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The Demand for Inputs to the Production Process

The demand for inputs to a production process within agriculture is dependent on a number of factors: (1) the price of the output being produced, (2) the price of the input, (3) the price of other substitute or complement inputs that are also in the production function, and (4) the technical coefficients or parameters of the production function itself, particularly production elasticities for each input. Under certain conditions, the quantity as well as the price of other inputs, and the availability of dollars for the purchase of inputs may affect the input demand function. This chapter shows how specific input demand functions can be derived that explicitly link the demand by a farmer for an input to the prices of other inputs and the technical parameters of the underlying production function.

Key terms and definitions:

Derived Demand
Input Demand Function
Elasticity of Input Demand
Logarithmic Differentiation
Output Price Input Demand Elasticity
Own Price Input Demand Elasticity
Cross Price Input Demand Elasticity
Technical Complement
Technical Competitiveness
Technical Independence

13.1 Introduction

The demand for inputs to the agricultural production process is a derived demand. That is, the input demand function is derived from the demand by buyers of the output from the farm. In general, the demand for an input or factor of production depends on (1) the price of the output or outputs being produced, (2) the price of the input, (3) the prices of other inputs that substitute for or complement the input, and (4) the parameters of the production function that describes the technical transformation of the input into an output. In some instances, the demand for an input might also depend on the availability of dollars needed to purchase the input.

For example, the demand by a farmer for seed, fertilizer, machinery, chemicals, and other inputs is derived from the demand by users for the corn produced by the farmer. The demand for each of these inputs is a function not only of their respective prices, but also the price of corn in the marketplace. The demand by a dairy farmer for grain and forage is dependent not only on the respective prices of grain and forage, but also on the price of the milk being produced.

13.2 A Single-Input Setting

In a single input setting, the derivation of a demand function for an input x makes use of (1) the production function that transforms the input x into the product y; (2) the price of the output y, called p, and (3) the own price of the input, called v. Since there are no other inputs, in a single input setting prices of other inputs do not enter.

A general statement of the problem is as follows. Given a production function $y = f(x$, ") where x is the quantity of input used and " represents the coefficients or parameters of the production function, a constant product price (p) and a constant input price (v), the corresponding input demand function can be written as $x = g($", $p, v)$. Notice that the function g, the input demand function, is a different function from f, the production function. The derivation of the input demand function for a specific production function and set of prices makes use of the firm's first order conditions for profit maximization.

Assume that the farm manager uses only one input in the production of a single output. The farmer is operating in a purely competitive environment, and the price of the input and the output is assumed to be fixed and given. The farmer is interested in maximizing profits. The first order conditions for maximum profit require that the farmer equate

†13.1 $$pMPP_x = VMP_x = v$$

where p is the output price and v is the input price.

Now suppose that the price of the input (v) varies. Figure 13.1 illustrates what happens. The intersection between VMP_x and v represents the demand for the input at that particular input price, which, in turn, traces out the demand curve or input demand function for the input x under a series of alternative input prices. If the price of the output increases, the VMP curve will shift upward, increasing the demand for x at any positive input price. Conversely, a decrease in the price of the output will reduce the demand for the input x at any given input price. The input demand function normally begins at the start of stage II and ends at the start of stage III.

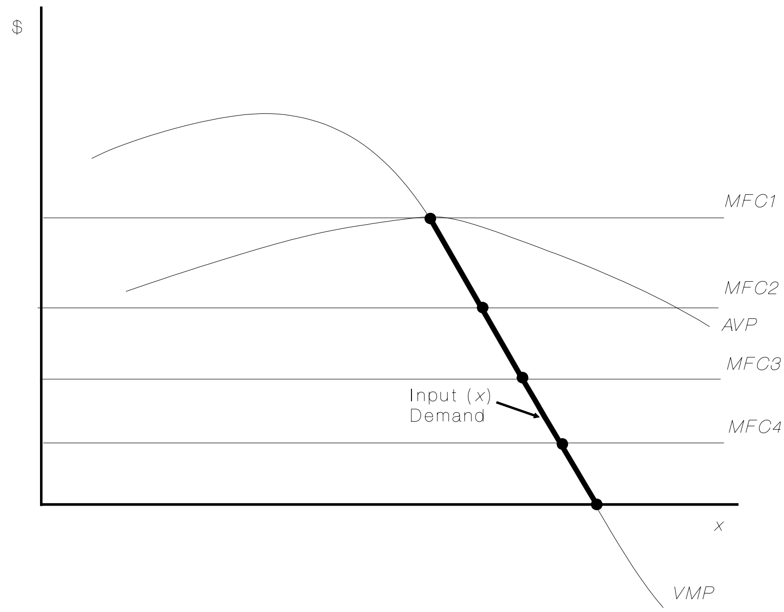

Figure 13.1 The Demand Function for Input *x*
(No Other Inputs)

As the productivity of the underlying production function increases, the MPP_x will increase. This, in turn, will increase the demand by farmers for input *x*. Conversely, a decrease in the productivity of the underlying production function will cause a reduction in the demand for *x* for a given input and output price.

Assume that the production function is

†13.2 $y = Ax^b$

Where *A* is a positive number and *b* is assumed to be greater than zero but less than 1.

The corresponding *MPP* of *x* is

†13.3 $MPP_x = dy/dx = bAx^{b!1}$

The first order conditions for maximum profit require that

†13.4 $PMPP_x = pbAx^{b!1} = v$

The demand for the input can be found by solving the first order conditions for *x*

†13.5 $x^{b!1} = v/pbA$

†13.6 $x = (v/pbA)^{1/(b!1)} = v^{1/(b!1)}p^{!1/(b!1)}(Ba)^{!1/(b!1)}$

Notice here that the demand for *x* is a function only of the price of the input (*v*), the price of the product (*p*), and the coefficient or parameter of the underlying production function (*b*) as suggested in the general case.

A numerical example is used to further illustrate these relationships. Assume that A is 1 and b is 0.5. Then

†13.7 $x = 0.25v^{-2}p^2 = 0.25p^2/v^2$

Table 13.1 provides four demand schedules for input x, when A is 1 and b is 0.5, and assuming output prices of \$2, \$4, \$6, and \$8. Each column represents a different demand function. As the price of x increases, the quantity demanded declines. An increase in the price of the output (y) causes a shift upward in the entire demand schedule or function.

Table 13.1 Demand for Units of Input *x* Under Various Assumptions about the Output Price, *p*

Price of *x* (*v*) in (dollars	Price of *y* in dollars			
	2	4	6	8
1	1.00	4.00	9.00	16.00
2	0.25	1.00	2.25	4.00
3	0.11	0.44	1.00	1.78
4	0.0625	0.25	0.5625	1.00
5	0.04	0.16	0.36	0.64

13.3 The Elasticity of Input Demand

In consumer demand, the elasticity of demand is defined as the percentage change in quantity of a good taken from the market divided by the percentage change in the price of that good. Using calculus, the point elasticity of demand is defined as

†13.8 $(dQ/dP)(P/Q)$

where P is the price of the good being demanded by the consumer, and Q is the quantity of the good

Now suppose that the specific demand function is

†13.9 $Q = P^a$

Taking natural logarithms of both sides of equation †13.1 , results in

†13.10 $\ln Q = a \ln P$

Now let r equal $\ln Q$ and s equal $\ln P$; equation †13.10 may be rewritten as

†13.11 $r = as$

Now differentiate equation †13.11 :

†13.12 $dr/ds = a$

But notice that

†13.13 $d \ln Q/d \ln P = a$

The elasticity of demand for Q can be shown to be equal to the coefficient or parameter a. In this example

†13.14 $dq/dp = aP^{a-1}$

†13.15 $(dq/dp)\,(P/Q) = (Ap^{a-1})(P/Q) = (Ap^{a-1})(P/P^{a}) = a$

which is the same result as that obtained in equation †13.13 . In general, any elasticity can be expressed as the derivative of the logarithm of one of the variables with respect to the derivative of the logarithm of the other variable.

Parallel formulas for input demand elasticities exist. The own price elasticity of demand for an input is defined as the percentage change in the quantity of the input taken from the market divided by the percentage change in the price of that input. Using calculus, the own price input demand elasticity is

†13.16 $(dx/dv)(v/x)$, or

†13.17 $d\ln x/d\ln v$.

The output-price elasticity can be similarly defined as the percentage change in the quantity of the input taken from the market divided by the percentage change in the price of the output. Using calculus, the output-price demand elasticity is defined either as

†13.18 $(dx/dp)(p/x)$

or as

†13.19 $d\ln x/d\ln p$.

If there were more inputs to the production process than one, both own-price and cross-price elasticities can be defined. The own price elasticity is the same as is the single input case, that is, the percentage change in the quantity of the input x_i taken from the market divided by the percentage change in the price of that input (v_i). The subscript i indicates that the price and quantity are for the same input. The formula using calculus would be either

†13.20 $(dx_i/dv_i)(v_i/x_i)$

or as

†13.21 $d\ln x_i/d\ln v_i$

The cross-price elasticity is defined as the percentage change in the quantity of input x_i taken from the market divided by the percentage change in the price of input x_j (v_j). The subscript i is not the same as j. Using calculus, the formula is

†13.22 $(dx_i/dv_j)(v_j/x_i)$

for all $i \neq j$
or as

†13.23 $d\ln x_i/d\ln v_j$

Now consider a production function

†13.24 $y = Ax^b$

The input price (v) and the output price (p) are assumed constant and the farmer is assumed to maximize profits. The input demand function is

†13.25 $x = (v/pbA)^{1/(b-1)} = v^{1/(b-1)}p^{-1/(b-1)}(Ba)^{-1/(b-1)}$

The own price elasticity of input demand is derived as follows

†13.26 $dx/dv = [1/(b-1)/v]x = [1/(b-1)](x/v)$

†13.27 $(dx/dv)(v/x) = [1/(b-1)](x/v)(v/x) = 1/(b-1)$

The own-price elasticity could be obtained by taking natural logarithms of the input demand function and then finding the derivative

†13.28 $d\ln x/d\ln v = 1/(b-1)$

The own price elasticity of demand for the input depends entirely on the parameter b from the underlying power production function. Given information about the elasticity of production for the input, the corresponding input demand elasticity can be calculated. For example, if b were 0.5, the own-price elasticity of demand for x is $1/(0.5-1) = -2$. There exists a close association between the elasticity of demand for an input and the underlying elasticity of production for that input. This analysis breaks down if b is greater than or equal to 1. If b is greater than 1, *VMP* cuts *MFC* (v) from below, and the second-order conditions for profit maximization do not hold for any finite level of use of x. If b is equal to 1, *VMP* = *MFC* everywhere and there is no demand function based on the profit-maximization assumption.

A similar analysis can be made for the output-price elasticity

†13.29 $dx/dp = [-1/(b-1)](x/p) = -x/[p(b-1)]$

†13.30 $(dx/dp)p/x = -px/[px(b-1)] = -1/(b-1)$

or

†13.31 $d\ln x/d\ln p = -1/(b-1)$

In the single-input case, the output-price elasticity of demand for input x is equal to the negative of the own-price elasticity of demand. In this case, the output-price elasticity of demand is 2. This suggests that a 1 percent increase in the price of the output will be accompanied by a 2 percent increase in the demand for the input x. Again, the output-price elasticity of demand is a function solely of the elasticity of production of the underlying production function.

13.4 Technical Complements, Competitiveness, and Independence

An input (x_2) can be defined as a technical complement for another input (x_1) if an increase in the use of x_2 causes the marginal product of x_1 to increase. Most inputs are technical complements of each other. Notice that inputs can be technical complements and still substitute for each other along a downward- sloping isoquant.[1]

A simple example of technical complements in agriculture would be two different kinds of fertilizer nutrients in corn production. For example, the presence of adequate quantities of phosphate may make the productivity of nitrogen fertilizer greater.

Technical complements can also be defined by

†13.32 $\qquad d(MPP_{x_1})/dx_2 > 0$

Consider a production function given by

†13.33 $\qquad y = Ax_1^a x_2^b$

MPP_{x_1} is

†13.34 $\qquad dy/dx_1 = aAx_1^{a!1}x_2^b$

†13.35 $\qquad d(dy/dx_1)/dx_2 = baAx_1^{a!1}x_2^{b!1} > 0$

By this definition, inputs are technical complements for a broad class of Cobb Douglas type of production functions. An increase in the use of x_2 causes the MPP_{x_1} to shift upward.

An input (x_2) is said to be technically independent of another input if when the use of x_2 is increased, the marginal product of x_1 (MPP_{x_1}) does not change. This requires that

†13.36 $\qquad d(MPP_{x_1})/dx_2 = 0$

Consider a production function given by

†13.37 $\qquad y = ax_1 + bx_1^2 + cx_2 + dx_2^2$

†13.38 $\qquad dy/dx_1 = a + 2bx_1$

†13.39 $\qquad d(dy/dx_1)/dx_2 = 0$

For additive production functions without interaction terms, inputs are technically independent.

Examples of technically independent inputs to a production process within agriculture are difficult to find. Even the marginal product of a laborer may be affected by the availability of other inputs such as seed and chemicals.

An input (x_2) is said to be technically competitive with another input (x_1) if when the use of x_2 is increased, the marginal product of x_1 (MPP_{x_1}) decreases. This requires that

†13.40 $d(MPP_{x_1})/dx_2 < 0$

An example of a production function in which this might occur is an additive function with a negative interaction term.

Consider a production function given by

†13.41 $y = ax_1 + bx_1x_2 + cx_2$

†13.42 $dy/dx_1 = a + bx_2$

†13.43 $d(dy/dx_1)/dx_2 = b$

If b were negative, the inputs would be technically competitive.

Examples of inputs that are technical substitutes for each other would include inputs that are very similar to each other. For example, suppose that x_1 represented nitrogen applied as ammonium nitrate and x_2 represented nitrogen applied as anhydrous ammonia. The presence of ample quantities of x_1 would reduce the marginal product of x_2.

13.5 Input-Demand Functions in a Two-Input Setting

Input demand functions in a two input setting can also be derived. Suppose that the farmer is again interested in maximizing profits, and that output and input prices are given. The production function is

†13.44 $y = Ax_1^a x_2^b$

The profit function corresponding to equation †13.44 is

†13.45 $\pi = py - v_1x_1 - v_2x_2$

$= pAx_1^a x_2^b - v_1x_1 - v_2x_2$

Suppose also that $a + b < 0$ (decreasing returns to scale). Then the first order conditions for profit maximization are

†13.46 $\partial\pi/\partial x_1 = apAx_1^{a-1}x_2^b - v_1 = 0$

†13.47 $\partial\pi/\partial x_2 = bpAx_1^a x_2^{b-1} - v_2 = 0$

One approach for finding the input demand function for x_1 would be to solve the first-order condition equation †13.46 for x_1 in terms of the remaining variables. This yields

†13.48 $x_1^{a-1} = v_1(apA)^{-1}x_2^{-b}$

†13.49 $x_1 = v_1^{1/(a-1)}(apa)^{-1/(a-1)}x_2^{-b/(a-1)}$

Equation †13.49 expresses the demand for x_1 in terms of its own price (v_1) the price of the output (p), and the quantity of the other input (x_2). This approach leads to a demand function made up of points of intersection between a single *VMP* function (that assumes a constant x_2) and the price of x_1 (v_1). But the quantity of x_2 used will probably change if the price of x_1 changes, so the assumption that x_2 can be assumed constant is untenable.

Figure 13.2 illustrates three cases. Diagram A illustrates the common case in which an increase in the price of x_1 causes the quantity of x_2 that is used to decrease. Diagram B illustrates a case in which the use of x_2 increases as a result of an increase in the price of x_1. Diagram C illustrates a special case in which the use of x_2 remains constant when the price of x_1 increases. Diagram C illustrates the only case in which this approach would yield the correct input demand function.

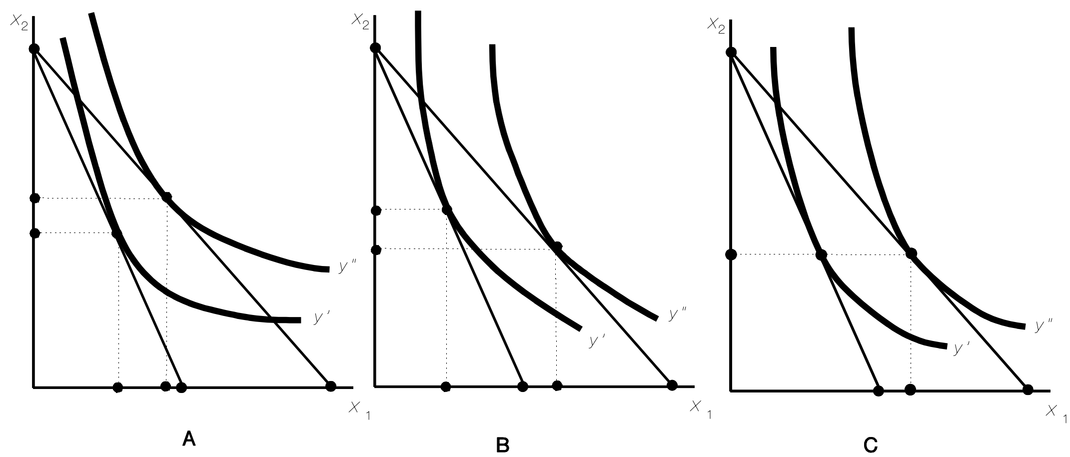

Figure 13.2 Possible Impacts of an Increase in the Price of x_1 on the Use of x_2

Only if inputs are technically independent will the marginal product and *VMP* of one input be unaffected by the quantity of the other input(s) that is(are) available. In other words, it is highly unlikely that the *VMP* for x_1 would be unaffected by the availability of x_2. As a result, the input demand function specified in equation †13.49 will probably make the demand function for the input x_1 appear less elastic than it really is.

As the price of input x_1 increases, the farmer will use less of it, because the level of x_1 that maximizes profits will shift to the left. This effect is captured by the own price elasticity in equation †13.49 . However, the farmer might also respond to the increased price for x_1 by substituting x_2 for x_1, and equation †13.49 ignores this substitution possibility. The quantity of x_2 used by the farmer is treated as fixed.

Another approach is clearly needed that will explicitly take into account the possibility of substitution x_2 for x_1 as the price of x_1 rises. The use of x_1 should be a function not of the quantity of x_2 but rather of the price of x_2. Such an approach would allow the farmer to move from one *VMP* function to another as the price of x_1 (v_1) changes. A change in the price of x_1 causes the use of x_2 to change, which in turn, results in a new *VMP* function for x_1 (Figure 13.3).

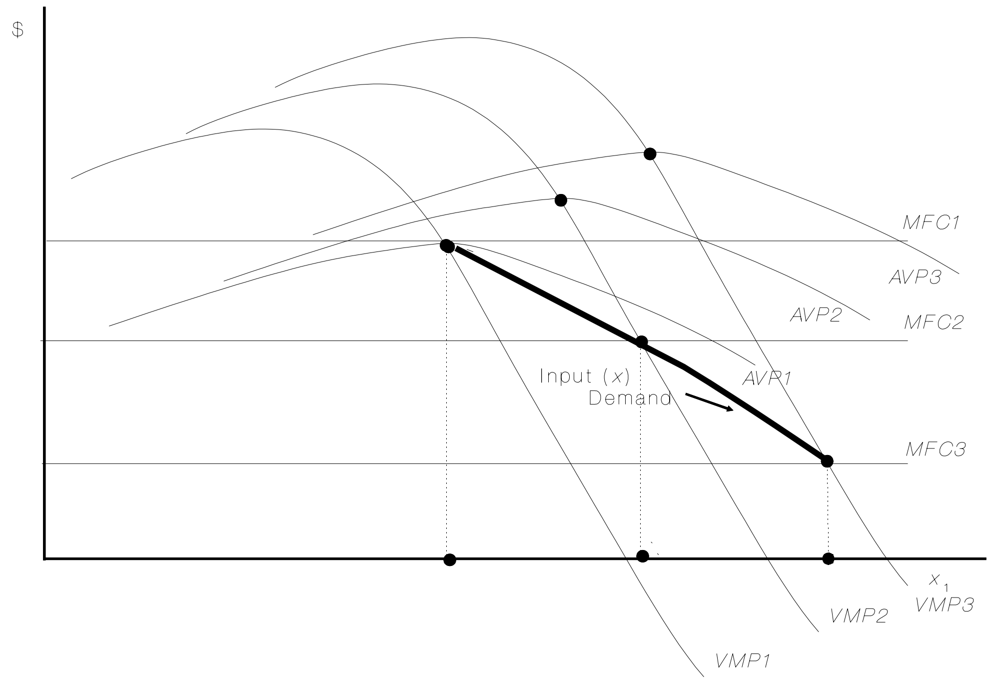

Figure 13.3 Demand for Input x_1 When a Decrease in the price of x_1 Increases the Use of x_2

The new approach makes use of the same first order conditions (equations †13.46 and †13.47) as those used in the first example. Prices and production function parameters are treated as knowns, the quantities of x_1 and x_2 are unknowns. Equations †13.46 and †13.47 thus represent two equations in two unknowns that are solved as a system. To solve the system, first-order condition †13.46 is divided by first-order condition †13.47 to yield

†13.50 $ax_2/bx_1 = v_1/v_2$

or

†13.51 $x_2 = v_1bx_1/av_2$

Equation †13.51 is then substituted into first-order condition †13.46 and solved for x_1

†13.52 $Apax_1^{a+b-1}v_1^{b}v_2^{-b}\, b^{b}\, a^{-b} = v_1$

†13.53 $x_1^{a+b-1} = v_1^{1-b}\, v_2^{b}(pA)^{-1}\, a^{b-1}\, b^{-b}$

†13.54 $x_1 = v_1^{(1-b)/(a+b-1)}v_2^{b/(a+b-1)}(pA)^{-1/(a+b-1)}a^{(b-1)/(a+b-1)}b^{-b/(a+b-1)}$

For equation †13.54 , the input own ! price demand elasticity is

†13.55 $(dx_1/dv_1)(v_1/x_1) = (1 - b)/(a + b - 1) < 0$

†13.56 $d\ln x_1/d\ln v_1 = (1 - b)/(a + b - 1) < 0$

If $a + b < 1$, then the input own ! price demand elasticity is negative. For any specific set of values for a and b, the input own ! price demand elasticity may be calculated.

The cross price demand elasticity between input x_1 and x_2 may be defined from equation †13.54 as

†13.57 $$(dx_1/dv_2)(v_2/x_1) = b/(a + b - 1) < 0$$

†13.58 $$d\ln x_1/d\ln v_2 = b/(a + b - 1) < 0$$

This elasticity is also negative when $a + b < 1$. As the price of x_2 increases, less of x_1 will be used.

The output price elasticity is

†13.59 $$(dx_1/dp)(p/x_1) = -1/(a + b - 1) > 0$$

†13.60 $$d\ln x_1/d\ln p = -1/(a + b - 1) > 0$$

This elasticity is positive when $a + b < 1$. This suggests that the demand for x_1 increases as the output price increases.

Notice also that the sum of the input own−price and cross-price elasticities equals the negative of the output price elasticity

†13.61 $$(1 - b)/(a + b - 1) + b/(a + b - 1) = -1[-1/(a + b - 1)]$$

The relationship defined in equation †13.61 between elasticities holds for production functions with decreasing returns to scale. This relationship would also hold in instances where there are more than two inputs. In general, the sum of the own−price and cross-price input demand elasticities equals the negative of the output-price input demand elasticity.

The own−price and product-price elasticities obtained from the second approach will in general be more strongly negative or elastic than those obtained from the first approach (see Figure 13.3). However, the exact relationship between elasticities will depend on the extent to which the farmer substitutes x_2 for x_1 in the face of rising prices and the impact that this substitution has on the *VMP* function for x_1. Estimates of elasticities from the second approach normally should more accurately portray the extent of the adjustment process by the farmer in response to changing input prices than those estimates obtained from the first approach.

13.6 Input-Demand Functions Under Constrained Maximization

Ordinarily, no attempt would be made to derive individual input demand functions for production functions that have constant or increasing returns to scale. If there were increasing returns to scale and input prices were constant (not a function of the demand for the input), profits to the farmer could be maximized by securing as much of both (or all) inputs as possible. Here, no demand function as such could exist. If there were constant returns to scale, the farmer would shut down if the cost of the inputs per unit of output exceeded the output price. If the cost of the inputs per unit of output was less than the product price, the farmer would again attempt to secure as much of each input as possible, and no demand function for the input would exist.

However, if the farmer has a constraint or limitation in the availability of dollars for the purchase of inputs, it may be possible to derive input demand functions even when the underlying production function has no global profit maximizing solution, or in other situations where a constraint exists in the availability of dollars for the purchase of inputs. Such demand functions are sometimes referred to as *conditional demand functions*, in that they assume that the specific budget constraint is met. The conditional demand function specifies the quantity of x_1 and x_2 that will be demanded by the farmer for a series of input prices v_1 and v_2, and assuming that $C°$ total dollars are spent on inputs.

Consider the production function

†13.62 $$y = x_1x_2$$

The function coefficient for this production function is 2. Now suppose that the farmer faces a budget constraint $C°$

†13.63 $$C° = v_1x_1 + v_2x_2$$

At the budget level defined by equation †13.63 , output y° can be produced.

The Lagrangean representing the constrained maximization problem is

†13.64 $$L = x_1x_2 + \lambda(C° - v_1x_1 - v_2x_2)$$

A key assumption of Lagranges formulation is that the farmer must spend exactly $C°$ dollars on x_1 and x_2. The corresponding first order conditions are

†13.65 $$\partial L/\partial x_1 = x_2 - \lambda v_1 = 0$$

†13.66 $$\partial L/\partial x_2 = x_1 - \lambda v_2 = 0$$

†13.67 $$\partial L/\partial \lambda = C° - v_1x_1 - v_2x_2 = 0$$

Dividing equation †13.65 by equation †13.66 and rearranging gives us

†13.68 $$x_2 = (v_1/v_2)x_1$$

Inserting equation †13.68 into equation †13.67 yields

†13.69 $$C° - v_1x_1 - v_1x_1 = 0$$

†13.70 $$C° - 2v_1x_1 = 0$$

†13.71 $$2v_1x_1 = C°$$

†13.72 $$x_1 = C°/2v_1$$

In this example, the demand for input x_1 is a function only of its own price and the dollars available for the purchase of x_1. However, this conclusion is a result of the particular set of coefficients or parameters chosen for the production function and does not hold in the general case.

The input demand function for x_2 could be derived analogously. The price of x_2 (v_2) would have appeared in the input demand function if both x_1 and x_2 appear in each *MPP*. The price of the output does not enter. The constrained maximization problem assumes that the output level defined by the isoquant tangent to the budget constraint will be produced regardless of the output price. The possibility that the farmer may wish to instead shut down is not recognized by the calculus.

13.7 Comparative Statics and Input Demand Elasticities

Consider a general profit function for the two-input case

†13.73) $$B = pf(x_1,x_2) - v_1x_1 - v_2x_2.$$

The first order profit-maximizing conditions are

†13.74) $$MB/Mx_1 = pf_1 - v_1 = 0$$

†13.75) $$MB/Mx_2 = pf_2 - v_2 = 0.$$

How does the use of the inputs x_1 and x_2 vary with prices of the inputs v_1 and v_2 and with the output price p. To determine this, it is necessary to take the total differential of (13.74) and (13.75), treating the input quantities and the prices of both the inputs and the outputs as constants.

The elasticity of demand for input x_1 with respect to its own price is $(dx_1/dv_1)(v_1/x_1) = d\ln x_1/d\ln v_1$; with respect to the price of the second input is $(dx_1/dv_2)(v_2/x_1) = d\ln x_1/d\ln v_2$; with respect to the product price is $(dx_1/dp)(p/x_1) = d\ln x_1/d\ln p$. The sign on each of these elasticities determines whether the firm will increase or decrease its use of the input or factor of production with respect to a change in each of the prices.

The prices and input quantities are always positive, and hence, do not affect the sign on each elasticity. However, the sign on dx_1 and dx_2 when either v_1, v_2, or p changes determines the sign on the corresponding elasticity. Hence, dx_1 and dx_2 must each be calculated assuming a change in v_1 (dv_1), a change in v_2 (dv_2) and a change in p (dp).

To do this, the total differential of equations (13.74) and (13.75) is calculated, allowing input quantities and the prices of inputs and the output to vary. The result is.

†13.76) $$pf_{11}dx_1 + pf_{12}dx_2 = dv_1 - f_1dp$$

$$pf_{21}dx_1 + pf_{22}dx_2 = dv_2 - f_2dp$$

First, equation 13.76 is solved. It is easier to employ matrix notation to do this.

(13.77) $$\begin{bmatrix} pf_{11} & pf_{12} \\ pf_{21} & pf_{22} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix} = \begin{bmatrix} (dv_1 - f_1dp) \\ (dv_2 - f_2dp) \end{bmatrix}$$

Solving for dx_1 and dx_2,

(13.78) $$\begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix} = \begin{bmatrix} pf_{11} & pf_{12} \\ pf_{21} & pf_{22} \end{bmatrix}^{-1} \begin{bmatrix} (dv_1 - f_1 dp) \\ (dv_2 - f_2 dp) \end{bmatrix}$$

Equation (13.78) can be solved for dx_1 or dx_2 by using Cramer's rule. For example, dx_1 is

(13.79) $$dx_1 = \frac{\begin{vmatrix} (dv_1 - f_1 dp) & pf_{12} \\ (dv_2 - f_2 dp) & pf_{22} \end{vmatrix}}{\begin{vmatrix} pf_{11} & pf_{12} \\ pf_{21} & pf_{22} \end{vmatrix}}$$

Since by Young's theorem, $f_{12} = f_{21}$, then

(13.80) $$dx_1 = \frac{pf_{22}(dv_1 - f_1 dp) - pf_{12}(dv_2 - f_2 dp)}{p^2(f_{11}f_{22} - f_{12}^2)}$$

Notice that p^2 is always positive. Furthermore, for second order conditions to be met for profit maximization, the quantity $f_{11}f_{22} - f_{12}^2$ must always be positive. Therefore, the bottom half of equation (13.80) must always be positive. Hence, the sign on dx_1 is conditional on the sign on the top half of equation (13.80).

First, suppose that the input's own price increases, while other prices are held constant. Thus, dv_1 increases, but dv_2 and dp are assumed to be zero. Equation (13.80) becomes

(13.81) $$dx_1 = \frac{pf_{22}(dv_1 - 0f_1) - p_{f_{12}}(0 - 0f_2)}{p^2(f_{11}f_{22} - f_{12}^2)}$$

Therefore,

(13.82) $$\frac{dx_1}{dv_1} = \frac{f_{22}}{p(f_{11}f_{22} - f_{12}^2)}$$

Since the bottom half of equation (13.82) is always positive, the sign on dx_1/dv_1 depends entirely on the sign on f_{22}. The second derivative f_{22} is the slope of MPP_{x_2}, which must be negative to fulfill the second order conditions for profit maximization derived by differentiating equations (13.74) and (13.75). Therefore, without exception, if the first and second order profit-maximizing conditions are fulfilled, then the firm will *always* use less of an input in response to an increase in the input's own price. Since the own-price input elasticity of demand is defined as $(dx_1/dv_1)(v_1/x_1)$, and v_1/x_1 is always positive, the input's own-price elasticity of demand is therefore always negative.

Now consider the demand for input x_1 in response to an increase in the price of the product, p. Equation (13.81) becomes

(13.83) $$dx_1 = \frac{pf_{22}(0 - f_1 dp) - pf_{12}(0 - f_2 dp)}{p^2(f_{11}f_{22} - f_{12}^2)}$$

Rearranging,

(13.84) $$\frac{dx_1}{dp} = \frac{-f_1 f_{22} + f_2 f_{12}}{p(f_{11}f_{22} - f_{12}^2)}$$

What is known about the sign on equation (13.84)? Once again the bottom half of the fraction must be positive in order to fulfill the second order conditions for profit maximization. We know that if the inputs have positive prices, then both f_1 and f_2 must be positive, since MPP_{x_1} and MPP_{x_2} are always positive at the point of profit maximization. The second derivative, f_{22} (the slope of MPP_{x_2}), is always negative for a maximum. Therefore the term $-f_1f_{22}$ is always positive. Since f_2 is also positive, the sign on equation (13.84) depends in part on the sign on f_{12}. Only if f_{12} is negative is there a *possibility* that dx_1/dp could be negative. If f_{12} is negative, then the sign on dx_1/dp will be negative if the absolute value of f_2f_{12} is greater than f_1f_{22}.

Clearly, we cannot conclude that the firm will *always* use more of x_1 in response to an increase in the output price. However, the circumstances under which f_{12} would be negative enough for the firm do decrease its use of x_1 in response to an increase in the product price are quite rare. To illustrate, it is helpful to understand the economic interpretation of the cross partial f_{12}. The cross partial f_{12} is the change in MPP_{x_1} with respect to an increase in the use of x_2. (By Young's theorem, f_{12} is also the change in the $MPPx_2$ with respect to an increase in the use of x_1.) In other words, if the use of x_2 is increased, f_{12} tells us by how much this increase will affect MPP_{x_1}.

Consider three production functions. The first is

(13.85) $$y = x_1^{"} + x_2^{\$}.$$

For equation (13.85), since there are no cross products (interaction terms containing the product of x_1 and x_2), f_{12} is zero. In general, this will be true for all additive functions that do not include interaction terms (cross products) between the two inputs.

Suppose, however, that equation (13.85) was modified such that

(13.86) $$y = x_1^{"} + x_2^{\$} + * x_1 x_2.$$

For equation (13.86), f_{12} could be negative if * were negative, but this would mean that an *increase* in the use of one of the inputs *decreased* the productivity of the other input. Even if * were negative, it would need to be quite negative if the absolute value of f_2f_{12} were to be greater than greater than f_1f_{22}. This means that for the elasticity of demand for input x_1 to be negative with respect to the price of the product, increases in the use of input x_2 would need to result in a *substantial* decline in MPP_{x_1}!

Finally, consider a Cobb-Douglas type function

(13.87) $$y = Ax_1^{"}x_2^{\$}.$$

Assuming that " and $ are positive, f_{12} will always be positive. That is, an increase in the use of x_2 *cannot* decrease the marginal productivity of x_1. The reader may verify the sign on f_{12} for other production functions in this book.

The firm's response to changes in the price of a second input depends only on the sign on f_{12}, that is, whether input x_2 is a substitute or a complement to input x_1. To illustrate, assume a positive change in the price of the second input v_2, and therefore that dv_2 is positive. Then,

$$(13.88) \qquad dx_1 = \frac{pf_{22}(0 - 0f_1) - pf_{12}(dv_2 - 0f_2)}{p^2(f_{11}f_{22} - f_{12}^2)}$$

Therefore,

$$(13.89) \qquad \frac{dx_1}{dv_2} = \frac{-f_{12}}{p(f_{11}f_{22} - f_{21}f_{12})}$$

If f_{12} is positive, the firm will *decrease* its use of input x_1 in response to an increase in the price of the second input (v_2). In this instance, the inputs are technical complements and increases in the use of x_2 increase MPP_{x_1}. If f_{12} is negative (however near zero) the firm will *increase* its use of x_1 in response to an increase in the price of the second input. In this instance, the inputs are technical substitutes. For the production function represented in equation (13.86), the inputs are technical complements if * > 0, but technical substitutes if * < 0.

By Young's theorem, f_{12} equals f_{21}, and as a consequence, dx_1/dv_2 equals dx_2/dv_1. As a result, the elasticity of demand for input x_1 with respect to a change in the price of input x_2 is always exactly equal to the elasticity of demand for input x_2 with respect to a change in the price of input x_1. This is the symmetry of the cross-price input demand elasticities.

13.8 Concluding Comments

This chapter has shown how demand functions for inputs or factors of production can be obtained from the production function for a product. A key assumption of the model of pure competition, that the prices for both inputs and outputs be constant and known with certainty, was made throughout the analysis. The demand for an input is then determined only by the input and output prices and the coefficients or parameters of the underlying production function.

Notes

1. The definitions for technical complements, technical substitutes, and technical independence proposed here are quite different from those suggested in Doll and Orazem (pp. 106 ! 107). Doll and Orazem argue that technical complements must be used in fixed proportion to each other, resulting in isoquants consisting of single points or possibly right angles. Downward sloping isoquants indicate that inputs are technical substitutes. By the Doll and Orazem definition, most inputs are technical substitutes, not complements. In all three cases specified in this text, isoquants can be downward sloping.

Problems and Exercises

1. Assume that the production function is $y = x^{0.5}$. The price of the input is \$2, and the price of the output is \$5. What is the profit-maximizing level of use of x? What is the own-price elasticity of demand for input x? What is the output-price elasticity of demand for input x?

2. Find the demand function for input x under an alternative set of prices for x. Graph the function. Now increase the price of y to \$7 per unit. Graph the function again. Now decrease the price of y to \$3 per unit. Again graph the function.

3. Suppose that the production function is given as $y = 0.3x$. Is there a demand function for input x? Explain.

4. Suppose that the production function is given as $y = x^2$. Is there a demand function for input x? Explain.

5. Suppose that the production function is given as

$$y = x_1^{0.3} x_2^{0.9}$$

Find the input demand function for x_1 assuming that input x_2 is allowed to vary. What happens to the demand for x_1 when the price of x_2 declines? What is the own-price elasticity of demand for input x_1? What is the cross-price elasticity of demand for x_1 (the elasticity of demand for input x_1 when the price of input x_2 changes)? What is the output or product-price elasticity of demand for input x_1?

6. Assume that the production function is

$$y = x_1^{0.5} x_2^{0.5}$$

The price of y is \$10 per unit, and the price of x_1 and x_2 are each \$2 per unit. How much of each of x_1 and x_2 would the manager demand if he or she had but \$100 to spend on x_1 and x_2? Now suppose that the price of x_1 increases to \$10 per unit, and the manager has the same \$100 to spend. How much of x_1 and x_2 would the manager demand?

7. Verify that for the profit maximizing firm, regardless of the specific production function employed, the sum of the elasticities of demand with respect to the input's own and the other input prices plus the elasticity of demand for the input with respect to the product price equals zero. That is, verify that all input demand functions must be homogeneous of degree zero with respect to product and all factor prices.

Hint: First multiply equation (13.82) by v_1/x_1, equation (13.84) by p/x_1 and equation (13.89) by v_2/x_1. Then remember that for the profit-maximizing firm, the *MPP* for each input equals the respective factor/product price ratio.

8. Suppose that the production function that generated the isoquants in Figure 13.2 was equation 13.86. For each case, what must be the value of *?

Reference

Doll, John P., and Frank Orazem. *Production Economics: Theory with Applications.* 2nd ed. New York: John Wiley, 1984.