12

The Elasticity of Substitution

This chapter develops the concept of an elasticity of substitution. An elasticity of substitution is a measure of the extent to which one input substitutes for another input along an isoquant. To the extent inputs substitute for each other, a farmer can respond to changing relative input prices by adjusting the combination or mix of inputs that are used. The constant elasticity of substitution, or CES production function, is used as a means for illustrating how the shape of isoquants change as the input mix changes. Examples of research using a translog production function to estimate elasticities of substitution for agricultural inputs are cited.

Key terms and definitions:

Isoquant Pattern
Right Angle Isoquant
Diagonal Isoquant
Elasticity of Substitution
Zero Elasticity of Substitution
Infinite Elasticity of Substitution
Constant Elasticity of Substitution (CES) Production Function
Translog Production Function
Translog Cost Function
Shephard's Lemma
Cost Share Equations

12.1 An Introduction to the Concept

Isoquants can vary widely in their patterns. Isoquants might form a series of right angles, or they might have constant slopes and look like iso-outlay lines. Isoquants for the Cobb Douglas production function appear to be hyperbolic. Isoquants for the transcendental production function under certain parameter assumptions appear to be a series of concentric rings, ovals, or lopsided ovals. The shape of the isoquants can tell a good deal about the nature of the production functions that underlie them.

The shape of an isoquant depends on the extent to which the two inputs being pictured substitute for each other, as changes in the mix or proportions of the two inputs are made. A specific isoquant produces a fixed amount of output (y). Along an isoquant, a diminishing marginal rate of substitution is usually a result of the law of diminishing returns that applies to the underlying production functions for each input.

Consider a production function

†12.1 $$y = ax_1 + bx_2$$

The marginal product of x_1 is a, and the marginal product of x_2 is b. Since both marginal products are constant, the slopes of each member of the family of single input production functions for x_1 and x_2 are also constant. The marginal rate of substitution of x_1 for x_2 = ! MPP_{x_1}/MPP_{x_2}, or ! a/b. The slope of each isoquant is everywhere ! a/b. Inputs are perfect substitutes for each other at the rate given by the marginal rate of substitution. An example is a production function for steers. Assume that x_1 is corn the farmer grew himself, and x_2 is corn purchased from a neighbor. If the corn is of comparable quality, or have constant *MPP*'s, corn grown at home and corn grown by the neighbor should be perfect substitutes for each other.

The production function in equation †12.1 indicates a constant marginal product of beef from incremental units of corn. Such a super steer has not yet been developed, and it is easy to see why such a production function is seldom used by agricultural economists. The expansion path conditions for such a production function can be derived by the reader.

Now consider a production function in which the two inputs must be used in a fixed proportion, such as tractors and tractor drivers. Two tractor drivers and one tractor produce no more output than one tractor and one driver. Two tractors and one driver produce no more output than do one tractor and one driver. Isoquants are right angles, and inputs can be thought of as not substituting with each other at all, or zero substitutability between inputs.

Between these extreme cases lie a myriad of other possible isoquant patterns or maps. Isoquants might be bowed in only slightly toward the origin, or they might look very nearly like, but not quite be, right angles. The hyperbolic isoquants for the Cobb Douglas production function that asymptotically approach each axis appear to be in between these extreme cases.

The need exists for a simple measure linked to the shape of the isoquants that would make it possible to determine the extent to which an one input substitutes for another. The ideal measure would be a pure or unitless number that could assume values between zero and infinity. The number should be unitless to make possible comparisons between isoquant maps representing widely varying pairs of inputs. Any elasticity is a unitless or pure number in that it represents the ratio of two percentages, and thus the units cancel. The ideal measure would assume a value of zero if inputs do not substitute for each other, but approach infinity as the inputs became perfect substitutes for each other.

Thus the concept of the elasticity of substitution came into being. Actually, several formulas were developed. For example, Heady proposed that the elasticity of substitution (e_{sh}) should be equal to the percentage change in the use of x_2 divided by the percentage change in the use of x_1

†12.2 $$e_{sh} = (\Delta x_2/x_2)/(\Delta x_1/x_1)$$

Assuming that the change in x_2 and x_1 is sufficiently small

†12.3 $$e_{sh} = (dx_2/dx_1)(x_1/x_2)$$

or

†12.4 $$e_{sh} = MRS_{x_1x_2}(x_1/x_2)$$

This elasticity of substitution is the slope of the isoquant at a particular point multiplied by the inverse ratio of input use defined by that point.

For a Cobb Douglas type of production function, $MRS_{x_1x_2} = \alpha_1x_2/\alpha_2x_1$, and therefore the elasticity of substitution between the input pairs is α_1/α_2, the ratio of the partial elasticities of production. Moreover, this elasticity of substitution for a Cobb Douglas type of function could vary widely even though the isoquant map for any Cobb–Douglas type function looks very similar in terms of the shape of the isoquants. So if being able to broadly determine the shape of the isoquant map on the basis of the elasticity of substitution was important, this measure failed.

The more generally accepted algebraic definition of the elasticity of substitution is somewhat more complicated, but the interpretation of the calculated values relative to the shape of the underlying isoquant map is clear. In the two input setting, the elasticity of substitution is defined as the percentage change in the input ratio divided by the percentage change in the marginal rate of substitution

†12.5 $$e_s = [\%\text{ change in }(x_2/x_1)]/[\%\text{ change in }MRS_{x_1x_2}]$$

$$= [\Delta(x_2/x_1)/(x_2/x_1)]/[\Delta MRS_{x_1x_2}/MRS_{x_1x_2}]$$

If the change is sufficiently small, the formula becomes

†12.6 $$e_s = [d(x_2/x_1)/(x_2/x_1)]/[dMRS_{x_1x_2}/MRS_{x_1x_2}]$$

$$= [d(x_2/x_1)/(x_2/x_1)]/[d(dx_2/dx_1)/(dx_2/dx_1)]$$

Equation †12.6 can be rearranged as

†12.7 $$[d(x_2/x_1)/d(dx_2/dx_1)][(dx_2/dx_1)/(x_2/x_1)]$$

The expression contained within the first pair of brackets represents the rate of change in the proportions of the two inputs being used as the marginal rate of substitution changes. The expression in the second pair of brackets is the marginal rate of substitution divided by the proportions of the two inputs.

This second definition for the elasticity of substitution can be presented graphically and is illustrated in Figure 12.1. Suppose that the elasticity of substitution is to be calculated over the finite range from point P_1 to point P_2. First calculate the percentage change in the input

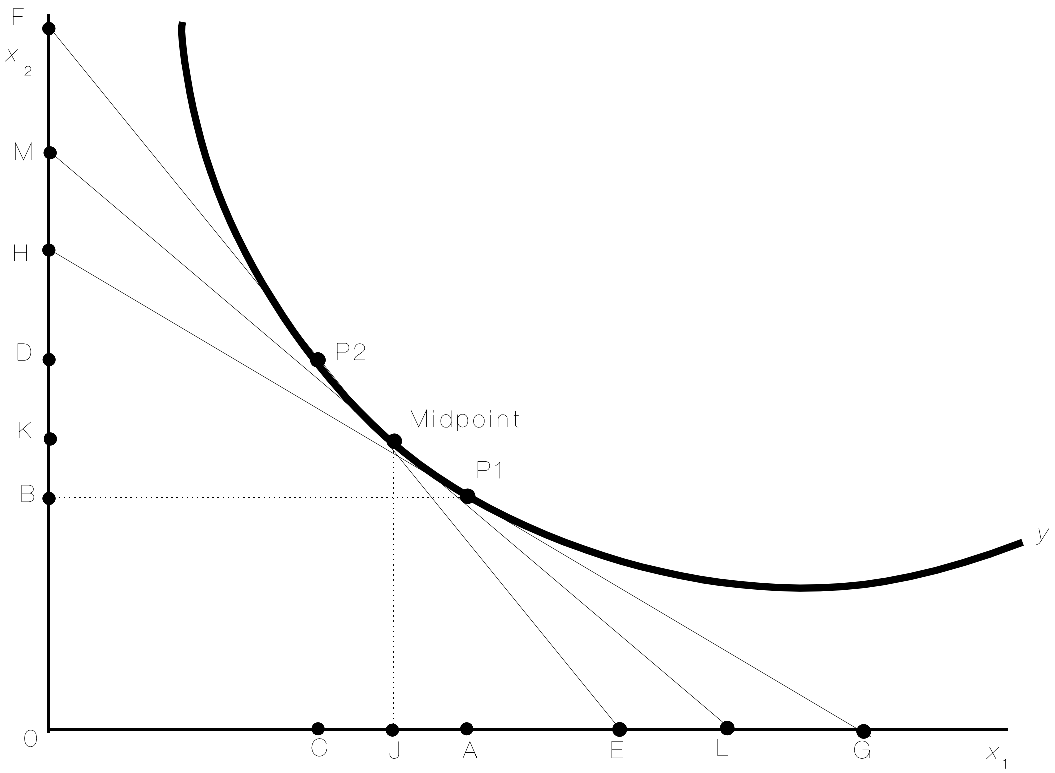


Figure 12.1 The Arc Elasticity of Substitution

ratio. The input ratio at point P_1 is equal to OB/OA. The input ratio at point P_2 is equal to OD/OC. The input ratio at some average point between P_1 and P_2 is OK/OJ. The percentage change in the input ratio is (OB/OA ! OD/OC)/(OK/OJ).

Now calculate the percentage change in the marginal rate of substitution, or the percentage change in the slope of the isoquant. The slope of the isoquant at point P_1 is OH/OG. The slope of the isoquant at point P_2 is OF/OE. The slope of the isoquant at a point midway between P_1 and P_2 is OM/OL. So the percentage change in the marginal rate of substitution is (OH/OG ! OF/OE)/(OM/OL).

The elasticity of substitution is the percentage change in the input ratio divided by the percentage change in the marginal rate of substitution. So the formula for the elasticity of substitution is

†12.8 [(OB/OA ! OD/OC)/(OK/OJ)]/[(OH/OG ! OF/OE)/(OM/OL)]

Assume that the isoquant is very nearly a line with a constant downward slope. As a result, the percentage change in the marginal rate of substitution between point P_1 and P_2 is very near zero. But the percentage change in the input ratio is a comparatively large number. A very small number will be divided into a large number and the result will be a very large elasticity of substitution.

Now suppose that the isoquant is a right angle, with P_1 on the horizontal portion of the angle and P_2 on the vertical portion of the right angle. The slope at P_1 is zero, the slope at P_2 is infinite. The percentage change in the *MRS* between P_1 and P_2 is infinite as well. The percentage change in the input ratio between P_1 and P_2 can be calculated as a very ordinary number, neither very large nor very small. As the percentage change in the *MRS* approaches infinity, a very ordinary number is divided by a very large number, which results in an elasticity of substitution that approaches zero.

In the two input case, the values for the elasticity of substitution lie between zero and infinity. Inputs that do not substitute at all with each other have a zero elasticity of substitution, while inputs that substitute for each other in fixed proportions at any point along an isoquant have an infinite elasticity of substitution. Values near zero indicate little potential input substitution. Very large values indicate a great potential of substituting one input for another within the production process.

Figure 12.1 illustrates what could be called an "arc" elasticity of substitution, since the difference between P_1 and P_2 is assumed to be finite. The point elasticity of substitution can be calculated with the aid of the calculus.

Henderson and Quandt provide a formula for calculating elasticities of substitution based solely on first and second derivatives of the production function. (See Henderson and Quandt for a detailed derivation of the formula.) Define

†12.9 $f_1 = \partial y/\partial x_1 = MPP_{x_1}$

†12.10 $f_2 = \partial y/\partial x_2 = MPP_{x_2}$

†12.11 $f_{11} = \partial^2 y/\partial x_1^{\,2} = \text{slope of } MPP_{x_1}$

†12.12 $f_{22} = \partial^2 y/\partial x_2^{\,2} = \text{slope of } MPP_{x_2}$

†12.13 $f_{12} = f_{21} \text{ (by Young's theorem)} = \partial^2 y/\partial x_1 \partial x_2$

Equation †12.13 is the change in the slope of MPP_{x_1} with respect to a change in the use of $x_2 = \partial^2 y/\partial x_2 \partial x_1$, or the change in the slope of MPP_{x_2} with respect to a change in the use of x_1.

Then the formula for calculating the elasticity of substitution is

†12.14 $e_s = [f_1 f_2 (f_1 x_1 + f_2 x_2)]/[x_1 x_2 (2 f_{12} f_1 f_2 - f_1^2 f_{22} - f_2^2 f_{11})]$

Equation †12.14 makes it possible to calculate the elasticity of substitution at a particular point on an isoquant for any production function for which the first and second derivatives exist.

Still other formulas for elasticities of substitution have been proposed by other authors. These include a definition called the Allen (or AES) measure, found in his 1938 book. McFadden proposed a definition he called the shadow elasticity of substitution. Yet another definition is called the Morishima measure, and is found in a paper by Koizumi. All these definitions are the same as equation †12.14 when there are but two inputs, but each measure differs slightly from the others when more than two inputs are used in the production process. A detailed discussion and comparison of the alternative measures can be found in the McFadden reference.

12.2 Elasticities of Substitution and the Cobb Douglas Function

Any Cobb Douglas type of production function will have an elasticity of substitution according to equation †12.14 of exactly 1. This means that as the percentage change in the ratio of the use of inputs x_1 and x_2 is changed along a specific isoquant, there will be the exact same percentage change in the marginal rate of substitution. The conclusion is not dependent on the magnitude of the individual production elasticities and occurs even if the elasticities do not sum to 1. The result holds for any production function in which the marginal rate of substitution is a linear function of the input ratio.

The proof need not rely on the Henderson and Quandt formula. The marginal rate of substitution for a Cobb Douglas type of function is

†12.15 $MRS_{x_1x_2} = -(\beta_1/\beta_2)(x_2/x_1)$

Now let b equal the negative ratio of the elasticities of production $(-\beta_1/\beta_2)$. Since β_1 and β_2 are constant, so is b. Let $x = x_2/x_1$. Therefore, the marginal rate of substitution is a linear function of the input ratio

†12.16 $MRS_{x_1x_2} = bx$

or

†12.17 $x = (1/b)MRS_{x_1x_2}$

Therefore

†12.18 $dx/dMRS_{x_1x_2}$ (the change in the input ratio with respect to a change in the marginal rate of substitution) = $1/b$

†12.19 $(MRS_{x_1x_2})/(x_2/x_1) = bx/x$

Hence, the elasticity of substitution for a Cobb-Douglas type of function is

†12.20 $[d(x_2/x_1)/dMRS_{x_1x_2}][MRS_{x_1x_2}/(x_2/x_1)] = (1/b)(bx/x) = 1.$

To reiterate, equation †12.20 holds for any two input multiplicative production function of the Cobb Douglas type and does not depend on the magnitude or the sum of the individual production elasticities.

12.3 Policy Applications of the Elasticity of Substitution

The elasticity-of-substitution concept has important applications to key issues linked to agricultural production. The recent liquid fuels energy crisis provides an illustration of the importance of the concept. Of concern is the extent to which other inputs can be substituted for liquid fuels energy in agricultural production. An example might be the potential substitutability between farm labor, farm tractors, and machinery and liquid fuels.

Agriculture in the United States as well as in most foreign countries has become increasingly mechanized. Hence tractors and machinery can and do substitute for farm labor. The reduction in the farm population that has taken place in the United States over the past century and more indicates that farm tractors and machinery can substitute for human labor, and this substitution can take place, at least in the aggregate, relatively easily. This suggests that the elasticity of substitution is comparatively high between human labor and farm tractors and machinery.

Massive changes in the mix of inputs required to produce agricultural products would not have taken place without clear economic signals. These economic signals are the relative prices for tractors and machinery and the fuel required to run versus farm labor. Farmers often complain about the prices for tractors and other farm machinery, but changes in the mix of inputs toward tractors and farm machinery would not have taken had it not been economic. Farmers look for the point of least-cost-combination today, much as they always have.

If the relative proportions of each input do not change, or change very little in the face of changing relative input prices, then there is evidence to suggest that the elasticity of substitution between the inputs is nearly zero. However, when relative prices change and are accompanied by a change in the input mix, there is evidence in support of a positive elasticity of substitution.

Liquid fuel prices increased very rapidly during the 1970s and the early 1980s. Since the price of fuel was increasing relative to other input costs, there again was concern with respect to whether there existed a positive elasticity of substitution between liquid fuels and other agricultural inputs. Some even argued that rising fuel prices would eventually lead back to a labor-oriented agriculture more broadly consistent with agriculture in the nineteenth century, but the mix of inputs used in agriculture changed very little as a result of the increased fuel prices.

There are some hypotheses as to why the input mix did not change significantly in response to increases in liquid fuel prices relative to other inputs. One possibility is that the elasticity of substitution between liquid fuels and other agricultural inputs is nearly zero. This would imply that there would be little if any changes in the input mix even in the face of changing relative prices. Farm tractors and the fuel to run them may be inputs that are required in nearly fixed proportions. Clearly, a tractor cannot run without fuel. Another possibility is that substitution is possible, but that it takes time, more time than a few years. A farmer cannot dramatically change the approach to the production of crops and livestock overnight. Elasticities of substitution may not remain forever constant, but change over time.

The economic motives for the replacement of a tractor might be examined. A farmer might replace an old tractor with a new one that is more fuel efficient per unit of output produced, thus substituting the new tractor (a form of capital) for liquid fuel energy. The replacement suggests a positive elasticity of substitution between a new tractor and liquid fuels. Rising relative labor costs (wage rates) and declining real fuel prices provided the economic signals that led to the substitution of tractors and machinery for labor during much of the twentieth century.

Consumers replaced their aging and fuel!wasting fleet of automobiles with a newer, more expensive, but energy!conserving fleet as a result of increasing real fuel prices during much of the last decade and a half. The result was a significant reduction in the demand for gasoline. The elasticity of substitution between the capital embodied in a new automobile and gasoline was clearly positive.

The elasticity of substitution between input pairs may differ significantly among various farm enterprises. There still appear to be few substitutes for human labor in tobacco production. Dairy remains labor-intensive, but possibilities are increasing for the substitution of capital for labor. Wheat, corn and soybean production are capital (tractors and machinery) intensive, and the possibility of substituting labor for capital are limited without a drastic reduction in output. A reduction in output suggests a movement across isoquants rather than along an isoquant. The extent to which labor, capital, and energy can be substituted in the production of horticultural crops varies with the specific type of crop. Some crops lend themselves to mechanization, but others remain labor intensive but liquid fuels conserving.

Agricultural economists in developing countries need to be vitally concerned with respect to the elasticities of substitution for the major agricultural commodities being produced. For example, the extent to which labor is free to move out of agriculture and into other sectors of the economy may be dependent on the elasticity of substitution between labor and the other inputs, given the resources and technology within the developing country.

12.4 The CES Production Function

Since the Cobb Douglas type of production function imposes an elasticity of substitution between input pairs of exactly 1, then if a Cobb Douglas type of production function were estimated, the elasticity of substitution between input pairs would be an assumption underlying the research rather than a result based on the evidence contained in the data. The problem with the Cobb−Douglas type of production function is widely known and is of particular interest to economists engaged in macro-oriented issues, such as the extent to which capital could substitute for labor within an economy.

The study published by Arrow, Chenery, Menhas, and Solow "Capital Labor Substitution and Economic Efficiency" in 1961 was a landmark. The study might also be considered a remake of the 1928 effort by Cobb and Douglas without the assumption that the elasticity of substitution between capital and labor was 1. In the study the authors first introduced the constant elasticity of substitution (CES) production function. The CES production function had two principal features. First, the elasticity of substitution between the two inputs could be any number between zero and infinity. Second, for a given set of parameters, the elasticity of substitution was the same on any point along the isoquant, regardless of the ratio of input use at the point: hence the name *constant elasticity of substitution production function*.

The CES production function is

†12.21 $$y = A[\delta x_1^{-\rho} + (1 - \delta)x_2^{-\rho}]^{-1/\rho}$$

The CES appears to be a very complicated function. The developers of the CES no doubt started with the result that they wished to obtain, a constant elasticity of substitution that could assume any value between zero and infinity, and worked toward a functional form that was consistent with this result. The elasticity of substitution (e_s) and the parameter ρ are closely related

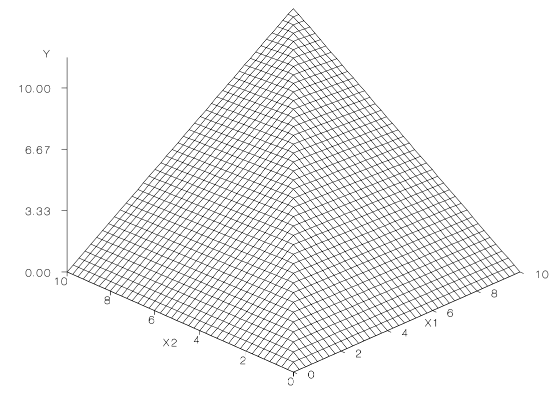
†12.22 $$e_s = 1/(1 + \rho)$$

†12.23 $$\rho = (1 - e_s)/e_s$$

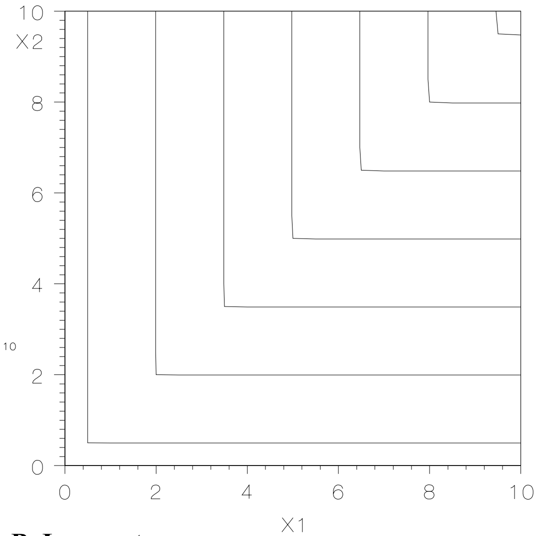
The authors retained the Cobb Douglas assumption of constant returns to scale in that $\delta + (1 - \delta) = 1$, but this assumption is not required.

In addition to having research application, the CES is a useful pedagogical tool in that it can be used to illustrate what happens to the shape of a series of isoquants as the elasticity of substitution changes. Henderson and Quandt suggest five possible cases. Figure 12.2 illustrates the production surfaces and corresponding isoquants generated under each of these cases.

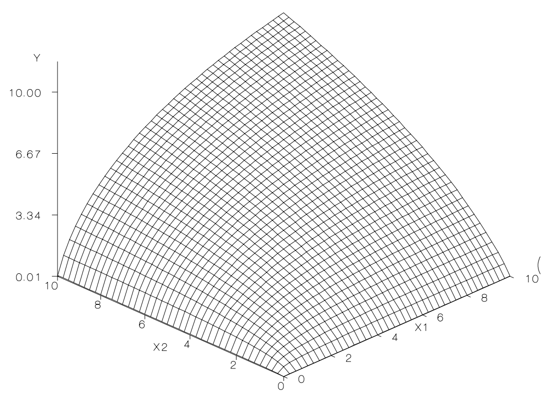
Case 1: $\rho \to +\infty$, $e_s \to 0$. At the limit, substitution between input pairs is impossible and isoquants form a right angle. Diagrams A and B illustrate what happens as ρ becomes a rather large number. The shape of the production surface becomes like a pyramid. The production surface and isoquants illustrated in diagrams A and B was drawn with the assumption that $\rho = 200$.



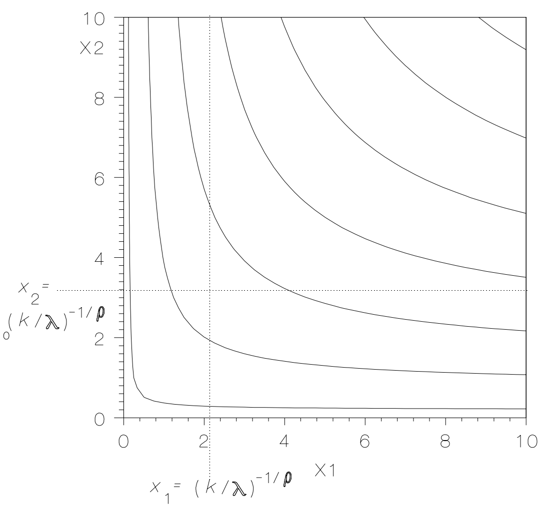
A Case 1 D = 200



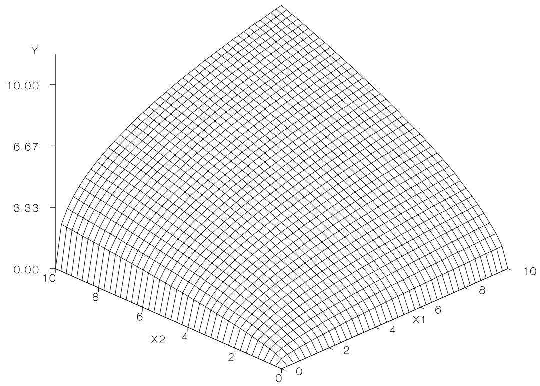
B Isoquants



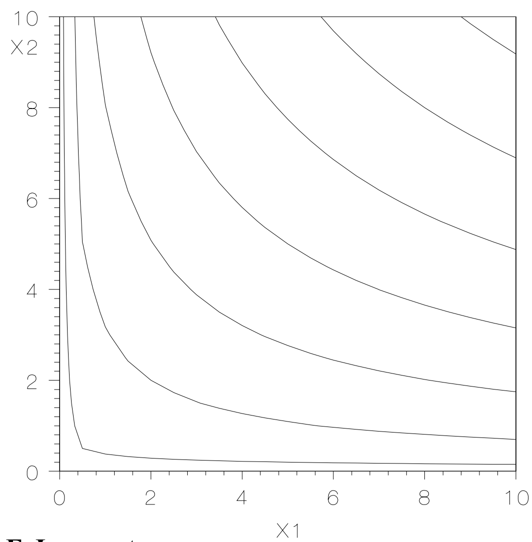
C Case 2 D = 0.5



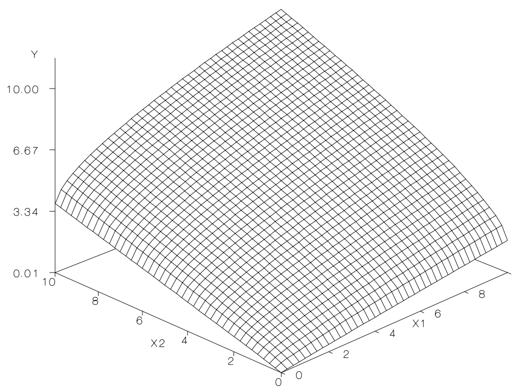
D Isoquants



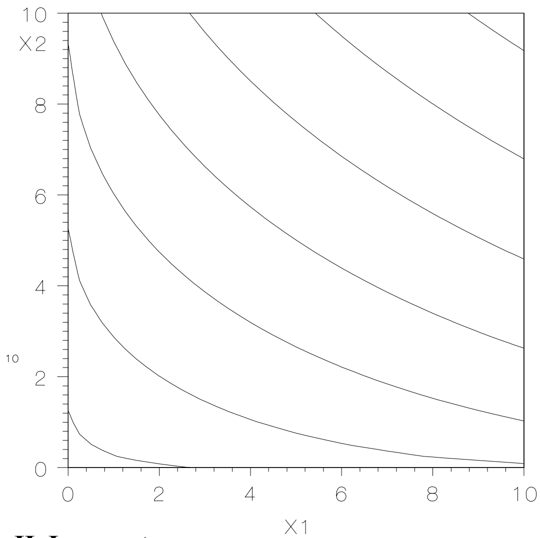
E Case 3 D = 0



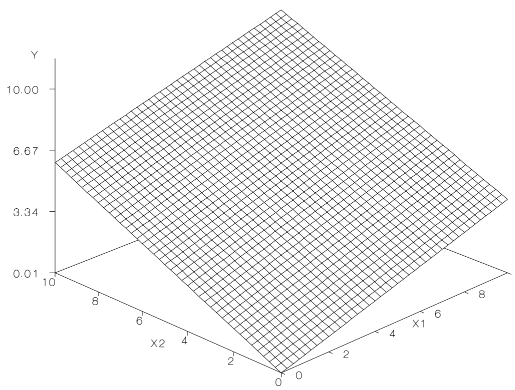
F Isoquants



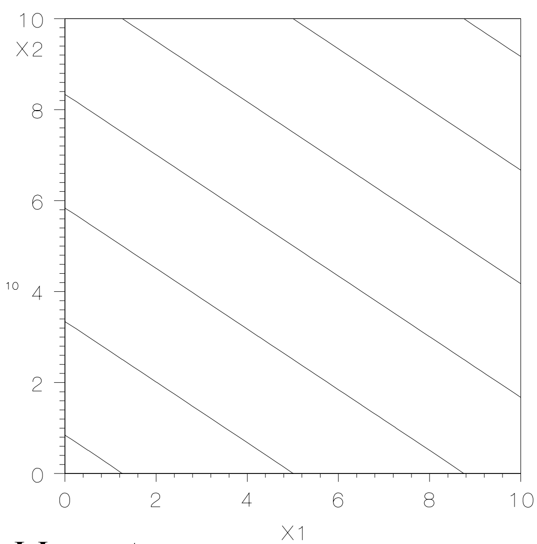
G Case 4 $\rho = -0.5$



H Isoquants



I Case 5 ρ approaches -1



J Isoquants

Figure 12.2 Production Surfaces and Isoquants for the *CES* Production Function under Varying Assumptions about ρ

Case 2: $0 < e_s < 1$; $\rho > 0$. Inputs substitute for each other, but not very easily. The isoquants are asymptotic to some value for x_1 and x_2 rather than the axes. The vertical line is at $x_2 = (k/\lambda)^{-1/\rho}$, and the horizontal line is at $x_1 = (k/(1-\lambda))^{-1/\rho}$. The number $k = (y/A)^{-\rho}$. The isoquants can be thought of as something in between the right angles in case 1 and those for a Cobb-Douglas type function. Diagrams C and D illustrate the production surface and isoquant map when $\rho = 0.5$. The production surface is undistinguished and looks similar to that for the Cobb Douglas.

Case 3: $e_s = 1$; $\rho = 0$. The CES becomes the Cobb Douglas illustrated in Diagrams E and F. The proof of this requires the use of L'Hopital's Rule and can be found in Henderson and Quandt.

Case 4: $e_s > 1$; ! 1 < D < 0. Isoquants cut both axes. In diagram G and H, for D = ! 0.5, e_s = 2, note the white area directly above the x_1 and x_2 axes. This suggests that output is possible in the absence of one of the two inputs.

Case 5: As e_s 6 +4, D 6 ! 1. At the limit the isoquants consist of lines of constant slope (with no curvature), and the production surface and isoquants are illustrated in diagram I and J. The CES reduces to the production function y = 8x_1 + (1 ! 8)x_2, and inputs substitute for each other in the fixed proportion 8/(1 ! 8).

The CES had some important advantages over the Cobb Douglas production function in that the same general functional form could be used to represent a variety of substitution possibilities and corresponding isoquant patterns, but the function had two important disadvantages. Like the Cobb Douglas, for a given set of parameter values, only one stage of production could be represented, usually stage II for both inputs. This problem was not unrelated to the fact that the elasticity of substitution was the same everywhere along the isoquant. Isoquant patterns consisting of concentric rings or ovals were not allowed.

The CES can be extended to allow for more than two inputs. However, there is but one parameter D in the multiple-input extensions. Thus only one elasticity of substitution value can be obtained from the production function, and this same value applies to all input pairs. For example, in agriculture, one might expect that the elasticity of substitution between chemicals and labor would differ markedly from the elasticity of substitution between fuel and tractors. But the CES would estimate the same elasticity of substitution between both input pairs. Despite its pedagogical charm for understanding the effects of changing elasticities of substitution on the shape of isoquants, the usefulness of the CES production function for serious research in agricultural economics in which more than two inputs were involved was limited.

12.5 Elasticities of Substitution and the Translog Production Function

Unlike the Cobb Douglas and the CES, most production functions do not have constant elasticities of substitution. The percentage change in the input ratio divided by the percentage change in the marginal rate of substitution is not constant all along the isoquant but varies from one point to another. To determine the elasticity of substitution for production functions such as these, it is necessary not only to know the parameters of the production function, but also to be aware of the precise point on the isoquant for which the elasticity of substitution is to be estimated and the input ratio (x_2/x_1) for that point.

Application of the Henderson and Quandt formula for calculating the elasticity of substitution can then be made. The elasticity of substitution as based on this formula for most production functions will contain the parameters of the function as well as x_1 and x_2.

If a production function has more than two inputs, partial elasticities of substitution for each pair of inputs can be calculated, but the algebra for doing this quickly becomes quite complicated. In the two-input setting, the elasticity of substitution will always be greater than zero. However, in the multiple-input setting, it is possible for some pairs of inputs to be substitutes and others complements. For the complement pairs, the elasticity of substitution will be negative. An example of a pair of inputs that are complements might be a tractor and the fuel required to run it.

A production function that has recently become popular with agricultural economists interested in estimating elasticities of substitution between input pairs is called the *translog production function*. A specification for the translog production function is

†12.24 $$\ln y = \ln \alpha + \beta_1 \ln x_1 + \beta_2 \ln x_2 + (1/2)\,\gamma \ln x_1 \ln x_2$$

Sometimes squared terms are also included

†12.25 $$\ln y = \ln \alpha + \beta_1 \ln x_1 + \beta_2 \ln x_2 + (1/2)\,\gamma \ln x_1 \ln x_2 + 1/2\varphi_1(\ln x_1)^2 + 1/2\varphi_2(\ln x_2)^2$$

The translog production function is a member of de Janvry's generalized power production function family. Equation †12.24 written as its antilog is

†12.26 $$y = \alpha x_1^{\beta_1} x_2^{\beta_2} e^{\frac{\gamma}{2}\ln x_1 \ln x_2}$$

Notice how similar the appearance of the translog production function is to the transcendental developed by Halter, Carter and Hocking. Moreover, the Cobb Douglas is a special case of the translog when γ equals zero.[1]

Equation †12.26 differs from the transcendental in that the parameter γ is usually assumed to be positive. The function is similar to the Cobb Douglas in that for most possible positive parameter values for γ, the function never achieves a maximum if the level of input use for x_1 and x_2 is finite. However, unlike the Cobb Douglas, the translog function does not always generate elasticities of substitution of 1. The translog function is easily generalized to problems involving more than two inputs.

The translog production function can be generalized to include any number of input categories, and each pair of inputs may have a different elasticity of substitution. The shape of the isoquants for the translog depend heavily on the parameter γ. If γ were zero, the function would generate isoquants like those for the Cobb Douglas. The marginal rate of substitution would be a linear function of the input ratio, and the elasticity of substitution would be 1 everywhere along each isoquant. As the value of γ increases, output increases markedly when both inputs are used in similar proportions to each other. As γ becomes larger and larger, the isoquants bow inward, become more nearly a right angle, and the elasticity of substitution becomes smaller and smaller.

The *MPP* of x_1 for equation †12.26 is

†12.27 $$\partial y/\partial x_1 = [\beta_1/x_1 + \gamma/2 \ln x_2(1/x_1)]y$$

The *MPP* can be set equal to zero and solved for x_2 in terms of x_1 is the equation for the ridge line for x_1.

The marginal rate of substitution for equation †12.26 is

†12.28 $$dx_2/dx_1 = -[\beta_1/x_1 + \gamma/2 \ln x_2(1/x_1)]/[\beta_2/x_2 + \gamma/2 \ln x_1(1/x_2)]$$

While parameters of the translog production function can be estimated using physical data on agricultural inputs, cost data on agricultural inputs generally more readily available than physical input data. Parameters of the production function are estimated indirectly from the cost function data. Thus, a more common research approach is to rely on duality to estimate important parameters of the underlying production function by working with a cost function having a translog form

†12.29 $ln\ C = \ln N + 2_1\ ln\ v_1 + 2_2 \ln v_2 + ½2_3\ ln\ v_1\ ln\ v_2$

where C = total cost

v_1, v_2 = input prices

$N, 2_1, 2_2, 2_3$ = parameters or coefficients

ln = the natural logarithm of

Partially differentiating the natural logarithm of †12.29 with respect to the natural logarithm of v_1 and v_2 results in

†12.30 $M\ln C/M\ln v_1 = 2_1 + ½2_3 \ln v_2$

†12.31 $M\ln C/M\ln v_2 = 2_2 + ½2_3 \ln v_1$

Notice that[2]

†12.32 $M\ln C/M\ln v_1 = (MC/Mv_1)(v_1/C)$

†12.33 $M\ln C/M\ln v_2 = (MC/Mv_2)(v_2/C)$

*Shephard's lemm*a can be used to convert equations †12.30 and †12.31 into cost-share equations. Shephard's lemma states that

†12.34 $MC/Mv_1 = x_1^*$

†12.35 $MC/Mv_2 = x_2^*$

where x_1^* and x_2^* are the amounts of x_1 and x_2 defined by the points of least-cost combination on the expansion path. Along the expansion path, the change in the cost function with respect to each input price is equal to the quantity of input that is used. Therefore

†12.36 $M\ln C/M\ln v_1 = v_1 x_1^*/C = S_1$

or the share or proportion of total cost for input x_1.

†12.37 $M\ln C/M\ln v_2 = v_2 x_2^*/C = S_2$

or the share or proportion of total cost for input x_2.

Substitution †12.36 and †12.37 into equations †12.30 and †12.31

†12.38 $S_1 = 2_1 + ½2_3 \ln v_2$

†12.39 $S_2 = 2_2 + ½2_3 \ln v_1$

Equations †12.38 and †12.39 are the cost-share equations for inputs x_1 and x_2. Estimates of 2_1, 2_2, and 2_3 can be used as the basis for deriving the elasticities of substitution and other parameters or coefficients for the underlying production function.[3]

Economists and agricultural economists have attempted to determine the elasticities of substitution for major input categories using the cost share approach outlined above. The focus of economists such as Berndt and Wood has recently been to determine whether capital and energy complement or substitute for each other. Some studies by economists have concluded on the basis of the estimates of the translog production function parameters that energy and capital are complements, whereas others have concluded that they are substitutes.

Webb and Duncan, Brown and Christensen, and Aoun all estimated elasticities of substitution for major input categories in U.S. agriculture using the translog production function as a basis. Aoun estimated partial elasticities of substitution between the input category energy and the input category tractors and machinery. In the 1950s and 1960s, tractors and machinery were complements, as indicated by a negative partial elasticity of substitution, but by the late 1970s, these two input categories had become substitutes. This provides evidence that farmers can now substitute improved tractors and machinery (that produce greater output per unit of fuel burned) for fuel. The belief that improvements in tractors and machinery can come only with increased fuel use may not now hold true.

12.5 Concluding Comments

The elasticity of substitution between pairs of inputs is among the most important concept in all of economics. Increasingly, production research both in and out of agriculture has focused on the estimation of elasticities of substitution between input pairs. The CES production function is a useful teaching tool for uncovering the linkage between the elasticity of substitution and the shape of the isoquants. Despite its usefulness as a teaching tool, because it could generate only a single estimate of an elasticity of substitution in the multiple-input case, its application to agriculture was limited.

The development of the translog production and cost functions in the early 1970s represented a major step forward in production theory. The translog form was not nearly as restrictive as the Cobb Douglas and CES forms that preceded it. The translog production and cost functions could be inverted, and recent theoretical developments related to the duality of cost and production could have application both in and out of agriculture. The application of translog cost functions using the cost share approach for estimating elasticities of substitution between inputs will have applications to many different agricultural sectors in the coming years.

Notes

1. One way of looking at production functions is in terms of Taylor's series expansions. The Cobb Douglas production function is a first-order Taylor's series expansion of ln y in ln x_1 and ln x_2, and the translog is a second order expansion of the same terms. The CES is a first order expansion of y^D in x_1^D and x_2^D. If the translog production function is treated as a Taylor's series expansion, squared terms are included:

$$\ln y = \ln " + \$_1 \ln x_1 + \$_2 \ln x_2 + 1/2 (\ln x_1 \ln x_2$$

$$+ 1/2 N_1 (\ln x_1)^2 + 1/2 N_2 (\ln x_2)^2$$

Squared terms can also be added to the translog cost function (equation †12.29 ; see also Christensen, Jorgenson and Lau).

2. A detailed proof can be found in Section 13.3.

3. A detailed derivation of the linkage between the parameters of the cost share equations and the elasticity of substitution can be found in the Brown and Christensen reference.

Problems and Exercises

1. Explain what is meant by the term *elasticity of substitution*. How does the elasticity of substitution differ from the marginal rate of substitution? How does the elasticity of substitution differs from the elasticity of production? Why is the elasticity of substitution between input pairs important in agriculture?

2. For the following production functions, what is the elasticity of substitution?

a. $y = ax_1 + bx_2$
b. $y = x_1^{0.33}x_2^{0.5}$
c. $y = A(bx_1^{-2} + (1 - b)x_2^{-2})^{-1/2}$

3. Draw the isoquants associated with each production function listed in Problem 2.

4. The elasticity of substitution is closely linked to both the marginal rate of substitution and the input ratio (x_2/x_1). Suppose that the marginal rate of substitution is given by the formula

$$MRS_{x_1x_2} = (x_2/x_1)^b$$

a. What is the corresponding elasticity of substitution?
b. What is known about the production function that produced such a marginal rate of substitution?

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