18

General Multiple-Product and Multiple-Input Conditions

The necessary and sufficient conditions for maximization and minimization developed for the factor-factor model in Chapter 8 and for the product-product model in Chapter 16, can be extended to accommodate any number of inputs and outputs. This chapter illustrates three models. The first model extends the two-input factor-factor model to more than two inputs. The second model extends the two-output product-product model to more than two outputs. The third model combines the factor-factor and product-product models using many different inputs and outputs to derive a general set of conditions for constrained revenue maximization and global profit maximization.

Key terms and definitions:

Categorization of Inputs
First-Order Conditions
Second-Order Conditions
Necessary ! Conditions
Sufficient-Conditions
Bordered Principal Minor
Resource Endowment
Input Requirements Function
Implicit Production Function
General Equimarginal Return Principle

18.1 Introduction

The models that have been developed can be extended to accommodate any number of inputs and outputs. Farmers usually operate in a situation where many inputs are used to produce many different outputs. A general set of rules for allocation of inputs and outputs is needed. In this chapter the factor-factor and product-product models are extended to accommodate more than two inputs and outputs. A general set of rules are developed that would apply in a multiple-product, multiple-input setting.

18.2 Multiple Inputs and a Single Output

Production economists frequently rely on models in which only two factors of production are used. However, there are few if any production processes within agriculture that use only two inputs. The inputs to a production process within agriculture are usually quite diverse.

For example, a production function for a particular crop might include inputs such as land, the farmer's labor, hired labor, fertilizer, seed, chemicals (insecticides and herbicides), tractors, other farm machinery, and irrigation water. A production function for a particular livestock enterprise might include as inputs such as land, the farmer's labor, hired labor, feeds such as grain and forage, buildings, veterinary services and supplies, and specialized machinery and equipment.

If the production economist were to rely on the two-input factor-factor model, the inputs used for the production of either crops or livestock would need to be combined into only two aggregate measures. Here problems arise, for the inputs listed above are very different from each other. A production function calls for inputs measured in physical terms. If such inputs as tractors and fertilizer are to be aggregated, they would have to be measured in dollar terms. Moreover, the tractor provides a stream of services over a number of years, while a high percentage of applied fertilizer is used up during the crop year and a question arises as to how the aggregation for the production function for a single cropping season should take place.

A better approach might be to categorize inputs as fixed or variable and then to extend the theory such that more than two variable inputs could be included in the production function. In such an approach, production and variable cost functions include only those inputs that the farmer would normally treat as variable within the production season. For crops, seed, fertilizer, part time hired labor paid an hourly wage, herbicides and insecticides would be included, but inputs such as tractors and machinery, full time salaried labor, and land would be treated as fixed within the production function and would not be included in the production function and variable-cost equation.

The categorization of inputs as fixed or variable depends on the use which the farmer might make of the marginal conditions proposed by the theory. For example, if the farmer wishes to make use of the marginal conditions only to determine the proper quantity of fertilizer, pesticides, herbicides and part-time hired labor to use in the production of a particular crop, the remaining inputs should be treated as fixed and not as part of the maximization process.

However, such a model would not provide the farmer with any information with respect to questions such as whether or not the renting of additional land would be profitable. If the farmer wanted the model to provide information with regard to the amount of land that could be rented at a profit, the acres of land could be treated as variable with a cash rent charge per acre as the price in the cost function.

Assume that a decision has been made with respect to which inputs are to be treated as variable such that the farmer can make an allocation decision, and the n different inputs in the production function represent only those categorized as variable. A production function using *n* different variable inputs can be written as

†18.1 $$y = f(x_1, ..., x_n)$$

where *n* is the inputs to the production process to be treated as variable and under the control of the farmer. Each *x* represents one of the specific inputs used in the production process, whereas *y* may be the output from either a specific crop or livestock enterprise.

The cost equation for *n* inputs treated as variable by the farmer in a purely competitive environment is

†18.2 $$C = v_1x_1 + ... + v_nx_n = \Sigma v_ix_i \text{ for } i = 1, ..., n$$

A general Lagrangean formulation for revenue maximization allowing for multiple inputs is

†18.3 $$L = pf(x_1,...,x_n) + \lambda(C^\circ - \Sigma v_ix_i)$$

where *p* is the output price.

n different inputs can be varied, and the farmer can control the amount to be used of each. Let

f_1 denote the *MPP* of x_1 holding all other inputs constant

f_i denote the *MPP* of x_i holding all other inputs constant

f_n denote the *MPP* of x_n holding all other inputs constant

Then the first-order conditions for constrained revenue maximization in a many input setting requires that

†18.4 $$pf_1/v_1 = ... = pf_i/v_i = ... = pf_n/v_n = \lambda$$

†18.5 $$pMPP_{x_1}/v_1 = ... = pMPP_{x_i}/v_i = ... = pMPP_{x_n}/v_n = \lambda$$

†18.6 $$VMP_{x_1}/v_1 = ... = VMP_{x_i}/v_i = ... = VMP_{x_n}/v_n = \lambda$$

First-order conditions for constrained revenue maximization in a many-input setting require that all ratios of *VMP* for each variable input to the respective variable input price be equal and equal λ, the imputed value of an additional dollar available for the purchase of *x*. If the Lagrangean multiplier λ is 1, a point of global profit maximization on the input side has been achieved. Conditions presented in equations †18.4 - †18.6 represent the general equimarginal return principle in the *n*-factor case and are consistent with those developed in the two factor case.

For every pair of inputs *i* and *j*,

†18.7 $$dx_j/dx_i = v_i/v_j$$

†18.8 $$MRS_{x_ix_j} = v_i/v_j$$

The slope of the isocost line must be equal to the slope of the isoquant for every pair of inputs. Equations †18.7 and †18.8 define a point of least-cost combination on the expansion path. These conditions are also consistent with those obtained in the two factor setting.

The second-order conditions for a constrained maximization in the *n* ! factor case differ somewhat from those derived in the two-factor setting. In the two-factor setting the determinant of the matrix of partial derivatives obtained by differentiating each of the first-order conditions with respect to x_1, x_2, and the Lagrangean multiplier was always positive. In the *n* ! factor case, the second-order conditions require that determinant of the following matrix have the sign associated with $(! 1)^n$, where *n* is the number of inputs

†18.9
$$\begin{array}{cccc}
f_{11} \cdots f_{1i} \cdots f_{1n} & & & ! v_1 \\
\vdots & \vdots & & \vdots \\
f_{i1} \cdots f_{ii} \cdots f_{in} & & & ! v_i \\
\vdots & \vdots & & \vdots \\
f_{n1} \cdots f_{n1} \cdots f_{nn} & & & ! v_n \\
! v_1 \cdots ! v_i \cdots ! v_n & & & 0
\end{array}$$

In the two-input setting, the determinant of this matrix must be positive, but negative for three inputs, positive for four inputs, and so on. If the number of inputs is even, the determinant will be positive. If the number of inputs is odd, the determinant will be negative.[1] Moreover, the bordered principal minors in the *n* ! input case alternate in sign. To illustrate, the bordered principal minors for the three-input case and the required signs for the determinants are

†18.10
$$\begin{array}{cc} f_{11} & ! v_1 \\ ! v_1 & 0 \end{array} < 0 \qquad \begin{array}{ccc} f_{11} & f_{12} & ! v_1 \\ f_{21} & f_{22} & ! v_2 \\ ! v_1 & ! v_2 & 0 \end{array} > 0 \qquad \begin{array}{cccc} f_{11} & f_{12} & f_{13} & ! v_1 \\ f_{21} & f_{22} & f_{23} & ! v_2 \\ f_{31} & f_{32} & f_{33} & ! v_3 \\ ! v_1 & ! v_2 & ! v_3 & 0 \end{array} < 0$$

The first-order conditions represent the necessary conditions for constrained revenue maximization in the many ! input setting. If the first order conditions hold, the second-order conditions as specified by the required signs on the determinants above are necessary and sufficient for constrained revenue maximization in a many input setting. Second-order conditions rule out points of revenue minimization as well as saddle-point solutions. If the first- and second-order conditions hold and the Lagrangean multiplier is equal to 1, the global point of profit maximization on the input side has been found.

18.3 Many Outputs and a Single Input

Most farmers do not restrict production to a single output, but are involved in the production of several different outputs. The endowment of resources or inputs available to a farmer may differ markedly from one farm to another. Usually, it is not the physical quantities of inputs that are restricted, but rather the dollars available for the purchase of inputs contained within the bundle.

An input requirements function using a single-input bundle to produce many different outputs can be written as

†18.11 $$x = g(y_1, ..., y_i, ..., y_m)$$

where m is the number of outputs of the the production process.

Multiplying by the weighted price of the input bundle v yields

†18.12 $$vx = vg(y_1, ..., y_i, ..., y_m)$$

where $vx = C°$, the total dollars available for the purchase of inputs used in the production of each output.

A general revenue equation for m different outputs produced in a purely competitive environment is

†18.13 $$R = p_1y_1 + ... + p_my_m = \sum p_iy_i \text{ for } i = 1, ..., m$$

A general Lagrangean formulation for revenue maximization allowing for multiple outputs is

†18.14 $$L = p_1y_1 + ... + p_iy_i + ... + p_my_m + \lambda[vx° - vg(y_1, ..., y_i, ...,y_m)]$$

where $vx° = C°$, the money available for the purchase of the input bundle x.

Let g_i denote one over the *MPP* of x in the production of y_i holding all other outputs constant. Then the first order conditions for constrained revenue maximization in a many output setting require that

†18.15 $$p_1/g_1v = ... = p_i/g_iv = ... = p_m/g_mv = \lambda$$

†18.16 $$p_1MPP_{xy_1}/v =... = p_iMPP_{xy_i}/v = ... = p_mMPP_{xy_m}/v = \lambda$$

†18.17 $$VMP_{xy_1}/v = ... = VMP_{xy_i}/v = ... = VMP_{xy_m}/v = \lambda$$

where v is the price of the input.

First-order conditions for constrained revenue maximization in a many output setting require that all ratios of the *VMP* of x to the price of the input bundle (v) be equal, and equal λ, the imputed value of an additional dollar available for the purchase of x. If the Lagrangean multiplier λ is 1, a point of global profit maximization on the output side has been achieved.

Dollars available to the farmer and used for the purchase of the input bundle must be allocated in such a way that the last dollar spent in the production of each output returns the same amount for all the possible different outputs. In other words, if the farmer has found the optimal solution in the constrained case, then the last dollar spent in the production of each output will generate the same return, whether the output is corn, beef, soybeans wheat or milk.

For every pair of outputs *i* and *j*,

†18.18 $$dy_j/dy_i = p_i/p_j$$

†18.19 $$RPT_{y_iy_j} = p_i/p_j$$

The slope of the isorevenue line must be equal to the slope of the product transformation function for every pair of outputs. This equation defines a point on the output expansion path.

Second-order conditions require that determinant of the following matrix have the sign associated with $(!\,1)^m$, where m is the number of outputs

†18.20
$$
\begin{array}{ccccccc}
!\,\mathrm{R}vg_{11} & \ldots & !\,\mathrm{R}vg_{1i} & \ldots & !\,\mathrm{R}vg_{1m} & & !\,vg_1 \\
. & & . & & . & & . \\
. & & . & & . & & . \\
. & & . & & . & & . \\
!\,\mathrm{R}vg_{i1} & \ldots & !\,\mathrm{R}vg_{ii} & \ldots & !\,\mathrm{R}vg_{im} & & !\,vg_i \\
. & & . & & . & & . \\
. & & . & & . & & . \\
. & & . & & . & & . \\
!\,\mathrm{R}vg_{m1} & \ldots & !\,\mathrm{R}vg_{m1} & \ldots & !\,\mathrm{R}vg_{mm} & & !\,vg_m \\
!\,vg_1 & \ldots & !\,vg_i & \ldots & !\,vg_m & & 0
\end{array}
$$

In the two-output setting, the determinant of this matrix must be positive, but negative for three outputs, positive for four outputs, and so on. This second-order condition rules out points of revenue minimization as well as saddle-point solutions. Again, the bordered principal minors must alternate in sign.

The first-order conditions comprise the necessary conditions for constrained revenue maximization in a many input setting. If the required signs for the determinant of equation †18.20 and the bordered principal minors also hold, the conditions are sufficient. If the Lagrangean multiplier is equal to 1 and these sufficient conditions have been met, the global point of profit maximization on the input side has been found. The farmer is globally maximizing profits if the last dollar spent for the input bundle returns exactly a dollar in each farm enterprise.

18.4 Many Inputs and Many Outputs

The most realistic setting is one in which the farmer uses many different inputs to be treated as variable in the production of many different products. The farmer faces a series of decisions. Normally, he or she is constrained by limitations in the availability of dollars that can be used for the purchase of inputs, so the total dollars used for the purchase of inputs must not exceed some predetermined fixed level. The farmer must decide how the available dollars are to be used in the production of various commodities such as corn, soybeans, wheat, beef, or milk. The mix of commodities to be produced must be determined. The farmer must also decide the allocation of dollars with respect to the quantities of variable inputs to be used in each crop or livestock enterprise. Therefore, the mix of inputs to be used in the production of each of the many enterprises must be determined.

Marginal analysis employing Lagrange's method can be used to solve the problem under conditions in which many different factors or inputs to the production process are used in the production of many different commodities. The rules developed in the many-input, many-output case are the same as those derived in the two-factor, two-product case presented in Chapter 17. However, the mathematical presentation becomes somewhat more complicated.

In the problem with two inputs and two products, the equality that must hold contained four expressions, each representing a ratio of *VMP* for an input used in the production of a product relative to the price of an input. In a general setting allowing for many more inputs and outputs, there will be many more expressions in the equality. If there are *m* different outputs produced and every possible output uses some of each of the *n* different inputs, there will be *n* times *m* expressions in the equality representing the first-order conditions. For example, if a farmer uses six inputs in the production of four different outputs, the 24 ratios of *VMP*'s to input prices must be equated.

Suppose that the farmer uses n different inputs in the production of m different outputs. The farmer wishes to maximize revenue subject to the constraints imposed by the technical parameters of the production function, as well as the constraints imposed by the availability of dollars for the purchase of inputs. The revenue function is

†18.21 $$R = p_1y_1 + ... + p_my_m$$

The production function linking inputs to outputs is written in its implicit form[2]

†18.22 $$H(y_1, ..., y_m; x_1, ..., x_n) = 0$$

In the implicit form, a function of both inputs and outputs (*H*) is set equal to zero. The inputs are treated as negative outputs, so each *x* has a negative sign associated with it.

The Lagrangean maximizes revenue subject to the constraint imposed by the technical parameters of the production function, and the availability of dollars for the purchase of inputs. The Lagrangean function is

†18.23 $$L = p_1y_1 + ... + p_my_m + \phi[0 - H(y_1, ..., y_m; x_1, ..., x_n)] + \lambda[C^\circ + v_1x_1 + ... + v_nx_n]$$

Since each input has a negative sign associated with it, it is appropriate that the second constraint be written as $C^\circ + \Sigma v_ix_i$ rather than as $C^\circ - \Sigma v_ix_i$.

Differentiating first with respect to outputs, the first order or necessary conditions are

†18.24 $\partial L/\partial y_1 = p_1 - \phi \partial H/\partial y_1 = 0$

$\vdots$

$\partial L/\partial y_i = p_i - \phi \partial H/\partial y_i = 0$

$\vdots$

$\partial L/\partial y_m = p_m - \phi \partial H/\partial y_m = 0$

For every pair of outputs, *i* not equal to *j*

†18.25 $dy_j/dy_i = p_i/p_j$

The slope of the product transformation function or rate of product transformation must equal the slope of the isorevenue line or inverse output price ratio. Moreover

†18.26 $(\partial H/\partial y_1)/p_1 = ... = (\partial H/\partial y_i)/p_i = ... = (\partial H/\partial y_m)/p_m = 1/\phi$

Differentiating with respect to inputs, the first order conditions are

†18.27 $\partial L/\partial x_1 = -\phi \partial H/\partial x_1 + \lambda v_1 = 0$

$\vdots$

$\partial L/\partial x_i = -\phi \partial H/\partial x_i + \lambda v_i = 0$

$\vdots$

$\partial L/\partial x_n = -\phi \partial H/\partial x_n + \lambda v_n = 0$

For every pair of inputs, *i* not equal to *j*

†18.28 $dx_j/dx_i = v_i/v_j$

The marginal rate of substitution must be equal to the corresponding inverse price ratio. Furthermore

†18.29 $\phi(\partial H/\partial x_1)/v_1 = ... = \phi(\partial H/\partial x_i)/v_i = ... = \phi(\partial H/\partial x_n)/v_n = \lambda$

But

†18.30 $$\mathrm{R} = p_1/(\mathrm{M}H/\mathrm{M}y_1) = ... = p_i/(\mathrm{M}H/\mathrm{M}y_i) = ... = p_m/(\mathrm{M}H/\mathrm{M}y_m)$$

The *m* different expressions for R from equation †18.30 can be substituted each time R appears in equation †18.29 . The multiplier R appears in equation †18.29 *n* different times, so the required *m* times *n* expressions are possible.

†18.31 $$p_i(\mathrm{M}H/\mathrm{M}x_j)/(v_j\ \mathrm{M}H/\mathrm{M}y_i) = (p_i\ \mathrm{M}y_i/\mathrm{M}x_j)/v_j = p_i MPP_{x_j y_i}/v_j$$

or

†18.32 $$p_1 MPP_{x_1 y_1}/v_1 = ... = p_i MPP_{x_1 y_i}/v_1 = ... = p_m MPP_{x_1 y_m}/v_1$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$= p_1 MPP_{x_j y_1}/v_j = ... = p_i MPP_{x_j y_i}/v_j = ... = p_m MPP_{x_j y_m}/v_j$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$= p_1 MPP_{x_n y_1}/v_n = ... = p_i MPP_{x_n y_i}/v_n = ... = p_m MPP_{x_n y_m}/v_n = 8$$

The ratios of the values of the marginal products to the respective input prices must be the same for each input in the production of each output and equal to the Lagrangean multiplier 8. The Lagrangean multiplier 8 is the imputed value of an additional dollar available for the purchase of inputs, allocated according to these conditions. A value for the Lagrangean multiplier 8 of 1 would imply global profit maximization in this setting.

Second-order conditions for the multiple-input, multiple-product case are not presented here, but would not be at variance with the second-order conditions presented earlier in the chapter. The final conclusion in the multiple-input, multiple-product setting is entirely consistent with each of the marginal conditions developed earlier in the text. The rules with respect to input allocation across various outputs can be looked upon as extensions to the simpler models rather than as something different.

18.5 Concluding Comments

This chapter has developed a general equimarginal return principle or rule that applies in a situation where a farmer uses many different inputs in the production of many different outputs. While the underlying conclusions in the case in which many factors are used to produce many different products do not differ from the conclusions reached in Chapter 17 for the two- input, two-output case, the derivation of these conclusions becomes somewhat more complicated. If *n* inputs are each used in the production of *m* different outputs, then *n* times *m* different terms will appear in the equimarginal return equation.

Since farmers usually use several different inputs in the production of a number of different outputs, the equimarginal return expressions developed in this chapter perhaps come closest to applying to the actual situation under which most farmers operate. A farmer will have found a constrained maximization solution if the ratio of *VMP* to input price is the same for every input in the production of every output. Global profit maximization occurs when this ratio is 1 for all inputs and all outputs.

Notes

[1] The slope of *MPP* or f_{11} is negative in stage II of the production function. If there are two inputs, f_{22} is also negative in stage II. If there are three inputs, f_{33} is also negative in stage II. Thus

$$f_{11} < 0$$

$$f_{11}f_{22} > 0$$

$$f_{11}f_{22}f_{33} < 0$$

and so on. The fact that the required sign on the determinant changes as the number of inputs increases is a direct result of the fact that *MPP* is declining within stage II of the production function, where the optimal solutions would be found that meet both necessary and sufficient conditions for a maximum.

[2] A function may be written in its implicit form. For example, the production function $y = f(x_1)$ can be written in its implicit form as $h(x_1, y) = 0$. However, if the implicit function $h(x_1, y) = 0$ is to be written as an explicit production function $y = f(x_1)$, or as the explicit cost function in physical terms $x_1 = f^{-1}(y)$, then the partial derivatives $\partial h/\partial x_1$ and $\partial h/\partial y$ must exist and be nonzero.

Problems and Exercises

1. Are the necessary and sufficient conditions for finding a point representing a solution to the constrained revenue maximization problem the same in an n-input, one-output setting, as in a two-input, one-output setting? Explain.

2. Are the necessary and sufficient conditions for finding a point representing a solution to the constrained revenue maximization problem the same in a one-input, n-output setting as in a one-input, two-output setting? Explain.

3. What do the necessary conditions for constrained revenue maximization require in an n-output, n-input setting? What are the required sufficient conditions?

4. Suppose that in an n-input, n-output problem, the Lagrangean multiplier was found to be 3 for all inputs used in the production of all outputs. Interpret this Lagrangean multiplier. What if the Lagrangean multiplier were instead found to be 1? What would be the interpretation of a Lagrangean multiplier of zero. Could the Lagrangean multiplier be negative? Explain.