16

Maximization in a Two-Output Setting

This chapter presents the marginal allocation conditions for a single input in the production of two outputs. First, a graphical and tabular presentation is used. Then the fundamental constrained maximization conditions on the output side are derived. Comparisons are made of solutions when the constraint is the physical quantity of the input versus dollars available for the purchase of the input. Global profit maximization conditions on the output side are outlined. Starting with the individual production functions for the two products, the product transformation and input demand functions are derived. The product-product model is applied to an output restriction problem.

Key terms and definitions:

Family of Product Transformation Functions
Output Maximization on the Product Side
Isorevenue Line
Constrained Revenue Maximization on the Product Side
Output Expansion Path
Output Pseudo Scale Line
Marginal Cost in Physical Terms
Output Restriction

16.1 The Family of Product Transformation Functions

A family of product transformation functions can be created by varying the assumptions with respect to the availability of the resource or input bundle x. Along each product transformation function, the amount of the resource or input bundle remains constant. Figure 16.1 illustrates a family of product transformation functions. Like isoquant families, an infinite number of product transformation functions can be drawn. No two product transformation functions will touch each other or intersect. Each successive product transformation function assumes a slightly different level of use of the input bundle.

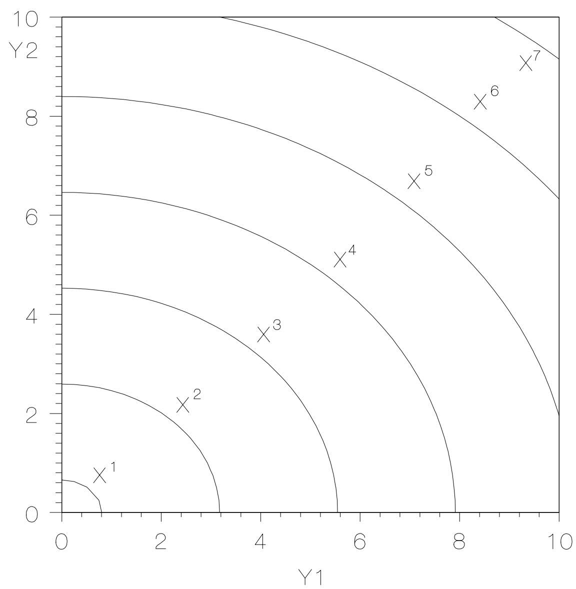

Figure 16.1 A Family of Product Transformation Functions

16.2 Maximization of Output

Assume that there is no limitation on the availability of the input bundle x. The equation describing the family of product transformation functions is

†16.1 $$x = g(y_1, y_2)$$

Suppose that the farm manager wishes to determine the amount of the input x that would be required such that the output of both y_1 and y_2 is at its maximum. The farm manager has available any amount of the input bundle x, and, at least for the moment, the cost of the input bundle is of no consequence.

One way is to look at the first derivatives of the product transformation equation dx/dy_1 and dx/dy_2. The expression dx/dy_1 is $1/(dy_1/dx)$ or $1/MPP_{xy_1}$. The expression dx/dy_2 is $1/(dy_2/dx)$ or $1/MPP_{xy_2}$. These expressions represent the marginal cost of producing an additional unit of y_1 or y_2, expressed in terms of physical quantities of the input bundle. If the farm manager is interested in maximizing the production of both y_1 and y_2, a level of input use where both y_1 and y_2 are at their respective maxima must be found.

If the amount of both outputs are at a global maximum, an additional unit of the input bundle will produce no additional output of either y_1 or y_2. In other words, the marginal product of x in the production of y_1 (MPP_{xy_1}) and the marginal product of x in the production of y_2 will be zero. As MPP_{xy_1} and MPP_{xy_2} approach zero, $1/MPP_{xy_1}$ and $1/MPP_{xy_2}$ become very large, and approach infinity. If MPP_{xy_1} and MPP_{xy_2} were exactly zero, $1/MPP_{xy_1}$ and $1/MPP_{xy_2}$ are undefined, although economists frequently treat them as infinite.

What happens to the appearance of an isoquant map as output approaches a maximum is clear. Isoquants become smaller and smaller concentric rings until the point of output maximum is achieved and the single point represents the isoquant for maximum output.

What happens to the appearance of a product transformation function as A global maximum for both outputs is approached is less clear. As more y_1 and y_2 is produced, each successive product transformation function becomes larger and larger and is drawn farther and farther from the origin of the graph. Exactly what happens to the shape of the product transformation function as the level of use of the input bundle x becomes large enough to achieve maximum output is not obvious, since at the point of output maximization for x in the production of both y_1 and y_2, the $1/MPPx$ in the production of either output is undefined.

When confronted with a problem such as this, economists frequently make assumptions such that they need not worry about the problem. Some arguments used to avoid thinking about such issues do make sense.

The assumption usually made to get around the problem is that the size of the resource or input bundle will always be constrained by something. Farmers nearly always face limitations in their ability to produce more because of the unavailability of land. An unlimited input bundle would imply that a single farmer owned all the farmland in the United States, not to mention all foreign countries. Then the constraint becomes the size of the earth. (Moreover, if a single farmer were to acquire all the world's farmland, the purely competitive assumptions would no longer hold!)

Every farmer faces capital constraints limiting the ability to borrow money for the purchase of more inputs. Perhaps the fact that a truly global point of output maximization cannot be achieved with the product-product model may not be such a serious problem after all. Important conclusions can be reached without looking at the case in which output is maximized without constraints.

16.3 The Isorevenue Line

The revenue function (R) for the farmer who produces two outputs is

†16.2 $$R = p_1y_1 + p_2y_2$$

Assume that a farmer needs \$1000 of revenue. The price of y_1 is \$5 and the price of y_2 is \$2. The farmer might choose to generate \$1000 by producing all y_1, in which case he or she would need to produce 200 units (\$1000/\$5). Or the farmer might choose to produce all y_2, and 500 units of output (\$1000/\$2) would be required. Perhaps some combination of the two outputs might be produced. The procedure for creating an isorevenue line is exactly the same as the procedure for creating an isocost line, with the following exceptions. Revenue replaces cost in the equation. Prices are now output prices rather than input prices. Table 16.1 illustrates some combinations of y_1 and y_2 that would yield \$1000 of revenue.

Table 16.1 Alternative Combinations of y_1 and y_2 that Result in $1000 of Revenue ($p_1$ = $5, p_2 = $2)

Combination	Units of y_1	Units of y_2	Revenue
A	200	0	$1000
B	150	125	1000
C	100	250	1000
D	50	375	1000
E	0	500	1000

There are many more (in fact, an infinite number) of combinations of y_1 and y_2 that would yield $1000 in revenue. The isorevenue line can be drawn on a graph with y_1 on the horizontal axis and y_2 on the vertical axis. The position where the isorevenue line cuts the horizontal axis can be found by assuming that the production of y_2 is zero, and solving the revenue function representing a fixed amount of revenue (R°) for y_1

†16.3 $$R° = p_1y_1 + 0p_2$$

†16.4 $$R° = p_1y_1$$

†16.5 $$y_1 = R°/p_1 = \$1000/\$5 = 200$$

where p_1 and p_2 are prices for y_1 and y_2, respectively.

A similar procedure can be use to find the point where the isorevenue line cuts the y_2 axis

†16.6 $$y_2 = R°/p_2 = 1000/2 = 500$$

The slope of an isorevenue line is $-y_2/y_1$, or

†16.7 $$(R°/p_2)/(R°/p_1) = -p_1/p_2 = (1000/2)/(1000/5) = -5/2$$

The slope of an isorevenue line is a constant ratio of the two output prices. If y_2 appears on the vertical axis and y_1 on the horizontal axis, the slope of the isorevenue line is the negative inverse output price ratio, $-p_1/p_2$.

The term *isorevenue* means equal revenue. At any point on an isorevenue line, total revenue is the same, but if total revenue is allowed to vary, a new isorevenue line can be drawn. The greater the total revenue, the farther the isorevenue line will be from the origin of the graph. If output prices are constant, the slope over every isorevenue line will be the same. No two isorevenue lines will ever touch or intersect. Families of isorevenue lines are drawn with each isorevenue line representing a slightly different revenue level.

16.4 Constrained Revenue Maximization

A family of isorevenue lines can be superimposed on a family of product transformation functions (Figure 16.2). Each isorevenue line has its own product transformation function that comes just tangent to it. The point of tangency represents the maximum revenue attainable from a given product transformation function. It is the point where the slope of the isorevenue line just equals the rate of product transformation. This point represents the position where the farmer would most like to be among the series of points along a product transformation

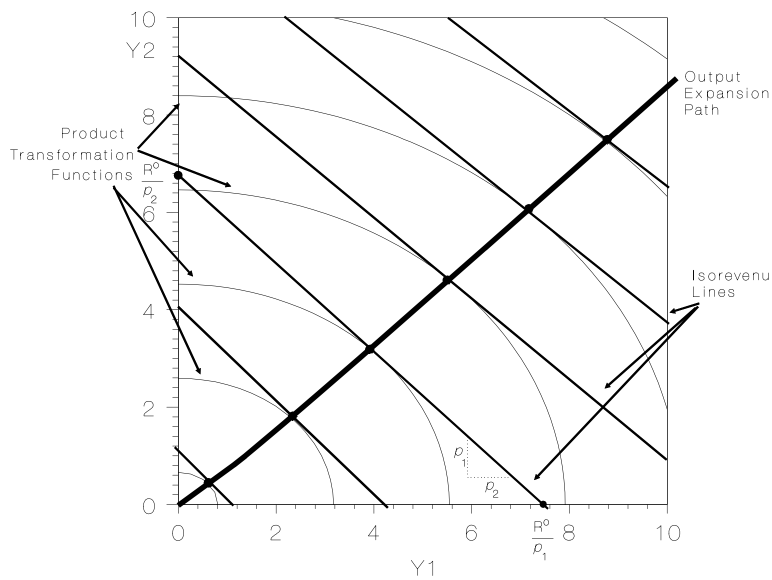

Figure 16.2 Product Transformation Functions, Isorevenue Lines, and the Output Expansion Path

function, for it represents maximum revenue from the given level of inputs which defines that particular product transformation function. The assumption is that the amount of the input bundle is fixed and given. These points of tangency can be defined by the following equations:

†16.8 $$-RPT_{y_1y_2} = -dy_2/dy_1$$
$$= (1/MPP_{xy_1})/(1/MPP_{xy_2})$$
$$= MPP_{xy_2}/MPP_{xy_1}$$
$$= p_1/p_2$$

Both the $RPT_{y_1y_2}$ and the isorevenue line are negative, as indicated by the sign. By multiplying both by -1, the result is

†16.9 $$RPT_{y_1y_2} = -dy_2/dy_1 = -p_1/p_2$$

An increase in the price of one of the outputs relative to the other will push the point of tangency toward the axis for the output that experienced the price increase. If the price of one output drops relative to the other, the production of the other output will be favored.

The path along which the farmer will expand his or her operation is a line that connects all points of tangency between the isorevenue lines and the corresponding product transformation curve. This line is called the output expansion path (Figure 16.2). To generate more revenue, the farmer must expand the resource base, or the availability of the input bundle *x*. As this happens, the farmer will move from one product transformation function to another along the output expansion path. If output prices are constant, most product transformation maps have underlying production functions that will result in an output expansion path with a constant slope.

Consider the data presented in Table 15.2 again, here presented in Table 16.2. Assume that soybeans sell for $9 per bushel and corn is $6 per bushel. The input combination where the rate of product transformation of corn for soybeans equals the price ratio is the combination between the combination 120 bushels of corn and 34 bushels of soybeans and the combination 111 bushels corn and 40 bushels soybeans. Total revenue for the first combination is 111·6 + 40·9 = $1026. Total revenue for the second combination is 120·6 + 34·9 = $1026.

Table 16.2 The Rate of Product Transformation of Corn for Soybeans from a Variable Input Bundle *X*

Units of *X* Applied to Corn	Yield per Acre (bushels)	*MPP* of *X* in Corn Production	Units of *X* Applied to Soybeans	Yield per Acre (bushels)	*MPP* of *X* in Bean Production	*RPT* of Corn for Soybeans
0	0		10	55		
		45			1	1/45 = 0.022
1	45		9	54		
		17			2	2/17 = 0.118
2	62		8	52		
		15			3	3/15 = 0.200
3	87		7	49		
		13			4	4/13 = 0.308
4	100		6	45		
		11			5	5/11 = 0.455
5	111		5	40		
		9			6	6/9 = 0.667
6	120		4	34		
		7			7	7/7 = 1.00
7	127		3	27		
		5			8	8/5 = 1.60
8	132		2	19		
		3			9	9/3 = 3.00
9	135		1	10		
		1			10	10/1 = 10.0
10	136		0	0		

Both combinations yield the same total revenue, but combinations on either side of these two combinations yield less total revenue. The exact point where revenue would be maximum lies between the two combinations yielding the same revenue. Tabular data can at best provide only an approximation of the true point where the rate of product transformation equals the inverse price ratio, as was the case here.

Not surprisingly, an increase in the price of one of the two outputs will tend to shift production toward the commodity that experienced the price increase and away from the other commodity. For example, if the price of corn remains unchanged, an increase in the price of soybeans will move the point of tangency between the product transformation function toward soybean production and away from corn production.

By observing what happens to the output of two products as the relative prices change, it is sometimes possible to discern the shape of the underlying product transformation function and the elasticity of product substitution. Suppose that the price of corn increased relative to soybeans. The expected result would be a significant but not total shift by farmers away from the production of soybeans and into the production of corn. The resources or inputs used in the production of soybeans can be used to produce corn, although inputs are not perfectly substitutable.

Now suppose that the two products are beef and hides. An increase in the price of hides would not cause the production of hides to increase relative to the production of beef at all. The technical relationship that requires each beef animal to have one and only one hide governs the shape of the product transformation function. No matter how high the price of hides, the farmer can still produce only one per animal. The elasticity of product substitution is as near zero as can be found in the real world anywhere.

Now assume that the two products are wool and lamb meat and that the price of wool relative to the price of lamb decreases. In a single production season, farmers with their existing flocks could increase lamb meat production relative to wool very little. They may be able to do so slightly by feeding out the lambs to larger weights. This suggests a single season elasticity of product substitution very near but not exactly zero.

However, if these price relationships persisted over time, farmers would sell the sheep capable of high wool production relative to lamb, and buy sheep capable of high lamb meat production relative to wool. The elasticity of product substitution is probably greater over several seasons than over a single production season.

Consider a situation where a farmer is producing two products y_1 and y_2. The $RPT_{y_1y_2}$ is constant and the product transformation functions have a constant negative slope. Hence the elasticity of product substitution is infinite. If the absolute value of p_1/p_2 is greater than the absolute value of the $RPT_{y_1y_2}$, the farmer will produce all y_1 and no y_2. If the absolute value of p_1/p_2 is less than the absolute value of $RPT_{y_1y_2}$, the farmer will produce all y_2 and no y_1. If the absolute value of p_1/p_2 is the same as the absolute value of $RPT_{y_1y_2}$, output of each product will be indeterminate. If the farmer is initially producing all y_2 and no y_1, an increase in the price of (p_1) relative to the price of y_2 (p_2) may not at first cause production to shift totally to y_1. As p_1 continues to increase, such that the price ratio p_1/p_2 exceeds the absolute value of $RPT_{y_1y_2}$, production will suddenly shift entirely out of y_2 and into y_1.

16.5 Simple Mathematics of Constrained Revenue Maximization

The problem of maximizing revenue subject to a resource or input constraint illustrated in Figure 16.2 can be cast as a constrained revenue maximization problem and be solved mathematically via Lagrange's method.

The objective function is

†16.10 $$\text{Maximize } p_1y_1 + p_2y_2$$

The constraint is the availability of the input bundle x, which is the equation for the product transformation function

†16.11 $$x^\circ = g(y_1, y_2)$$

where x° is a fixed available amount of the input bundle x.

The Lagrangean is

†16.12 $$L = p_1y_1 + p_2y_2 + \lambda[x^\circ - g(y_1, y_2)]$$

The corresponding first order or necessary conditions are

†16.13 $$\partial L/\partial y_1 = p_1 - \lambda \partial g/\partial y_1 = 0$$

†16.14 $$\partial L/\partial y_2 = p_2 - \lambda \partial g/\partial y_2 = 0$$

†16.15 $$\partial L/\partial \lambda = x^\circ - g(y_1, y_2) = 0$$

By dividing equation †16.13 by equation †16.14 , the result is

†16.16 $$p_1/p_2 = (\partial g/\partial y_1)/(\partial g/\partial y_2)$$

Since g is x,

†16.17 $$p_1/p_2 = (1/MPP_{xy_1})/(1/MPP_{xy_2})$$

†16.18 $$-MPP_{xy_2}/MPP_{xy_1} = -p_1/p_2$$

†16.19 $$RPT_{y_1y_2} = p_1/p_2$$

Equation †16.19 represents the same conclusion reached in section 16.4. First-order conditions find the point where the slope of the isorevenue line is the same as the slope of the product transformation function. Both the isorevenue line and the product transformation function will be downward sloping.

Equations †16.13 and †16.14 may be rearranged in other ways. Some possibilities are

†16.20 $$p_1/(\partial g/\partial y_1) = \lambda$$

†16.21 $$p_2/(\partial g/\partial y_2) = \lambda$$

†16.22 $$p_1/(\partial g/\partial y_1) = p_2/(\partial g/\partial y_2) = \lambda$$

†16.23 $$p_1MPP_{xy_1} = p_2MPP_{xy_2} = \lambda$$

†16.24 $$VMP_{xy_1} = VMP_{xy_2} = \lambda$$

Equation †16.24 represents the equimarginal return principle from the output side. The farmer should use the input bundle such that the last physical unit of the bundle returns the same *VMP* for both enterprises. The analysis assumes that the resource or input bundle is already owned by the farmer, and therefore the decision to produce will cost no more than the decision not to produce.

The assumption that the input bundle is free or worth nothing if sold by the farmer seems unrealistic. More likely, the input bundle has a price. Assume that the price for a unit of the bundle is v. The constrained revenue-maximization problem then becomes one of maximizing revenue from the sale of the two products subject to the constraint imposed by the availability of dollars for the purchase of the input bundle.

The restriction in the availability of funds might be in the form of both owned dollars as well as the credit availability from the local bank, Production Credit Association, or other lending agency. Any interest charges for borrowed funds might be subtracted from $C°$ before the problem is set up, so that $C°$ represents funds actually available for the purchase of the physical input bundle. This cost constraint can be written as

†16.25 $$C° = vx$$

The Lagrangean is reformulated with the same objective function

†16.26 maximize $p_1y_1 + p_2y_2$

The constraint is the availability dollars for the purchase of the input bundle x. Equation †16.27 is the product transformation function multiplied by the price of the input bundle v

†16.27 $$C° = vx° = vg(y_1, y_2)$$

The Lagrangean is

†16.28 $$L = p_1y_1 + p_2y_2 + \lambda[C° - vg(y_1, y_2)]$$

The corresponding first order (necessary) conditions are

†16.29 $$\partial L/\partial y_1 = p_1 - \lambda v\partial g/\partial y_1 = 0$$

†16.30 $$\partial L/\partial y_2 = p_2 - \lambda v\partial g/\partial y_2 = 0$$

†16.31 $$\partial L/\partial \lambda = C° - vg(y_1,y_2) = 0$$

By dividing equation †16.29 by equation †16.30 , the result is

†16.32 $$p_1/p_2 = (\partial g/\partial y_1)/(\partial g/\partial y_2)$$

†16.33 $$RPT_{y_1y_2} = p_1/p_2$$

Equation †16.33 is the same conclusion reached in equation †16.19 . First-order conditions find the point where the slope of the isorevenue line is the same as the slope of the product transformation function. The price of the input bundle does not affect the point of tangency between the product transformation function and the isorevenue line.

Equations †16.29 and †16.30 may also be rearranged in other ways. One possibility is

†16.34 $$p_1/v(\partial g/\partial y_1) = \lambda$$

†16.35 $$p_2/v(\partial g/\partial y_2) = \lambda$$

†16.36 $$p_1/v(\mathrm{M}g/\mathrm{M}y_1) = p_2/v(\mathrm{M}g/\mathrm{M}y_2) = \mathrm{N}$$

†16.37 $$VMP_{xy_1}/v = VMP_{xy_2}/v = \mathrm{N}$$

Equation †16.37 is the first order condition for revenue maximization subject to a cost constraint, assuming that the input bundle x has a price v. Equation †16.37 is the equimarginal return relationship that holds if the input bundle has a cost to the farmer. Equation †16.37 differs from equation †16.24 in that both sides of equation †16.37 has been divided by the price of the input bundle v.

Since the price of the input bundle is the same in the production of both outputs, these conditions suggest no change in the allocation of the input bundle between the production of y_1 and y_2 relative to the conclusions derived in the last example. Equation †16.37 states that the farmer should allocate the input bundle in such a way that the last dollar spent on the input bundle yields the same ratio of *VMP* to the cost of the bundle for both outputs.

This derivation does have an important advantage over the example in equation †16.24 . The values for the Lagrangean multiplier (N) that would result in maximum net revenue to the farmer now become apparent. The farmer would not spend an extra dollar on the input bundle x if it did not return the extra dollar. Profit maximization on the output side thus occurs when

†16.38 $$VMP_{xy_1}/v = VMP_{xy_2}/v = 1$$

Equation †16.38 is the global point of profit maximization on the output side, and can occur only when N equals 1. A value for N of greater than 1 suggests that the farmer has insufficient dollars for the purchase of enough x to globally maximize profits. Any point where the equality holds is a point on the output expansion path. The point of global profit maximization also lies on the output expansion path, and here the Lagrangean multiplier assumes a value of 1. Notice also that N equals 2/v.

A pseudo scale line for each output can also be defined. An output pseudo scale line for y_1 would be a line on the map of product transformation curves connecting points where profits are maximum for y_1, but not necessarily for y_2. In other words, VMP_{xy_1}/v equals 1, but VMP_{xy_2}/v may not necessarily be 1.

Each pseudo scale line is derived from the profit maximization point on a member of the family of the production functions transforming x into y_1, assuming that a portion of the input bundle x has been already allocated to the production of y_2. A similar derivation could be done to generate an output pseudo scale line for y_2. These output pseudo scale lines intersect at the global point of profit maximization, where

†16.39 $$VMP_{xy_1}/v_1 = VMP_{xy_2}/v_2 = 1$$

16.6 Second-Order Conditions

In the product-product model, the point where the manager would prefer to be found is a point of tangency between the product transformation function and the isorevenue line. In factor-factor or input space, the point where the manager would prefer to be found is a point of tangency between the isocost line and the isoquant.

The point of tangency between the isorevenue line and the product transformation function does not look the same as the point of tangency between the isocost line and an isoquant. Isoquants are normally bowed inward or convex to the origin of the graph. Product transformation functions are normally bowed outward or concave to the origin of the graph.

The first-order conditions for revenue maximization subject to an input constraint are repeated here

†16.40 $$L = p_1y_1 + p_2y_2 + \lambda[x° - g(y_1, y_2)]$$

†16.41 $$\partial L/\partial y_1 = p_1 - \lambda \partial g/\partial y_1 = 0$$

†16.42 $$\partial L/\partial y_2 = p_2 - \lambda \partial g/\partial y_2 = 0$$

†16.43 $$\partial L/\partial \lambda = x° - g(y_1,y_2) = 0$$

Equations †16.41 , †16.42 and †16.43 are each differentiated with respect to y_1, y_2 and with respect to λ.

$$\partial \dagger 16.41/\partial y_1 = -\lambda \partial^2 g/\partial y_1^2 = -\lambda g_{11}$$

$$\partial \dagger 16.41/\partial y_2 = -\lambda \partial^2 g/\partial y_1 \partial y_2 = -\lambda g_{12}$$

$$\partial \dagger 16.41/\partial \lambda = -\partial g/\partial y_1 = -g_1$$

$$\partial \dagger 16.42/\partial y_1 = -\lambda \partial^2 g/\partial y_2 \partial y_1 = -\lambda g_{21} = -\lambda g_{12} \text{ (by Young's theorem)}$$

$$\partial \dagger 16.42/\partial y_2 = -\lambda \partial^2 g/\partial y_2^2 = -\lambda g_{22}$$

$$\partial \dagger 16.42/\partial \lambda = -\partial g/\partial y_2 = -g_2$$

$$\partial \dagger 16.43/\partial y_1 = -\partial g/\partial y_1 = -g_1$$

$$\partial \dagger 16.43/\partial y_2 = -\partial g/\partial y_2 = -g_2$$

$$\partial \dagger 16.43/\partial \lambda = 0$$

The partial derivatives g_1 and g_2 are the marginal costs for the production of an additional unit of y_1 and y_2, respectively, expressed in physical rather than dollar terms. Had these second derivatives been found for the revenue-maximization problem constrained by dollars available for the purchase of x rather than physical units of x, then g_1 and g_2 would have been multiplied by the price of the input v. The term vg_1 is the marginal cost of an additional unit of y_1. The term vg_2 is the marginal cost of an additional unit of y_2.

Marginal cost is negative in stage III, since *MPP* is negative in stage III but is never negative in stages I and II. In stages I and II, an incremental unit of output can never be produced without any additional cost in terms of the input bundle. Lagrange's method would not find a solution in stage III where the Lagrangean multiplier is negative.

The partial derivative g_{11} can be interpreted as the slope of the marginal cost function for y_1. The derivative g_{22} has the same interpretation for y_2. Marginal cost is again expressed in terms of physical input requirements rather than in dollar terms. The slope of marginal cost can be converted to dollars by multiplying by the input bundle price v, which would occur if the constraint were expressed in dollar and not in physical terms. Marginal cost is normally rising, except in stage III and perhaps in the early stages of stage I for the input bundle x.

This means that additional units of either y_1 or y_2 cannot be produced without incurring more and more additional cost or an increasing marginal cost. The cross partial derivatives ($g_{12} = g_{21}$) are needed to rule out production surfaces that appear as saddle points.

The Lagrangean multiplier 2 is again interpreted as a shadow price, or imputed value of the input bundle x. The number 2 is the increase in revenue associated with an additional unit of the input bundle. When *MPP* is positive (except for stage III for each input that is beyond the point of maximum output), the Lagrangean multiplier 2 will also be positive.

Every component of the second-order conditions for constrained output and revenue maximization has an economic meaning. This economic meaning will lead to conclusions with regard to the probable sign on each component of the second order conditions.

The second-order conditions for a constrained revenue maximization require that

†16.44 $$2(g_1^2 g_{22} + g_2^2 g_{11} \; ! \, 2g_{12}g_2g_1) > 0$$

Equation †16.44 is the determinant of the matrix

†16.45
$$\begin{matrix} !\,2g_{11} & !\,2g_{12} & !\,g_1 \\ !\,2g_{12} & !\,2g_{22} & !\,g_2 \\ !\,g_1 & !\,g_2 & 0 \end{matrix}$$

Since a negative value for 2 would not be found in the solution, then

†16.46 $$g_1^2 g_{22} + g_2^2 g_{11} \; ! \; 2g_{12}g_2g_1 > 0$$

Equation †16.46 ensures that the product transformation functions are concave or bowed outward from the origin.

The first- and second-order conditions, taken together, are the necessary and sufficient conditions for the maximization of revenue subject to the constraint imposed by the availability of the input bundle x.

The price of the input bundle is positive. If the input prices are constant, the required sign on the second-order condition is not altered if the constraint is constructed based on the availability of funds for the purchase of x rather than the availability of x itself. The required second-order conditions would then be based on the determinant of the matrix

†16.47
$$\begin{matrix} !\,2\mathrm{v}g_{11} & !\,2\mathrm{v}g_{12} & !\,\mathrm{v}g_1 \\ !\,2\mathrm{v}g_{12} & !\,2\mathrm{v}g_{22} & !\,\mathrm{v}g_2 \\ !\,\mathrm{v}g_1 & !\,\mathrm{v}g_2 & 0 \end{matrix}$$

16.7 An Additional Example

Starting with production functions for y_1 and y_2, the product transformation function is constructed. The first order conditions for revenue maximization subject to the constraint imposed by the availability of x are solved to determine the optimal amounts of y_1 and y_2 to be produced. The manager is then assumed to have the right amount of x needed to globally maximize profits in the production of both y_1 and y_2. The same level is needed irrespective of whether the problem is solved for the output or the input side.

The production functions for y_1 and y_2 are assumed to be

†16.48 $$y_1 = x_{y_1}^{0.33}$$

†16.49 $$y_2 = x_{y_2}^{0.5}$$

where x_{y_1} and x_{y_2} are assumed to be the quantities of x used in the production of y_1 and y_2, respectively.

The total availability of x is

†16.50 $$x = x_{y_1} + x_{y_2}$$

The inverse production functions are

†16.51 $$x_{y_1} = y_1^3$$

†16.52 $$x_{y_2} = y_2^2$$

Substituting equations †16.51 and †16.52 into equation †16.50 , the equation for the product transformation function is

†16.53 $$x = y_1^3 + y_2^2$$

The constraint imposed by the availability of funds for the purchase of x is

†16.54 $$C^\circ = vx = v(y_1^3 + y_2^2)$$

The Lagrangean that maximizes revenue subject to the constraint imposed by the availability of dollars for the purchase of x is

†16.55 $$L = p_1y_1 + p_2y_2 + \lambda[vx - v(y_1^3 + y_2^2)]$$

The first-order conditions for the constrained maximization of equation †16.55 are

†16.56 $$p_1 - \lambda 3vy_1^2 = 0$$

†16.57 $$p_2 - \lambda 2vy_2 = 0$$

†16.58 $$vx - v(y_1^3 + y_2^2) = 0$$

Now solve equations †16.56 and †16.57 of the first order conditions for y_1 and y_2 respectively

†16.59 $$p_1 = \lambda 3vy_1^2$$

†16.60 $$y_1 = (0.33)^{0.5}(\lambda v)^{-0.5}p_1^{0.5}$$

†16.61 $$p_2 = \lambda 2vy_2$$

†16.62 $$y_2 = (0.5)(\lambda v)^{-1}p_2$$

Production of y_1 and y_2 will decrease when the price of the input bundle v increases. Production of y_1 and y_2 will increase when the price of the output increases. The change in both cases will depend on the technical parameters of the underlying single input production functions. The farmer's elasticity of supply with respect to input prices for y_1 is -0.5, and for y_2 is -1. The farmer's elasticity of supply with respect to output prices for y_1 is 0.5, and for y_2 is 1.

Second order conditions for constrained revenue maximization will be met if the underlying production functions for y_1 and y_2 are homogeneous of a degree less than 1.

Now substitute for y_1 and y_2 the corresponding values for x_{y_1} and x_{y_2}, and assume that the manager has enough x available so that profits with respect to the production of both y_1 and y_2 are maximum. This implies that the Lagrangean multiplier 2 will be 1. Therefore

†16.63 $$y_1 = x_{y_1}^{0.33} = (0.33)^{.5}\, v^{-0.5}\, p_1^{0.5}$$

†16.64 $$y_2 = x_{y_2}^{0.5} = (0.5)v^{-1}p_2$$

†16.65 $$x_{y_1} = 0.33^{1.5}v^{-1.5}\mathrm{p}_1^{1.5}$$

†16.66 $$x_{y_2} = .5^2v^{-2}p_2^2$$

Insertion of prices for the input bundle v and the two output prices p_1 and p_2 into equations †16.65 and †16.66 yields the amount of x to be applied to y_1 and y_2 in order to globally maximize profits.

The own-price elasticity of demand by the farmer for the input bundle x in the production of y_1 is -1.5 and in the production of y_2 is -2. These are $1/(1 - e_p)$, where e_p is the production elasticity associated with the input bundle x in the production of each output.

The product price elasticity of demand by the farmer for the input bundle x in the production of y_1 is 1.5 and in the production of y_2 is 2. These are obtained from the formula $-1/(1 - e_p)$. Each of these elasticities can be interpreted as the percentage increase in the demand for the input bundle x that accompanies a 1 percent increase in the output prices for y_1 or y_2. For both production functions, the input bundle own-price elasticity is the negative of the input bundle output-price elasticity.

The quantity of x to be used in the production of y_1 and y_2 could be obtained from a pair of input-side profit-maximization equations as well, and the same results with respect to how x should be allocated would be found.

Let

†16.67 $$y_1 = x_{y_1}^{0.33}$$

†16.68 $$y_2 = x_{y_2}^{0.5}$$

†16.69 $$\mathrm{A}_{y_1} = p_1y_1 - vx_{y_1}$$

†16.70 $$\mathrm{A}_{y_1} = p_1\, x_{y_1}^{0.33} - vx_{y_1}$$

†16.71 $\pi_{y_2} = p_2 y_2 - v x_{y_2}$

†16.72 $\pi_{y_2} = p_2 x_{y_2}^{0.5} - v x_{y_2}$

To find first-order conditions for maximum profits, set the first derivatives of both profit equations with respect to x_{y_1} or x_{y_2} equal to zero

†16.73 $\partial\pi/\partial x_{y_1} = 0.33 p_1 x_{y_1}^{-0.67} - v = 0$

†16.74 $\partial\pi/\partial x_{y_2} = 0.5 p_2 x_{y_2}^{-0.5} - v = 0$

Solving equations †16.73 and †16.74 for x_{y_1} and x_{y_2} we obtain

†16.75 $x_{y_1} = 0.33^{1.5} v^{-1.5} p_1^{1.5}$

†16.76 $x_{y_2} = 0.5^2 v^{-2} p_2^2$

which is the same result as obtained as from equations †16.65 and †16.66 for the derived demand elasticities with respect to input and product prices. The result again provides the quantity of x_1 and x_2 needed to maximize profits at the point where the Lagrangean multiplier equals 1.

16.8 Minimization of Input Use Subject to a Revenue Constraint

Any constrained maximization problem has a corresponding dual or constrained minimization problem. This dual problem can also be solved via Lagrange's method. The objective function in this case requires that input use be minimized for a specific amount of total revenue R

†16.77 Minimize $g(y_1, y_2)$ or x

Subject to the constraint that

†16.78 $R° = p_1 y_1 + p_2 y_2$

The Lagrangean is

†16.79 $L = g(y_1, y_2) + \lambda(R° - p_1 y_1 - p_2 y_2)$

The corresponding first order conditions are

†16.80 $g_1 - \lambda p_1 = 0$

†16.81 $g_2 - \lambda p_2 = 0$

†16.82 $R° - p_1 y_1 - p_2 y_2 = 0$

By rearranging and dividing equation †16.80 by equation †16.81, the familiar point of tangency is found where

†16.83 $RPT_{y_1 y_2} = dy_2/dy_1 = p_1/p_2$

Solving equations †16.80 and †16.81 from the first-order conditions for R yields

†16.84 $$g_1/p_1 = R$$

†16.85 $$g_2/p_2 = R$$

†16.86 $$g_1/p_1 = g_2/p_2 = R$$

or

†16.87 $$1/VMP_{xy_1} = 1/VMP_{xy_2} = R$$

Compared with the conclusions derived in equation †16.24 , equation †16.87 appears upside down. In fact, the Lagrangean multiplier R is 1/2 found in equation †16.24 . If the problem is set up to maximize revenue subject to the availability of the input bundle x, then the Lagrangean multiplier (2) is interpreted as the increase in revenue associated with one additional unit of the input bundle. (Or the Lagrangean multiplier could be expressed as the decrease in revenue associated with a 1 − unit decrease in the size of the input bundle.)

If the problem is set up to minimize input use subject to a revenue constraint, the Lagrangean multiplier R is the increase in input use needed to produce \$1 of additional revenue. (Or the Lagrangean multiplier could also be expressed as the decrease in the use of the input bundle associated with \$1 less revenue.)

The second-order conditions for input bundle minimization subject to a revenue constraint require that

†16.88 $$2p_1p_2g_{12} - g_{22}p_1^2 - g_{11}p_2^2 < 0$$

Equation †16.88 is the determinant of the matrix formed by again differentiating each equation in the first-order conditions with respect to y_1, y_2 and the Lagrangean multiplier R

†16.89
$$\begin{array}{ccc} g_{11} & g_{12} & -p_1 \\ g_{21} & g_{22} & -p_2 \\ -p_1 & -p_2 & 0 \end{array}$$

Remembering Young's theorem, and multiplying both sides of the determinant by −1, we have

†16.90 $$g_{22}p_1^2 + g_{11}p_2^2 - 2p_1p_2g_{12} > 0$$

Now from the first order conditions †16.80 and †16.81 , substitute

†16.91 $$p_1 = g_1/R$$

†16.92 $$p_2 = g_2/R$$

†16.93 $$(1/R^2)[g_1^2g_{22} + g_2^2g_{11} - 2g_1g_2g_{12}] > 0$$

Since R is normally positive, these second order conditions impose the same requirements on g_1, g_2, g_{12}, g_{22}, and g_{11} as before.

16.9 An Output-Restriction Application

The example presented here illustrates the application of the product-product model to a problem in which the government restricts the quantity of a product that can be produced and marketed by the farmer. The federal government might attempt establish a policy to support the price of certain crops by limiting the amount of output produced by the farmer. An output restriction is quite different from an acreage allotment. An acreage allotment restricts the amount of the input land to be used in the production of a commodity. An output restriction limits the quantity of the commodity that can be placed on the market.

The analysis presented here is an application similar to the acreage allotment application presented in Chapter 8. Output restrictions have been used less often than acreage restrictions by the government to control the production of commodities. The federal tobacco program provides a unique example. The government previously controlled the production of tobacco simply by limiting the acreage of tobacco that could be planted. Tobacco was treated by the government just like wheat. Farmers readily adapted to the acreage restriction as the earlier model would predict. Only the very best land was used for tobacco production. Farmers made intensive use of chemical fertilizers and pesticides, and production per acre soared. However, the tobacco program was changed, and in recent years, farmers were allowed to only place a certain quantity of tobacco on the market. As of 1985, each farm now had a tobacco poundage rather than acreage allotment.

The impacts of a tobacco poundage allotment can be illustrated by using a model in product-product space. Let Y represent the commodity or commodities other than tobacco that a farmer might grow, and T represent tobacco. A series of product transformation curves between tobacco and other commodities are illustrated in Figure 16.3. In the absence of any restrictions on output of tobacco, the farmer is operating on the output expansion path where

†16.94 $$VMP_{XT}/V_X = VMP_{XY}/V_X = \mathsf{R}$$

where VMP_{XT} and VMP_{XY} are the respective *VMP*'s of the input bundle *X* in the production of tobacco and other commodities respectively. Let this point be represented by A in Figure 16.3. Now suppose that the government imposes a poundage restriction. Let the poundage restriction be represented by the horizontal line labeled T*. To comply with the restriction, the farmer must move back along the output expansion path to point B, which lies at the intersection of the output expansion path and the poundage constraint. Point B is represented by a point where

†16.95 $$VMP_{XT}/V_X = VMP_{XY}/V_X = \mathsf{0}$$

Both R and $\mathsf{0}$ are probably greater than 1, but $\mathsf{0}$ is larger than R. With the poundage restriction, the farmer has additional dollars available for the purchase of the input bundle X, but these dollars can only be used to produce commodities other than tobacco. The farmer will again move to the right along the constraint T*. The farmer will probably not move to the point where the last product transformation function intersects the constraint T*. If sufficient revenue for the purchase of the input bundle is available, the farmer will move to the product transformation function where

†16.96 $$VMP_{XY}/V_X = 1$$

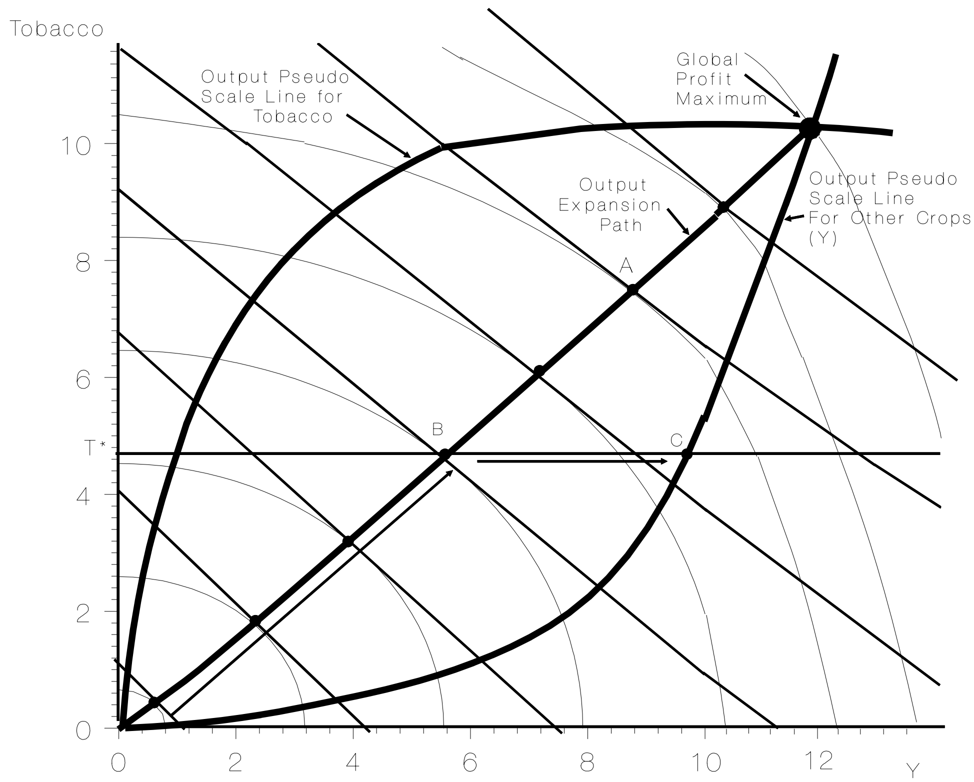

Figure 16.3 An Output Quota

This point is not on the output expansion path, but is the point of global profit maximization for the input bundle X used in the production of other commodities Y. Tobacco production will remain constant at T*. This point is C and represents a point on the output pseudo scale line for the production of the commodities represented by Y.

When the tobacco program was changed from an acreage allotment to a poundage allotment, tobacco production per acre declined, as would have been predicted by the earlier model. It is more difficult to determine if the production of other crops increased as a direct result of the tobacco poundage allotment, since tobacco has not in the recent past been grown in the absence of a government program.

The expected impact of the tobacco poundage program based on this model should be to increase the output of those crops requiring a similar bundle of inputs to tobacco but not affected by the quantity restrictions. The tobacco producing areas of Kentucky have recently seen an increase in the production of labor intensive horticultural crops planted on small acreages in much the same manner as tobacco and requiring very similar inputs. This is the expected impact of an output restriction based on the product-product model.

Tobacco farmers have also used the tobacco poundage allotment system as a method of dealing with output uncertainty. Overproduction is good years can be stored and used to meet the output quota in years when nature is uncooperative and production is low.

16.10 Concluding Comments

Farmers respond to changes in relative prices for commodities by adjusting production levels toward the commodity that is experiencing the relative price increase and away from the commodity for which the price is decreasing in relative terms. If there is but a single input, or an input bundle already owned by the farmer, the optimal conditions for constrained revenue maximization require that the farmer equate the respective *VMP*'s for each output.

The shape of the product transformation function determines the extent to which a farmer will adjust the output mix in the face of changing relative prices. If the elasticity of product substitution is near zero, the product transformation function is nearly a right angle, and the farmer will not adjust the mix of outputs in response to changing relative prices. However, to the extent that the elasticity of product substitution is positive, the farmer will respond to changing relative prices by adjusting the output mix.

The constrained maximization conditions on the product side look very similar to those on the input side. In both instances, the equimarginal return principle still applies. Farmers should allocate dollars in such a way that the last dollar used in the production of each product produces the same return. The product-product model can be applied to problems when the government implements policy to support prices by restricting the output of a particular commodity.

Problems and Exercises

1. Suppose that y_1 and y_2 sold for the same price. Using the data contained in Problem 1, Chapter 15, how much x would be applied to y_1 and y_2?

2. What would happen to your results in question 1 if y_1 were three times as expensive as y_2?

3. Show 10possible combinations of output that could produce \$1000 of revenue in y_1 sold for \$5 and y_2 sold for \$10. On the basis of these data alone, should the production of y_2 be favored over the production of y_1? Explain.

4. Suppose that the product transformation function is given by

$$x = 2y_1^2 + 3y_2^3$$

The price of y_1 is \$5 and the price of y_2 is \$4. Ten units of x are available. How much x should be applied to y_1 and y_2?

5. Compare the interpretations of the Lagrangean multipliers for the following problems in a multiple-product setting.

a. Output maximization subject to an input availability constraint.

b. Revenue maximization subject to an input availability constraint.

c. Global profit maximization on the output side in product space.

d. Resource or input use minimization subject to a revenue constraint.

6. Suppose the government restricts the amount of a product that a farmer might sell. Will the farmer always continue to produce at a point where $RPT_{y_1y_2} = p_1/p_2$? Explain.

7. Will the output of other commodities always increase if the government restricts the amount of a particular commodity that might be sold by the farmer? Explain.