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Two Outputs and Two Inputs

This chapter illustrates how the factor-factor model and the product-product model can be combined into a single model encompassing both the factor space and product space. First-order conditions for global profit maximization and for revenue maximization subject to the constraint imposed by the availability of dollars for the purchase of inputs are derived. An example of an intermediate product model is used to illustrate a possible application within agriculture.

Key terms and definitions:

Multiple-Input Multiple-Product Model
Equal Marginal Returns
Imputed Value
Intermediate Product Model
Quasi ! general Equilibrium
Rate of Product Transformation Equals Marginal Rate of Substitution
Shutdown Condition

17.1 Introduction

Until now, production has been presented with models in which one or two inputs were used to produce a single output, or a single-input bundle was used to produce two outputs. These models could easily be illustrated using graphics, because no more than three axes were required (x_1, x_2 and y or y_1, y_2, and x), and the resultant graph contained no more than three dimensions. However, most farmers use many different inputs in order to produce many different outputs. Despite the fact that these models often cannot be illustrated using graphics, the same rules for maximization and minimization found to exist in factor-product, factor-factor, and product-product models still apply. It is now appropriate to formulate some general rules that would apply to farmers who use many inputs in the production of many different outputs.

17.2 Two Inputs and Two Outputs: A Basic Presentation

Assume that the farmer uses two inputs, phosphate fertilizer (x_1) and potash fertilizer (x_2), in the production of two outputs, corn (y_1) and soybeans (y_2). Corn sells for \$4.00 per bushel and soybeans sell for \$8.00 per bushel. Units of phosphate fertilizer and potash fertilizer each cost \$10.00. Data contained in Table 17.1 describe the yields and *VMP* values for each input in the production of each output.

The data presented in Table 17.1, although useful in illustrating the basic logic behind the equimarginal return principle, oversimplify the problem. The marginal product of each unit of 1 type of fertilizer is assumed to be independent of the availability of the other type of fertilizer. Thus the underlying production function for each output exhibits no interaction between the two inputs and therefore is additive rather than multiplicative.

Suppose that the farmer has available only \$100, or enough to purchase a total of 10 units of fertilizer. Table 17.2 describes how each unit will be allocated. Units 1 and 2 produce the same *VMP* as do units 3, 4, and 5, and units 8,9 and 10. So it does not matter which allocation is done first within these groups.

The general equimarginal return rule requires that

†17.1 $$p_1MPP_{x_1y_1}/v_1 = p_2MPP_{x_1y_2}/v_1 = p_1MPP_{x_2y_1}/v_2 = p_2MPP_{x_2y_2}/v_2 = K$$

The *VMP* of each input in the production of each output will be the same and equal to some number K . The number K is actually a Lagrangean multiplier or an imputed value of an additional dollar available in the case for the purchase of fertilizer to be used in corn or soybean production.

In this example, the price of both inputs, v_1 and v_2, were the same at \$10.00 per unit. The last unit of fertilizer applied in this example produced \$40.00, except for the last unit of potash on soybeans, which produced \$48.00. The correct allocation would have resulted in the same ratio of *VMP* to the price of the input in the production of each output. However, this is often not possible from a tabular data presentation. With this exception, K in our example was \$4.00. The last dollar spent on each input gave back \$4.00 in the production of each output.

Table 17.1 Two Inputs in the Production of Two Outputs[a]

Units of Fertilizer	Phosphate on Corn	$VMP_{x_1y_1}$	Phosphate on Soybeans	$VMP_{x_1y_2}$	Potash on Corn	$VMP_{x_2y_1}$	Potash on Soybeans	$VMP_{x_2y_2}$
0	70		30		80		20	
		80		40		60		80
1	90		35		95		30	
		60		40		60		64
2	105		40		110		38	
		40		24		40		48
3	115		43		120		44	
		20		16		20		24
4	120		45		125		47	
		8		16		12		8
5	122		47		128		48	
		0		16		8		0
6	122		49		130		48	
		!8		8		4		!8
7	120		50		131		47	
		!8		!8		0		!16
8	118		49		131		45	
		!16		!16		!4		!24
9	114		47		130		42	
		!20		!24		!8		!32
10	109		44		128		38	

[a] The price of corn is $4.00, the price of soybeans is $8.00. Units of either phosphate or potash cost $10.00.

Table 17.2 Allocation of Two Fertilizers to Two Crops

Unit	Fertilizer	Crop
1	Phosphate	Corn
2	Potash	Soybeans
3	Potash	Soybeans
4	Phosphate	Corn
5	Potash	Corn
6	Potash	Corn
7	Potash	Soybeans
8	Phosphate	Corn
9	Phosphate	Soybeans
10	Phosphate	Soybeans

The general profit-maximization relationship requires that

†17.2 $$p_1 MPP_{x_1y_1}/v_1 = p_2 MPP_{x_1y_2}/v_1 = p_1 MPP_{x_2y_1}/v_2 = p_2 MPP_{x_2y_2}/v_2 = 1$$

On the input side

†17.3 $$MRS_{x_1x_2} = v_1/v_2$$

in the production of each output. On the output side

†17.4 $RPT_{y_1y_2} = p_1/p_2$

for each input.

17.3 Some General Principles

Suppose that the production of two outputs is governed by two production functions, each with two inputs. Let the production functions for y_1 and y_2 be

†17.5 $y_1 = h(x_{11}, x_{21})$

†17.6 $y_2 = j(x_{12}, x_{22})$

where y_1 and y_2 denote outputs and h and j are production functions for y_1 and y_2, respectively. The first subscript on each x denotes the input, and the second subscript denotes the product to which it is applied. For example, x_{21} is input x_2 that is applied to y_1.

The total amount of x_1 and x_2 are used in the production of y_1 and y_2 are

†17.7 $x_1 = x_{11} + x_{12}$

†17.8 $x_2 = x_{21} + x_{22}$

Total revenue from the sale of y_1 and y_2 is

†17.9 $R = p_1y_1 + p_2y_2$

†17.10 $= p_1h(x_{11}, x_{21}) + p_2j(x_{12}, x_{22})$

where p_1 and p_2 are prices of y_1 and y_2, respectively. The total cost is the sum of the quantities of x_1 and x_2 multiplied by their respective prices

†17.11 $C = v_1x_1 + v_2x_2$

†17.12 $= v_1(x_{11} + x_{12}) + v_2(x_{21} + x_{22})$

Profit (π) is revenue minus cost

†17.13 $\pi = R - C$

$= p_1y_1 + p_2y_2 - v_1x_1 - v_2x_2$

$= p_1h(x_{11}, x_{21}) + p_2j(x_{12}, x_{22}) - v_1(x_{11} + x_{12}) - v_2(x_{21} + x_{22})$

Now let

†17.14 $h_1 = \partial h/\partial x_{11}$

†17.15 $h_2 = \partial h/\partial x_{21}$

†17.16 $j_1 = \partial j/\partial x_{12}$

†17.17 $j_2 = \partial j/\partial x_{22}$

The first-order conditions for maximum profit entail setting the first derivative of the profit function †17.13 equal to zero with respect to each input used in the production of each output

†17.18 $\partial A/\partial x_{11} = p_1h_1 - v_1 = 0$

†17.19 $\partial A/\partial x_{21} = p_1h_2 - v_2 = 0$

†17.20 $\partial A/\partial x_{12} = p_2j_1 - v_1 = 0$

†17.21 $\partial A/\partial x_{22} = p_2j_2 - v_2 = 0$

Equations †17.18 -†17.21 can be rearranged in a number of ways. One way is

†17.22 $p_1h_1/v_1 = p_2j_1/v_1 = p_1h_2/v_2 = p_2j_2/v_2 = 1$

The partial derivative h_1 is the marginal product of x_1 in the production of y_1 or $MPP_{x_1y_1}$; j_1 is the marginal product of x_1 in the production of y_2 or $MPP_{x_1y_2}$; h_2 is the marginal product of x_2 in the production of y_1 or $MPP_{x_2y_1}$; j_2 is the marginal product of x_2 in the production of y_2 or $MPP_{x_2y_2}$. So equation †17.22 can be rewritten as

†17.23 $p_1MPP_{x_1y_1}/v_1 = p_2MPP_{x_1y_2}/v_1 = p_1MPP_{x_2y_1}/v_2 = p_2MPP_{x_2y_2}/v_2 = 1$

The farmer should allocate inputs between outputs in such a way that the last dollar invested in each input in the production of each output returns exactly a dollar. The Lagrangean multiplier in the profit maximization example is 1.

Another way of writing equations †17.18 – †17.21 is

†17.24 $-h_1/h_2 = dx_2/dx_1 = v_1/v_2$ in the production of y_1

†17.25 $-j_1/j_2 = dx_2/dx_1 = v_1/v_2$ in the production of y_2

The marginal rate of substitution of x_1 for x_2 must equal the inverse price ratio in the production of both outputs.

Yet another way of rearranging equations †17.18 – †17.21 is

†17.26 $(p_1h_1/v_1)/(p_2j_1/v_1) = 1$

†17.27 $(h_1/j_1)(p_1/p_2) = 1$

†17.28 $j_1/h_1 = p_1/p_2$

†17.29 $dy_2/dy_1 = p_1/p_2$ for input x_1

†17.30 $RPT_{y_1y_2} = p_1/p_2$ for input x_1

Similarly

†17.31 $j_2/h_2 = p_1/p_2$

†17.32 $dy_2/dy_1 = p_1/p_2$ for input x_2

†17.33 $RPT_{y_1y_2} = p_1/p_2$ for input x_2

The rate of product transformation must be the same for both inputs in the production of the two outputs and equal the inverse product-price ratio.

Of course

†17.34 $h_1 = MPP_{x_1y_1} = v_1/p_1$

†17.35 $h_2 = MPP_{x_2y_1} = v_2/p_1$

†17.36 $j_1 = MPP_{x_1y_2} = v_1/p_2$

†17.37 $j_2 = MPP_{x_2y_2} = v_2/p_2$

In equations †17.34 - †17.37 , the marginal product of each input in the production of each output must be equal to the corresponding factor/product price ratio.

17.4 The Constrained Maximization Problem

The problem might also be set up in a constrained maximization framework. The objective function is the maximization of revenue subject to the constraint imposed by the availability of dollars for the purchase of x_1 and x_2.

Revenue is

†17.38 $R = p_1y_1 + p_2y_2 = p_1h(x_{11}, x_{21}) + p_2j(x_{12}, x_{22})$

Cost is

†17.39 $C° = v_1x_{11} + v_1x_{12} + v_2x_{21} + v_2x_{22}$

All notation is the same as in the example in section 17.3. The Lagrangean is

†17.40 $L = p_1h(x_{11}, x_{21}) + p_2j(x_{12}, x_{22}) + \lambda(C° - v_1x_{11} - v_1x_{12} - v_2x_{21} - v_2x_{22})$

The corresponding first-order conditions for a constrained revenue maximization are

†17.41 $\partial L/\partial x_{11} = p_1h_1 - \lambda v_1 = 0$

†17.42 $\partial L/\partial x_{21} = p_1h_2 - \lambda v_2 = 0$

†17.43 $\partial L/\partial x_{12} = p_2j_1 - \lambda v_1 = 0$

†17.44 $\partial L/\partial x_{22} = p_2j_2 - \lambda v_2 = 0$

Equations †17.41 − †17.44 can also be rearranged in a number of ways. One way is

†17.45 $p_1h_1/v_1 = p_2j_1/v_1 = p_1h_2/v_2 = p_2j_2/v_2 = \lambda$

Again, the partial derivative h_1 is the marginal product of x_1 in the production of y_1, or $MPP_{x_1y_1}$; j_1 is the marginal product of x_1 in the production of y_2 or $MPP_{x_1y_2}$; h_2 is the marginal product of x_2 in the production of y_1 or $MPP_{x_2y_1}$; j_2 is the marginal product of x_2 in the production of y_2 or $MPP_{x_2y_2}$. So equation †17.45 can be rewritten as

†17.46 $p_1MPP_{x_1y_1}/v_1 = p_2MPP_{x_1y_2}/v_1 = p_1MPP_{x_2y_1}/v_2 = p_2MPP_{x_2y_2}/v_2 = \lambda$

The Lagrangean multiplier λ is the imputed value of an extra dollar available for inputs to be used in the production of y_1 and y_2 and allocated in the correct manner. These first-order conditions define a point on both the input and output expansion path.

17.5 An Intermediate Product Model

The intermediate product model is not quite a multiple-input multiple-product model, but it does have the key feature that factor and product space are brought together in a single model. Suppose that a farmer uses available inputs for the production of two products, grain or forage. The grain and the forage are in turn used in the production of beef. Grain and forage can be thought of as two outputs in product space but as two inputs in factor space. A product transformation function can be drawn that represents the farmer's possible combinations of grain and forage that can be produced from the set of inputs or resources available.

Superimposed on this product transformation function are a series of isoquants representing alternative levels of beef production, and each isoquant might represent a steer of a different weight (800, 900, 1000, 1100 pounds, and so on). The simple solution to the problem of maximizing beef production subject to the availability of inputs for the production of grain and forage is to find the point where the isoquant for beef production comes just tangent to the product transformation function. Here the output of beef is maximum, and the marginal rate of substitution of grain for forage in beef production equals the rate of product transformation of grain for forage production (Figure 17.1).

The solution illustrated in Figure 17.1 can be derived using Lagrange's method. Since grain and forage are inputs in one context but outputs in another context, call grain z_1 and forage z_2. The product transformation function for grain and forage is

†17.47 $$x^\circ = g(z_1, z_2)$$

where x° is the bundle of inputs available for grain or forage production, g represents the product transformation function, z_1 is grain, and z_2 is forage.

The production function describing the transformation of grain and forage into beef is

†17.48 $$b = f(z_1, z_2)$$

where b is the quantity of beef produced and f is the specific production function that describes the transformation of grain and forage into beef.

The Lagrangean can be set up as a constrained maximization problem. Beef production is maximized subject to the constraint imposed by the availability of the input bundle x used in the production of grain and forage. The Lagrangean is

†17.49 $$L = f(z_1, z_2) + \lambda[x^\circ - g(z_1, z_2)]$$

Define

$$f_i = \partial f/\partial z_i$$

$$g_i = \partial f/\partial z_i$$

for $i = 1, 2$

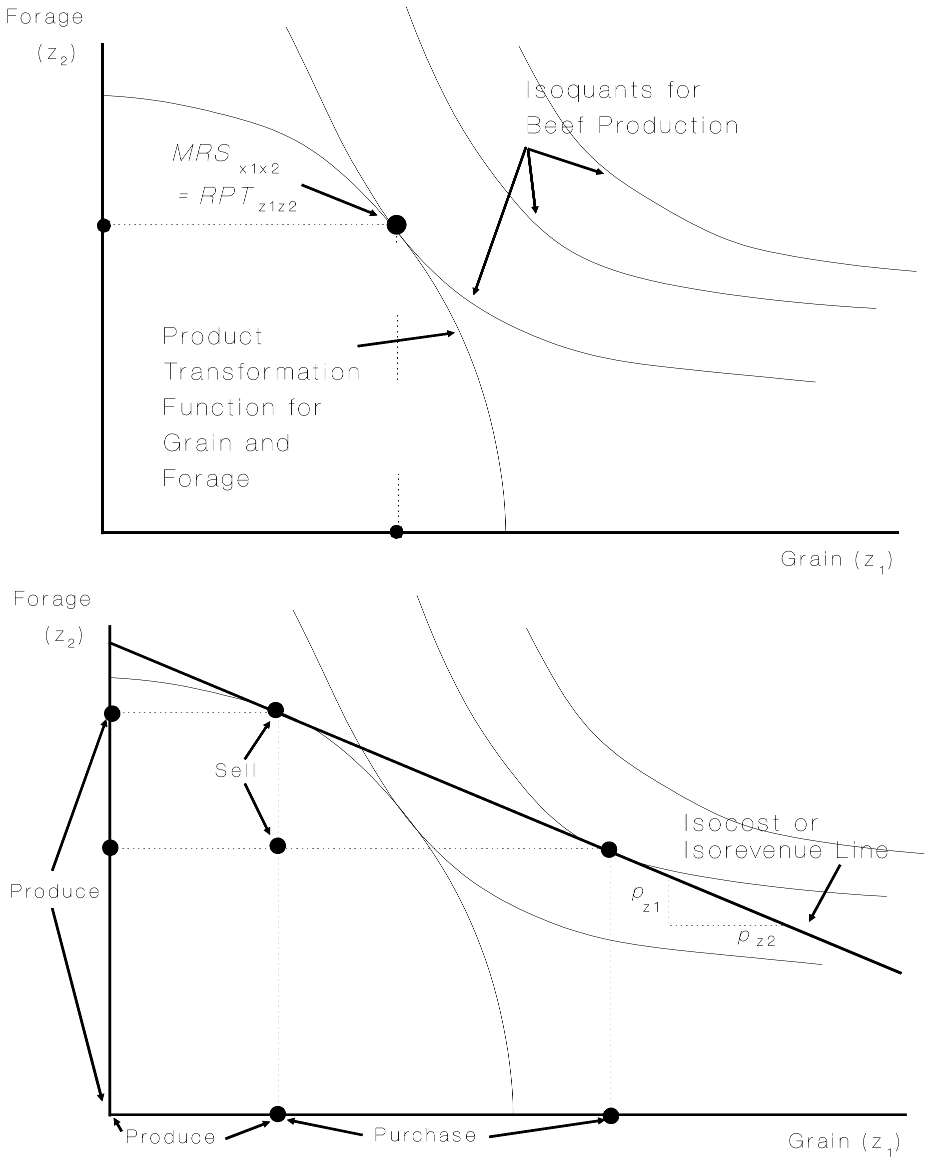

Figure 17.1 An Intermediate Product Model

Then the corresponding first order conditions for a maximum are

†17.50 $\partial L/\partial z_1 = f_1 - \lambda g_1 = 0$

†17.51 $\partial L/\partial z_2 = f_2 - \lambda g_2 = 0$

†17.52 $x - g(z_1, z_2) = 0$

The first order or necessary conditions for the maximization of beef production subject to the constraint imposed by the availability of the input bundle have economic interpretation.

For example, the Lagrangean multiplier (<) is the imputed value of an additional unit of the input bundle *x* in the production of beef *b*. The units on the Lagrangean multiplier are in pounds of beef resulting from the incremental or last unit of the input bundle used. The Lagrangean multiplier represents the shadow price or implicit worth of an additional unit of the input bundle *x* to be used in beef production. The Lagrangean multiplier tells how much additional beef would be produced from an additional unit of the input bundle *x*.

The partial derivatives f_1 and f_2 are the familiar marginal products of grain and forage to be used as inputs in the production of beef. The negative ratio of f_1/f_2 is also the familiar marginal rate of substitution of grain for forage (dz_2/dz_1).

The partial derivatives g_1 and g_2 are marginal factor costs of grain and forage expressed in terms of physical units of the input or resource bundle *x* used in their production. The negative ratio of g_1/g_2 is the familiar rate of product transformation of grain for forage (dz_2/dz_1).

The entire problem is solved relying only on physical or technical relationships governing the transformation of the input bundle into grain and forage and grain and forage into beef. Product and factor prices have not yet entered.

The first-order conditions can be rearranged. By dividing the equation †17.50 by equation †17.51 , we have

†17.53 $! f_1/f_2 = dz_2/dz_1 = MRS_{z_1z_2} = \; ! g_1/g_2 = dz_2/dz_1 = RPT_{z_1z_2}$

Equation †17.53 represents a point of tangency between the isoquant and the product transformation function, as illustrated in figure 17.1. The marginal rate of substitution of grain for forage in beef production must equal the rate of product transformation of grain for forage.

Another possible statement of the first-order conditions is

†17.54 $f_1/g_1 = f_2/g_2 = <$

The marginal product of grain in the production of beef (f_1) divided by its marginal cost in terms of the input bundle *x* (g_1) must equal the marginal product of forage in the production of beef (f_2) divided by its marginal cost in terms of the input bundle *x* (g_2). These ratios should all be equal to the Lagrangean multiplier <.

Suppose that all farmers faced the same product transformation function for grain and forage, the same isoquant map for beef production from grain and forage, and that grain and forage could only be used in the production of beef. Then the relative prices for grain that would prevail would be the prices defined by the ratio p_{z1}/p_{z2}, which would be equal to the marginal rate of substitution of grain for forage in factor space and the rate of product transformation of grain for forage in product space. If the technical parameters governing production by one farmer also apply to all farmers, the firm level marginal conditions will lead to a market level determination of relative prices for each input or intermediate product.

In such a quasi ! general equilibrium setting, all factor prices except one would be determined inside the model. The one price not so determined would become the price to which every other factor price would be compared. The relative prices of grain and forage could thus be determined internal to or endogenous to the model rather than as information coming from the marketplace determined outside the model.

The farmer has options not always recognized by the algebra of the marginal conditions. The market price for beef (p) will be determined not only by the technical parameters governing its production, but by other factors as well. Consumer utility functions for beef relative to other goods must enter. Eventually, consumer utility functions for beef will have an effect on prices in the marketplace, which, in turn, will effect the prices that farmers are able to pay for grain and forage in the production of beef. As a result, grain and forage prices will increase or decrease, since the demand for an input is a function of the product price.

However, in the short run, the price of beef may either exceed or be below its cost of production. Because all farmers may not have the same technical parameters governing grain, forage and beef production, and because grain and forage can be put into uses other than beef production, there is no particular reason to believe that the market will at any particular point in time have found its long run equilibrium as defined by the technical parameters governing the production of beef by each farmer, with input prices governed by the prices for beef in the marketplace.

One option the farmer has is to forget about beef production and sell the forage and grain. Total revenue from the sale of beef is

†17.55 $$TR = pb$$

where p is the price of beef and b is the quantity produced. The input bundle required to produce beef (x_b) can be broken into several cost components

†17.56 $$TC = \text{v}x_b = p_{z1}z_1 + p_{z2}z_2 + \sum v_j x_j + \sum v_k x_k$$

for all $j = 1, ..., m$

$k = m + 1, ..., n$

where z_1 is grain and z_2 is forage. The subscript j represents each of m variable inputs used in the production of beef excluding the cost of the grain and forage, and k represents each of $n - m - 1$ fixed inputs used in beef production. The first set of inputs represents costs incurred only if beef is produced, while the second set of inputs represents those costs incurred regardless of whether or not beef is produced.

The farmer will produce grain and forage and use that to produce beef in the short run if

†17.57 $$TR > p_{z1}z_1 + p_{z2}z_2 + \sum v_j x_j$$

where z_1 = grain

z_2 = forage

p_{z1} = price of grain

p_{z2} = price of forage

$\sum v_j x_j$ = all other variable costs except grain and forage

In other words, the farmer will produce beef in the short run only if variable costs are covered. In the long run, all costs are variable and therefore must be covered if production is to occur.

The farmer also faces a decision with respect to whether to produce grain and forage and sell them as commodities or to shut down entirely. For the farmer to shut down, the total revenue from the sale of beef must be less than the variable costs of production including the market value of the grain and forage. Production of beef is then ruled out.

Costs incurred in the production of grain and forage can again be categorized as fixed or variable depending on whether each cost item would be incurred regardless of production. The farmer would shut down entirely if the total revenue from the sale of the grain and forage in the marketplace did not exceed the costs for variable inputs used in their production. In the long run, all costs are variable, and all must be covered for production to take place.

The farmer faces another option not recognized by the marginal conditions presented so far. Suppose that in the short run, the product transformation function for an individual farmer favors forage production relative to grain production. Market conditions also result in a higher relative price for forage than for grain. The farmer will be able to produce at the level indicated by the point of tangency between the product transformation function and the isorevenue line. The forage is sold on the market and the money is used to purchase grain.

The farmer can reach any point on the isorevenue line for grain and forage as long as the isorevenue comes tangent to a single point on the product transformation function. Any point can be reached by buying one of the products and selling the other, in this case, selling forage and buying grain. By definition, any point on an isorevenue line produces the same total revenue. By selling forage and buying grain, the farmer may be able to produce more beef than would have been the case if he or she had relied solely on the point of tangency between the product transformation function and the isoquant (Figure 17.1). The isorevenue line for grain and forage production is the isocost line for beef production.

17.6 Concluding Comments

This chapter has derived the necessary marginal conditions for global profit maximization and for constrained revenue maximization in a two-factor two-product setting. The value of the marginal product of both inputs in the production of both outputs divided by the respective prices of each input must be the same for both inputs in the production of both outputs in the constrained maximization solution. In addition, global profit maximization (maximization of the difference between revenues and costs) requires that the equality be equal to 1.

The intermediate product model illustrates a situation in which prices for inputs and products might be determined by equating the rates of product substitution between products and the marginal rates of substitution between factors. Prices in such a model are determined within the model rather than taken as givens. If prices are assumed to be determined outside the model, a farmer may be able to take advantage of market conditions and produce a greater amount of output than would
otherwise be the case.

Problems and Exercises

1. Assume that the following conditions exist. What should the farmer do in each instance?

Case	$VMP_{x_1y_1}/v_1$	$VMP_{x_1y_2}/v_1$	$VMP_{x_2y_1}/v_2$	$VMP_{x_2y_2}/v_2$
a	3	3	2	2
b	3	3	3	3
c	3	1	3	1
d	2	2	1	1
e	0	2	0	2
f	1	0	1	0
g	1	1	1	1
h	0	0	0	0
i	!1	0	!1	0
j	!1	!2	!2	!1

2. In a multiple-input, multiple-output setting, how does the solution differ if the farmer is interested in global profit maximization versus constrained revenue maximization? In Problem 1, which conditions represent points of global profit maximization? Which conditions represent solutions to the constrained revenue maximization problem?

3. Explain what is meant by the term *intermediate product*.

4. In what instances might a farmer be able to produce a greater amount of beef than would be suggested by the amount of feed that the farmer could produce. If it is technically possible to produce beef from farm grown feed, should a farmer always do so? Explain. What role does the distinction between fixed and variable costs play in determining whether or not a farmer should sell grain and forage in the market or produce beef?