_1 = \alpha_1 + \gamma_1 x_1 + \gamma_3 x_2 x_1$$

Along the ridge line for x_1, the production elasticity for x_1 is zero. This implies that

†11.23 $$\alpha_1 + \gamma_1 x_1 + \gamma_3 x_2 x_1 = 0$$

or

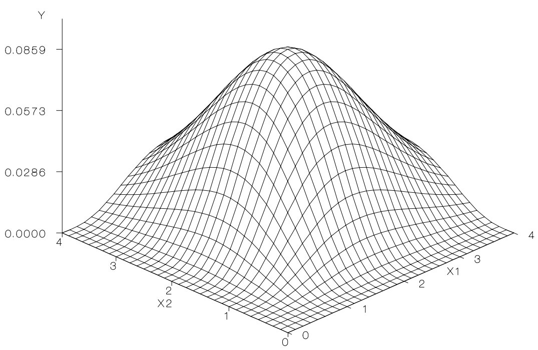
†11.24 $$x_1(\gamma_1 + \gamma_3 x_2) = -\alpha_1$$

†11.25 $$x_1 = -\alpha_1/(\gamma_1 + \gamma_3 x_2)$$

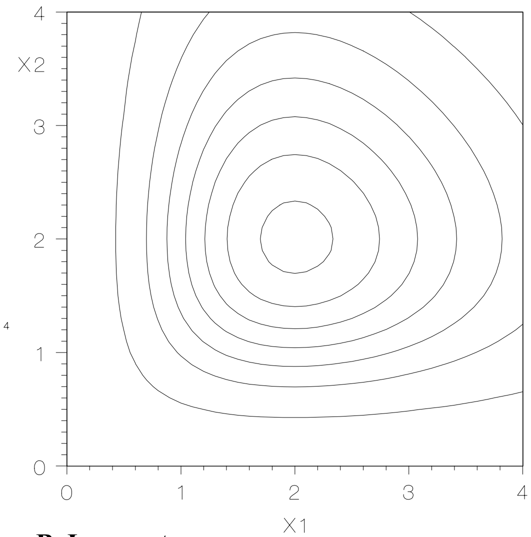
The amount of x_1 required to maximize output is clearly a function of the quantity of x_2 that is available. Ridge lines no longer form right angles with each other parallel to the x_1 and x_2 axes. If γ_3 is positive, ridge lines will slope upward and to the right. Moreover, ϵ_1 and ϵ_2 are functions of the amount of both inputs that are used.

11.5 Illustrations and Applications of the Transcendental

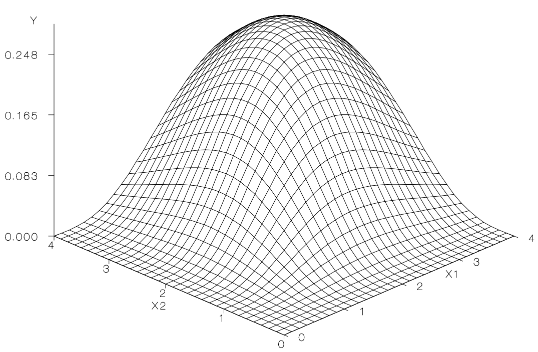
Figure 11.3 illustrates production surfaces and isoquants under varying assumptions with respect to the parameters of the two input transcendental. Diagrams A and B illustrate the "original" two-input transcendental with $\alpha_1 = \alpha_2 = 4$ and $\gamma_1 = \gamma_2 = -2$.



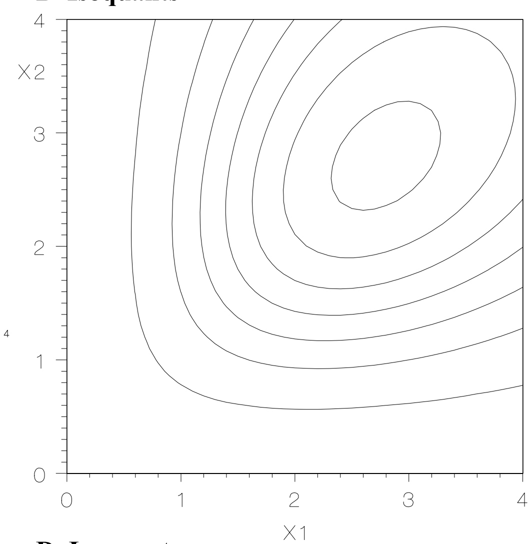
A Surface $(_{1} = (_{2} = -2; "_{1} = "_{2} = 4; (_{3} = 0$



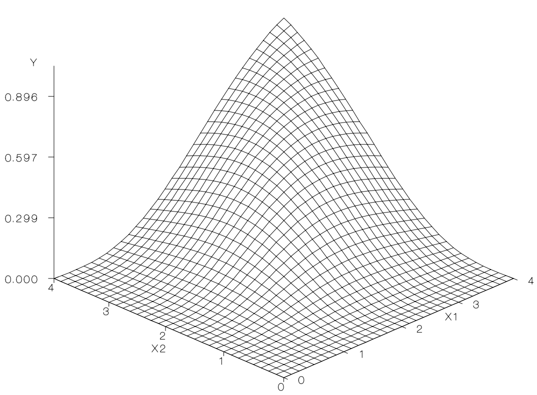
B Isoquants



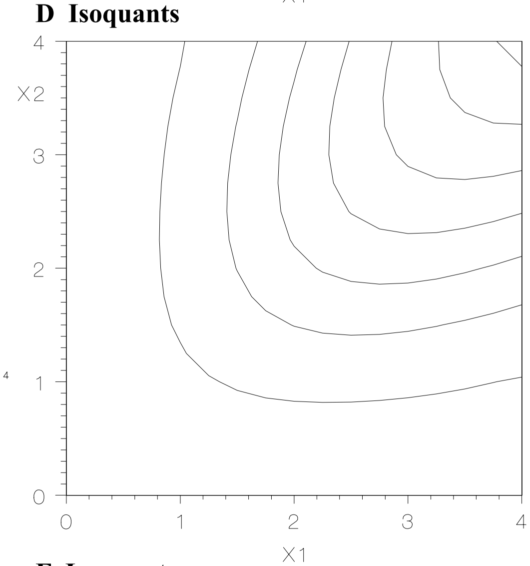
C Surface $(_{1} = (_{2} = -2; "_{1} = "_{2} = 4; (_{3} = 0.2$



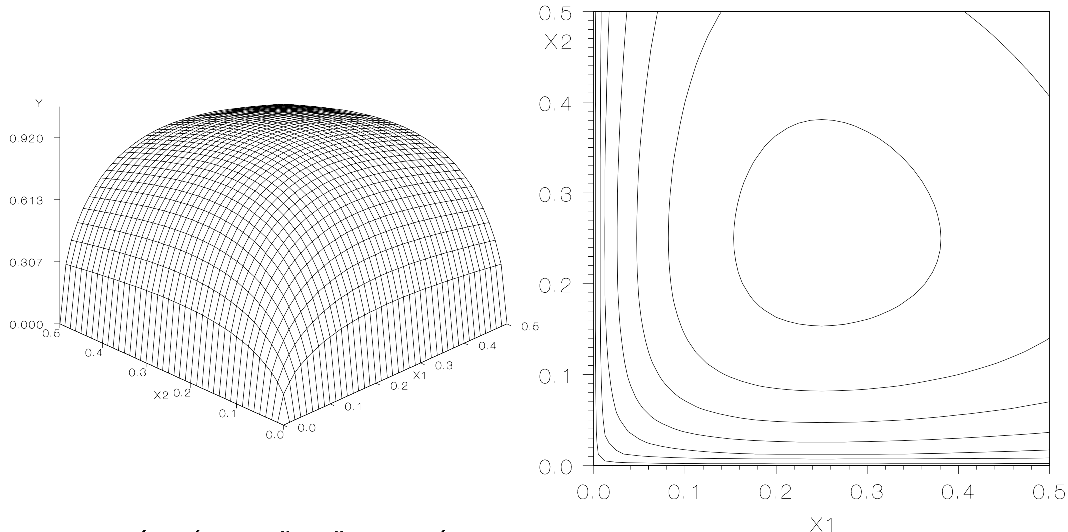
D Isoquants



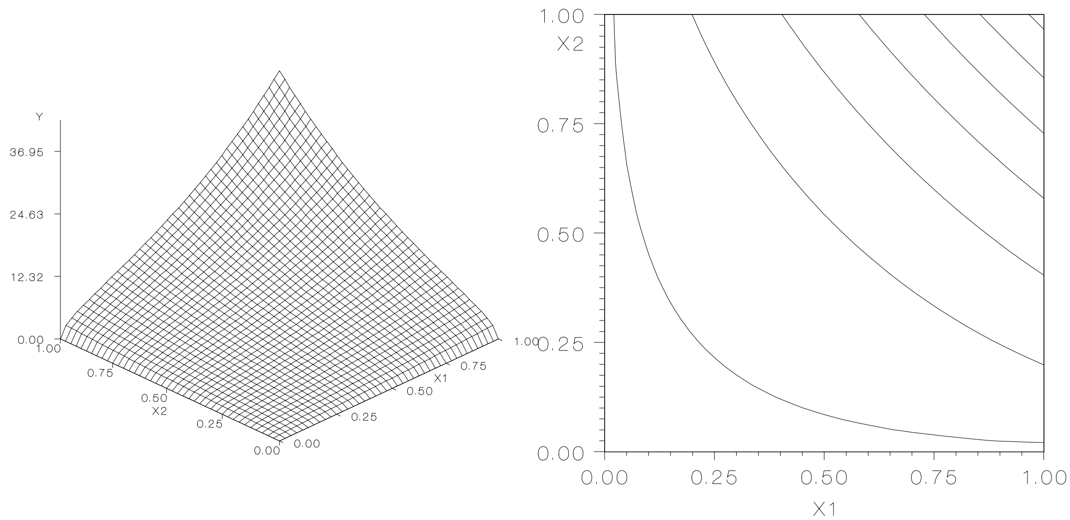
E Surface $(_{1} = (_{2} = -2; "_{1} = "_{2} = 4; (_{3} = 0.3$



F Isoquants



G Surface $\gamma_1 = \gamma_2 = -2$; $\alpha_1 = \alpha_2 = 0.5$; $\gamma_3 = 0$ **H Isoquants**



I Surface $\gamma_1 = \gamma_2 = 1$; $\alpha_1 = \alpha_2 = 0.5$; $\gamma_3 = 0$ **J Isoquants**

Figure 11.3 The Transcendental Production Function Under Varying Parameter Assumptions

The three stages of production are clearly visible, as is the fact that the maximum for each single input production function for x_1 generated by assuming x_2 held fixed at a varying level occurs at the same level of input use for x_1. Diagrams C D, E and F illustrate what happens as an interaction term with the parameter γ_3 is added. Diagrams C and D assume that γ_3 is 0.2, whereas diagram E and F assume that γ_3 is 0.3. Each successive production function for x_1 has a maximum to the right of the one below it. The same holds for input x_2. The shape of the production surface is highly sensitive to changes in the value of the parameter γ_3.

Diagrams G and H illustrate the surface and isoquants when α_1 and α_2 are positive but less than 1 (0.5), γ_1 and γ_2 are negative (-2), and γ_3 is zero. The function increases at a decreasing rate, and then decreases at $x_1 = -\alpha_1/\gamma_1$, $x_2 = -\alpha_2/\gamma_2$.

Diagram I and J illustrate what happens when γ_1 and γ_2 are positive (1.0) and α_1 and α_2 are positive (0.5). The surface looks not unlike a total cost function in three dimensions, first increasing at an increasing rate, and then increasing at a decreasing rate. The transcendental production function can be viewed as a generalization of the Cobb Douglas production function that can depict the three stages of production and has variable production elasticities. The transcendental is easily transformed to natural logs to yield

†11.26 $$\ln y = \ln A + \alpha_1 \ln x_1 + \alpha_2 \ln x_2 + \gamma_1 x_1 + \gamma_2 x_2 + \gamma_3 x_1 x_2$$

This function is linear in the parameters, and is again easily estimated via ordinary least squares regression techniques.

The first attempt to estimate parameters of a transcendental production function was published by Halter and Bradford in 1959. They estimated a *TVP* function with gross farm income as the dependent variable and dollar values for owned and purchased inputs as *x* variables. The dependent variable was adjusted by a weather measure based on the number of drought free days during the growing season. Data were collected from 153 individual farms in 1952 and 1956.

The function was estimated both as a Cobb Douglas specification and as a transcendental specification. Based on the statistical results including a comparison of actual values for the dependent variable with those predicted by the equation, the transcendental specification did give slightly improved results than the Cobb Douglas specification.

11.6 Cobb Douglas with Variable Input Elasticities

Another approach was to develop a Cobb Douglas type of function in which the powers on each input were assumed to vary. The function was

†11.27 $$y = Ax_1^{\beta_1(X)} x_2^{\beta_2(X)}$$

The β_i are functions of one or more inputs represented by *X*. These inputs may include x_1 and x_2, but they also may include inputs not incorporated in the function directly. One proposal suggested that X should incorporate the skills of the manager, and that production functions for skilled managers should have greater partial elasticities of production than production functions for unskilled managers.

11.7 de Janvry Modifications

de Janvry recognized the linkages between the Cobb Douglas production function with variable input elasticities and the two input transcendental. He proposed the generalized power production function (GPPF), which had as special cases the Cobb Douglas, the Cobb Douglas with variable input elasticities, and the transcendental.

The general form of the GPPF is

†11.28 $$y = x_1^{g(x_1,x_2)} x_2^{h(x_1,x_2)} e^{j(x_1,x_2)}$$

where *g*, *h* and *j* are each functions of the inputs. If $j = 0$; $g = \alpha_1$; and $h = \alpha_2$, the function is the traditional Cobb Douglas type. If *g* and *h* are constants and *j* is nonzero, the function is a general two input transcendental, without any particular restriction of the form of *j*. If $j = \gamma_1 x_1 + \gamma_2 x_2$, the function is the standard transcendental. The Cobb Douglas function with variable input elasticities results where j is zero but g and h vary according to x_1 and x_2.

The major contribution of de Janvry was to develop a general functional form that included as special cases many of the other production functions used by agricultural economists.

11.8 Polynomial Forms

The production functions described so far in Chapters 10 and 11 require that a positive amount of each input be present for output to be produced. Isoquants come asymptotic to, but do not intersect, the axes. When isoquants intersect an axis, output is possible even in the absence of the input represented by the other axis.

A polynomial form is inherently additive rather than multiplicative. if interaction terms are not included, there will be an additive but not synergistic impact on output as a result of an increase in the level of input use.

Consider the polynomial

†11.29 $$y = a + bx_1 + cx_1^2 + dx_2 + ex_2^2$$

where a, b, c, d, and e are constant parameters. The marginal product of x_1 is $b + 2cx_1$. The marginal product of x_2 is $d + 2ex_2$. The marginal product of x_1 is not linked to the quantity of x_2 that is present. The marginal product of x_2 is not linked to the quantity of x_1 that is present. The function achieves a maximum (or possibly minimum) when $b + 2cx_1 = 0$ and $d + 2ex_2 = 0$. Ridge lines again form right angles that intersect at the global output maximum. Second order conditions for a maximum require that c be negative and ce be positive. (The proof is left with the reader.) This implies that both c and e must be negative or that the *MPP* with respect to both inputs must slope downward to the right. The parameters b and d must be positive, or there will be no point at which an increase in the use of the input will produce a positive marginal product.

Now consider the polynomial

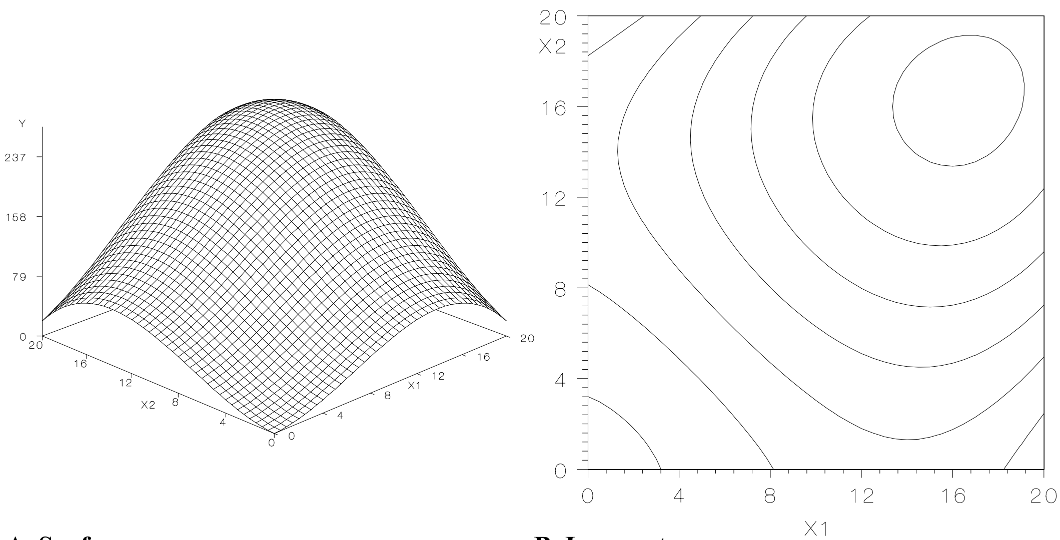
†11.30 $$y = a + bx_1 + cx_1^2 + dx_2 + ex_2^2 + fx_1x_2$$

The marginal product of x_1 is $b + 2cx_1 + fx_2$. The marginal product of x_2 is $d + 2ex_2 + fx_1$. The marginal product of each input is linked to the quantity of the other input that is present, as long as f is nonzero. The first order conditions for maximum output require that each marginal product be zero. Ridge lines no longer intersect at right angles, but if f is positive, each successive single-input production function achieves its maximum to the right of the one below it. Second order conditions for a maximum require that $2c$ be negative and $2c2e - f^2$ be positive. These polynomials and any other polynomial that is linear in its parameters could be estimated via ordinary least squares.

Figure 11.4 illustrates the polynomial

†11.31 $$y = x_1 + x_1^2 - 0.05x_1^3 + x_2 + x_2^2 - 0.05x_2^3 + 0.4x_1x_2$$

The three stages of production are clearly evident, and output is possible even in the absence of one of the two inputs. Note the white area between each axis and the production surface, indicating that the isoquants intersect both axes.



A Surface **B Isoquants**

Figure 11.4 The Polynomial $y = x_1 + x_1^2 - 0.05x_1^3 + x_2 + x_2^2 - 0.05x_2^3 + 0.4x_1x_2$

11.9 Concluding Comments

Agricultural economists have made use of a wide array of production functions over the last 50 years and more. Some of these efforts have represented attempts to make explicit linkages between the mathematical specification and the traditional neoclassical three stage production function. The effort conducted by Halter and his colleagues was clearly aimed at that objective, as have been the attempts to estimate polynomial forms.

Other agricultural economists saw the problem somewhat differently. Efforts in the early 1970s by de Janvry and others focused on the development of general functional forms that would encompass a number of explicit specifications as special cases.

In the 1960s and 1970s, the direction of research both in general and in agricultural economics increasingly turned to the problem of determining the extent to which inputs to a production process substituted for each other. This led to the development of functional forms that are not necessarily linked to the neo classical three-stage form, but rather were useful in estimating elasticities of substitution between input pairs. Chapter 12 discusses some of these functional forms.

Problems and Exercises

1. For input levels between zero and 10 units, graph the following production functions and compare their shape.

Single-input power (Cobb Douglas like):

a. $y = x^{0.5}$

Single-input (Spillman like):

b. $y = (1 - 0.5^x)$

Single-input transcendental:

c. $y = x^4 e^{-2x}$

where e is the base of the natural log 2.71828...

2. For part (c) in Problem 1, find the level of x corresponding to:

   a. The inflection point.
   b. Maximum *MPP*.
   c. Maximum *APP*.
   d. Maximum *TPP*.

3. If the production function is a polynomial consistent with the neoclassical three stage production function (see Problem 5, Chapter 2), show that the level of x that maximizes *MPP* will be two thirds of the level that maximizes *APP*.

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