3

Profit Maximization with One Input and One Output

This chapter introduces the fundamental conditions for profit maximization in the single input single output or factor- product case. The concept of the total value of the product and the value of the marginal product is introduced. The value of the marginal product and the marginal factor cost are equal at the point of profit maximization. Profits are normally maximum when the implicit value of the last dollar spent on an input is one dollar. Stages of production are described, and an explanation of why a farmer would choose to operate in stage II is given.

Key terms and definitions:

Total Value of the Product (*TVP*)
Profit
Revenue
Cost Function
Value of the Marginal Product (*VMP*)
Total Factor Cost (*TFC*)
Marginal Factor Cost (*MFC*)
Average Value of the Product (*AVP*)
First Order Condition
Second Order Condition
Necessary Condition
Sufficient Condition
Maximum Profits
Minimum Profits
Stages of Production (I, II, and III)
Rational Stage
Irrational Stage
Implicit Worth
Imputed Value
Shadow Price

3.1 Total Physical Product Versus Total Value of the Product

As indicated in Chapter 2, the output (y) from a production function can be also called total physical product (*TPP*). If a firm such as a farm is operating under the purely competitive conditions, the individual farm firm can sell as little or as much output as desired at the going market price. The market price, p, does not vary. A constant price might be called $p°$. Since

†3.1 $TPP = y$,

both sides of equation †3.1 can be multiplied by the constant price $p°$. The result is

†3.2 $p°TPP = p°y$.

The expression $p°y$ is the total revenue obtained from the sale of the output y and is the same as $p°TPP$. The expression $p°TPP$ is sometimes referred to as the *total value of the product* (*TVP*). It is a measure of output (*TPP*) transformed into dollar terms by multiplying by $p°$. For a farmer, it represents the revenue obtained from the sale of a single commodity, such as corn or beef cattle. If the output price is constant, the *TVP* function has the same shape as the *TPP* function, and only the units on the vertical axis have changed (Figure 3.1).

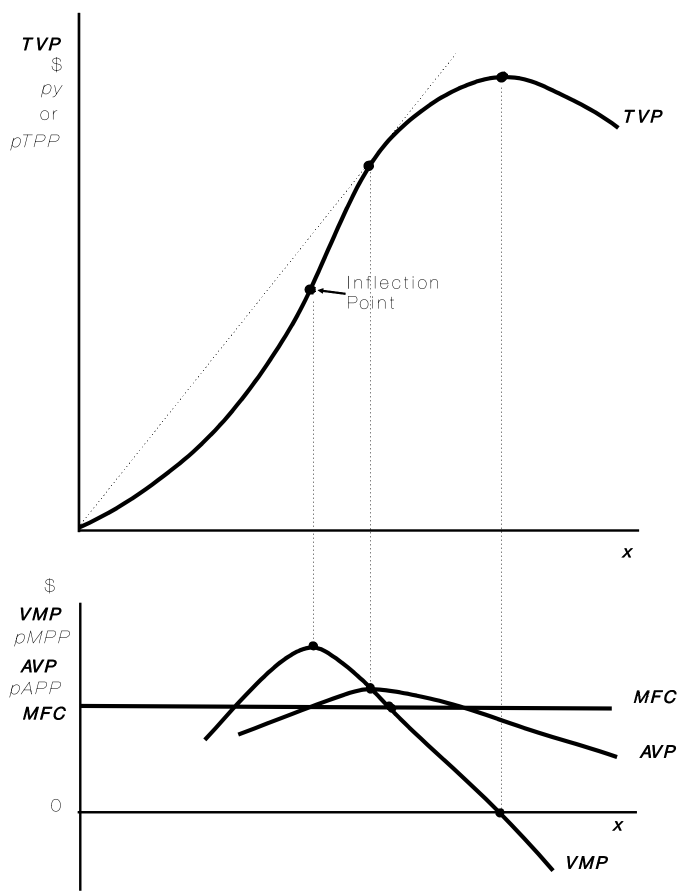

Figure 3.1 The Relationship Between *TVP, VMP, AVP,* and *MFC*

3.2 Total Factor or Resource Cost

Suppose that production requires only one input. Suppose also that a farmer can purchase as much of this input as is needed at the going market price *v*. The purely competitive environment is again assumed to exist. The market price for the input, factor, or resource does not vary with the amount that an individual farmer purchases. Thus the market price might be designated as *v*°. The term *v*°*x* can be referred to as *total factor cost* or *total resource cost*. These terms are sometimes abbreviated as *TFC* or *TRC*. Hence

†3.3 *TRC* = *TFC* = *v*°*x*.

The *TFC* function has a constant slope, in this case equal to *v*°. Another way of looking at *v*° is that it is the increase in cost associated with the purchase of an additional unit of the input. The increase in cost is equal to the price of the input *v*°.

3.3 Maximizing the Difference between Returns and Costs

A farmer might be interested in maximizing net returns or profit. Profit (A) is the total value of the product (*TVP*) less the total factor cost (*TFC*). The profit function for the farmer can be written as

†3.4 A = *TVP* ! *TFC*.

Or, equation †3.4 might be written as

†3.5 A = *p*°*y* ! *v*°*x*

Figure 3.2 illustrates the *TVP* function, the *TFC* function, and the profit function, assuming that the underlying production function is of the neoclassical form as described in detail in chapter 2. The profit function is easily drawn, since it is a graph representing the vertical difference between *TVP* and *TFC*. If *TFC* is greater than *TVP*, profits are negative and the profit function lies below the horizontal axis. These conditions hold at both the very early stages as well as the late stages of input use. Profits are zero when *TVP* = *TFC*. This condition occurs at two points on the graph, where the profit function cuts the horizontal axis. The profit function has a zero slope at two points. Both of these points correspond to points where the slope of the *TVP* curve equals the slope of the *TFC* curve. The first of these points corresponds to a point of profit minimization, and the second is the point of profit maximization, which is the desired level of input use.

The slope of the profit function can be expressed (using) notation) as)A/)*x*. Hence

†3.7)A/)*x* =)*TVP*/)*x* !)*TFC*/)*x*

The slope of the function is equal to zero at the point of profit maximization (and at the point of profit minimization ! more about this later). Therefore, the slope of the *TVP* function ()*TVP*/)*x*) must equal the slope of the *TFC* function ()*TFC*/)*x*) at the point of profit maximization.

3.3 Value of the Marginal Product and Marginal Factor Cost

The value of the marginal product (*VMP*) is defined as the value of the incremental unit of output resulting from an additional unit of *x*, when *y* sells for a constant market price *p*°.

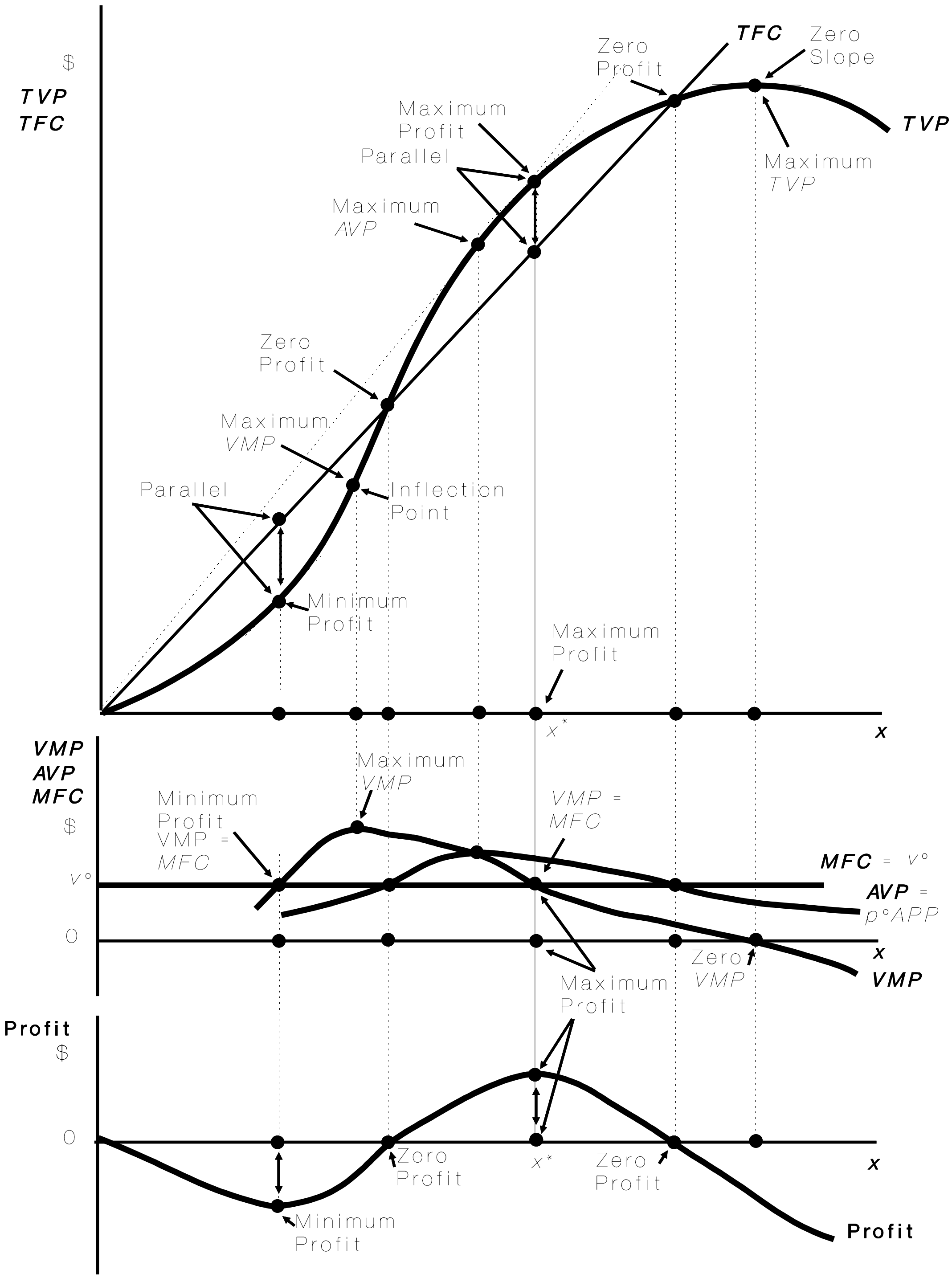

Figure 3.2 ***TVP*, *TFC*, *VMP*, *MFC*, and Profit**

The *VMP* is another term for the slope of the *TVP* function under a constant product price assumption. In other words, *VMP* is another name for $\partial TVP/\partial x$. Since $TVP = p^\circ TPP$, the *VMP* must equal $p^\circ\ \partial TPP/\partial x$. But $\partial TPP/\partial x = MPP$. Therefore, *VMP* must be equal to $p^\circ MPP$.

The *marginal factor cost* (*MFC*), sometimes called *marginal resource cost* (*MRC*), is defined as the increase in the cost of inputs associated with the purchase of an additional unit of the input. The *MFC* is another name for the slope of the *TFC* function. Note that if the input price is assumed to be constant at v°, then $MFC = v^\circ$These relationships might also be expressed by

†3.6 $$\pi = TVP - TFC$$

3.4 Equating *VMP* and *MFC*

The points where the slope of *TVP* equals the slope of *TFC* corresponds either to a point of profit minimization or a point of profit maximization. These points are also defined by

†3.8 $$p^\circ\ MPP = VMP = MFC = v^\circ$$

Figure 3.2 also illustrates these relationships. *MFC*, being equal to a constant v°, is a straight line. Notice that *APP* can be multiplied by the price of the product p°, and is sometimes referred to as average value of the product (*AVP*). It is equal to $p^\circ APP$ or $p^\circ y/x$, or in this case \$4.00·(*APP*).

There are many ways of rearranging the equation $p^\circ\ MPP = v^\circ$. One possibility is to divide both sides of the equation by the output price p°. Then at the point of maximum profit, *MPP* must be equal to v°/p°, the factor/product price ratio. Another possibility is to divide both sides of the equation by average physical product (*APP*) or y/x. The profit maximizing condition would then be given by

†3.9 $$MPP/APP = (v^\circ x)/(p^\circ y)$$

However, *MPP*/*APP* is the elasticity of production for x. The term v°x represents total factor cost. The term $p^\circ y$ represents total revenue to the farm, since it is the price of the output times output. At the point of profit maximization, the elasticity of production will be exactly equal to the ratio of total factor cost to total revenue for the farm.

The data contained in Table 2.5 can be used to determine how much nitrogen fertilizer should be applied to the corn. To do this, prices must be assigned both to corn and to the nitrogen fertilizer. Assume that the price of corn is \$4.00 per bushel and that nitrogen costs \$0.15 per pound. These data are presented in Table 3.1.

Several comments can be made with regard to the data contained in Table 3.1. First, at a nitrogen application level of 180 pounds per acre, the *MPP* of nitrogen is calculated to be 0.0264. The number is very close to zero and suggests that maximum yield is at very close to an application rate of 180 pounds per acre. The *MPP* is calculated by first differentiating the *TPP* or production function to find the corresponding *MPP* function

†3.10 $$y = 0.75x + 0.0042x^2 - 0.000023x^3$$

†3.11 $$dy/dx = 0.75 + 0.0084x - 0.000069x^2$$

Table 3.1 Profit Maximization in the Application of Nitrogen to Corn

Quantity of Nitrogen	Corn Yield (bu/acre)	*MPP* of Nitrogen	$p°$ ($)	*VMP* ($p°MPP$)	*MFC* ($v°$) ($)	Profit (π) ($)
0	0.0	0.7500	4.00	3.0000	0.15	0.0
20	16.496	0.8904	4.00	3.5616	0.15	62.98
40	35.248	0.9756	4.00	3.9024	0.15	134.99
60	55.152	1.0056	4.00	4.0224	0.15	211.61
80	75.104	0.9804	4.00	3.9216	0.15	288.42
100	94.000	0.9000	4.00	3.6000	0.15	361.00
120	110.736	0.7644	4.00	3.0576	0.15	424.94
140	124.208	0.5736	4.00	2.2944	0.15	475.83
160	133.312	0.3276	4.00	1.3104	0.15	509.25
180	136.944	0.0264	4.00	0.1056	0.15	520.78
200	134.000	−0.3300	4.00	−1.3200	0.15	506.00
220	123.376	−0.7416	4.00	−2.9664	0.15	460.50
240	103.968	−1.2084	4.00	−4.8336	0.15	379.87

Then the *MPP* at $x = 180$ is

$$MPP = 0.75 + 0.0084(180) - 0.000069(180)^2 = 0.0264$$

However, since at the point where $x = 180$, *MPP* is still positive, the true yield maximum must be at a nitrogen application level of slightly greater than 180 pounds per acre, where $dy/dx = MPP = 0$.

Profits appear to be greatest at a nitrogen application rate of 180 pounds per acre. However, at 180 pounds per acre, the return from the incremental unit of nitrogen (the *VMP* of x) is $0.1056, whereas its cost is $0.15. The results suggest that the last unit of nitrogen that was used returned less than it cost. The profit-maximizing level of nitrogen use must be at slightly less than 180 pounds per acre. If the input is not free, the profit-maximizing level of input use will always be somewhat less than the level of input use that maximizes the production function. In many instances, however, the difference between the profit-maximizing level of input use and the yield-maximization level of input use may not be very large. In this case the incremental pound of nitrogen must return corn worth only $0.15 in order to cover its cost. If corn sells for $4.00 per bushel, this is but $0.15/$4.00 = 0.0375 bushel of corn from the incremental pound of nitrogen.

The difference between the level of nitrogen needed to maximize profits versus the amount needed to maximize output and total revenue does not appear to be very great. If nitrogen were free, there would be no difference at all. As the price of nitrogen increases, the level of nitrogen required to maximize profits is reduced. For example, if nitrogen sold for $1.00 per pound, the last pound of nitrogen applied would need to produce 0.25 bushel of corn at $4.00 per bushel. In general, the distinction between the point representing maximum profit and the point representing maximum revenue becomes more and more important as input prices increase.

If the price of fertilizer is very cheap, the farmer will lose little by fertilizing at a level consistent with maximum yield rather than maximum profit. However, if fertilizer is expensive, the farmer needs to pay close attention to the level of input use that maximizes profits. The same analysis holds true for other inputs used in agricultural production processes for both livestock and crops.

Profits per acre of corn in this example appear to be extraordinarily high, but remember that the production function describing corn yield response to the application of nitrogen assumes that all other inputs are fixed and given. The cost per acre for these inputs could be calculated. Suppose that this turns out to be \$450 per acre. This value could be subtracted from each value in the profit column. Conclusions with regard to the profit maximizing level of nitrogen use would in no way be altered by doing this.

3.5 Calculating the Exact Level of Input Use to Maximize Output or Profits

The exact level of input use required to maximize output (y) or yield can sometimes be calculated. Several examples will be used to illustrate problems in doing this with various production functions. From the earlier discussion it is apparent that if output is to be at its maximum, the *MPP* of the function must be equal to zero. The last unit of input use resulted in no change in the output level and requires that $MPP = dy/dx = 0$ at the point of output maximization.

Suppose the production function

†3.12 $y = 2x$

In this case

†3.13 $MPP = dy/dx = 2$ (and not zero!)

The *MPP* is always 2, and 2 cannot be equal to zero, and the production function has no maximum. A more general case might be the production function

†3.14 $y = bx$

†3.15 $MPP = dy/dx = b = 0$?

If b were zero, regardless of the amount of x that was produced, no y would result. For any positive value for b, the function has no maximum. Now suppose the production function

†3.16 $y = x^{0.5}$

†3.17 $MPP = dy/dx = 0.5\, x^{-0.5} = 0$?

The only value for x is zero for which the *MPP* would also be equal to 0. Again, this function has no maximum. In general, any function of the form

†3.18 $y = ax^b$

where a and b are positive numbers, has no maximum.

Now suppose a production function

†3.19 $y = 10 + 8x - 2x^2$

†3.20 $dy/dx = 8 - 4x = 0$

†3.21 $4x = 8$

†3.22 $x = 2$

Equation †3.19 has a maximum at $x = 2$. In general, a production function of the form

†3.23 $y = a + bx + cx^2$

where

$$a \geq 0$$
$$b > 0$$
$$c < 0$$

will have a maximum at some positive level of x.

Finally, the output-maximizing level of input use can be found for the production function used in Chapter 2

†3.24 $y = 0.75x + 0.0042x^2$! $0.000023x^3$

First, differentiate to find MPP, and then set MPP equal to 0

†3.25 $MPP = dy/dx = 0.75 + 0.0084x$! $0.000069x^2 = 0$

Now recall from basic algebra that a polynomial of the general form

†3.26 $y = ax^2 + bx + c$

has two solutions for x. These solutions are

†3.27 $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For this production function (Equation †3.24), a = ! 0.000069, $b = 0.0084$ and $c = 0.75$. One solution generates a negative value for x, which can be ruled out as economically impossible. The second solution is 181.595 units of x, which is the output-maximizing level of nitrogen use (or a slightly greater value than 180, where MPP was 0.0264).

The exact amount of nitrogen required to maximize profits in corn production can be calculated by using a similar approach. A few production functions that do not have an output maximum do have a profit maximizing solution. First, if profits are maximum or minimum, the slope of the profit function must be equal to zero.

The total value of the product (TVP) is equal to

†3.28 $TVP = p^\circ y$

where p° = \$4.00 per bushel

y = yield of corn in bushels per acre

The relationship between corn yield and nitrogen use is again given by the production function written in the general form as

†3.29 $y = f(x)$

where x is the amount of nitrogen fertilizer applied in pounds per acre. Thus

†3.30 $TVP = p°f(x)$

The *total factor cost* is

†3.31 $TFC = v°x$

where $v°$ = \$0.15 per pound of nitrogen. The profit function is

†3.32 A $= TVP$! TFC

or

†3.33 A $= 4.00\,f(x)$! $0.15x$

To find the maximum or minimum of the profit function, it is necessary to locate the points on the profit function in which the slope is zero. If the slope of a function is equal to zero, its first derivative must also be equal to zero, because the first derivative of any function is an equation that represents the slope of the function. The first derivative of the profit function can be set equal to zero

†3.34 dA$/dx = 4.00(df/dx)$! $0.15 = 0$

or

†3.35 $4.00(df/dx) = 0.15$

The term on the left-hand side of equation †3.35 is $p°MPP$. The price of the product is multiplied times the amount produced by the incremental unit in order to obtain the value of the marginal product (VMP). The term on the right hand side of the expression is MFC. The conclusion reached here is the same conclusion that was reached from the tabular data. Profits can be maximized at the point where the slope of TVP = the slope of TFC, or $VMP = MFC$.

Several examples are again used to illustrate these ideas for specific production functions. Suppose that $f(x) = bx$, where b is any positive number. Then the production function is

†3.36 $y = bx$

and

†3.37 $TVP = p°bx$

†3.38 $TFC = v°x$

profit = A $= TVP$! $TFC = p°bx$! $v°x$

If profit is to be maximized, then the slope of the profit function must be equal to zero. That is.

†3.39 dA$/dx = p°b$! $v° = 0$

or

†3.40 $p°b = v°$

but $p°$, $v°$, and b are all constants. the value for $p°b$ either equals $v°$ or it does not equal $v°$. If $p°b$ does not equal $v°$, the profit maximizing position has not been found. The value for $p°b$ can be looked upon as the return from the incremental unit of x, and of course, $v°$ is the cost or price of x. If $p°b$ is greater than $v°$, profit could be increased by increasing the use of x by ever larger amounts. If $p°b$ is less than $v°$, any incremental increase in the use of x will not cover the incremental cost, and the farmer would be better off to shut down the operation. If $p°b$ equals $v°$, this is true for any level of input use, since *VMP* is a constant and not a function of x. *VMP* is equal to *MFC* everywhere and the farmer is indifferent as to the level of production.

Now consider a case where the production function is given by

†3.41 $$y = ax^b$$

The corresponding profit function is

$$\text{Profit} = \pi = p°ax^b - v°x$$

The profit maximizing condition is

†3.42 $$d\pi/dx = bp°ax^{b-1} - v° = 0$$

Suppose first that b is greater than 1. Each incremental unit of x produces more and more additional y. Thus *MPP* is increasing, and as a result, *VMP* must also be becoming larger and larger. As a result, the more input that is used by the farmer, the greater the incremental return. As a result the farmer will make the most profit by increasing the use of the input without limit.

Now suppose that b is less than 1 but greater than zero. In this case, *MPP* will decline as the amount of x used is increased. The exact amount of input that will be used to maximize profits can be determined by solving the equation for x

†3.43 $$bp°ax^{b-1} = v°$$

†3.44 $$x^{b-1} = v°/(bp°a)$$

†3.45 $$x = [v°/(bp°a)]^{1/(b-1)}$$

For example, if $b = 0.5$, then $1/(b - 1) = -2$ and $x = [(0.5bp°a)/v°]^2$. If a is positive and b is positive but less than 1, and with constant input and output prices, there will be a finite profit-maximizing level of input use. This is true despite the fact that the underlying production function has no maximum.

Now consider the case for the neoclassical production function used earlier to represent corn yield response to nitrogen as a polynomial (equation †3.10 . From equation †3.10 , it is also possible to determine specifically how much nitrogen would be required to exactly maximize profits.

†3.46 $$y = 0.75x + 0.0042x^2 - 0.000023x^3$$

†3.47 $$\text{Profit} = \pi = p°(0.75x + 0.0042x^2 - 0.000023x^3) - v°x$$

†3.48 $$d\pi/dx = p°(0.75 + 0.0084x - 0.000069x^2) - v° = 0$$

Now suppose that p° = \$4.00, and v° = \$0.15. The first derivative of the profit equation (equation 3.47) can be rewritten as

(3.49) $4.00(0.75 + 0.0084x - 0.000069x^2) = 0.15$

or

$$3 + 0.0336x - 0.000276x^2 = 0.15$$

or

$$2.85 + 0.0336x - 0.000276x^2 = 0$$

Using again the formula for finding solutions to polynomials

(3.50) $$x = \frac{-0.0336 \pm \sqrt{0.0336^2 - 4(-0.000276)(2.85)}}{2(-0.000276)}$$

= 179.32 pounds of nitrogen per acre to maximize profits.
(The only root solution with economic meaning is one that generates a positive value for the input x.)

To ensure that profits are maximized rather than minimized, the second-derivative test, or second-order conditions are sometimes used. The first derivative of the profit function is again differentiated. In this case

(3.51) $d\pi/dx = 2.85 + 0.0336x - 0.000276x^2$

(3.52) $d^2\pi/dx^2 = 0.0336 - 0.000552x$

If $x = 179.322$, the value for the second derivative is

(3.53) $0.0336 - 0.000552\,(179.322) = -0.0653857$

The negative number indicates that profits are at a maximum. A positive number implies a point of profit minimization.

Figure 3.3 illustrates the *TVP*, *VMP*, *AVP*, *MFC*, and profit curves illustrated from the data contained in Table 3.1. In this example, profit actually represents returns to all inputs other than the nitrogen fertilizer. The profit maximizing point can be found at the point where the slope of *TVP* equals the slope of *TFC*, or, equivalently, where *VMP* equals *MFC*. The first panel illustrates the results for an input (*MFC*) price of \$0.15 per pound. As indicated by the data, the level of input use that maximizes profits (179.322 pounds of nitrogen, for a *TVP* of \$547.69 and a profit level of \$520.79) is not very different from the level of input use that maximizes *TVP* (and *TPP*, 181.595 pounds of nitrogen for a *TVP* of \$547.86 and a profit of \$520.62).

In the second panel of Figure 3.3, the price of the input, nitrogen, is increased to \$0.45 per pound. There are two outcomes from the price increase. First, profit is reduced. Second, the maximum of the profit function shifts to the left, to a lower level of input use. In this example, assuming the *MFC* is \$0.45, the profit maximizing level of input use is reduced to 174.642 pounds, and profit is reduced to \$467.69.

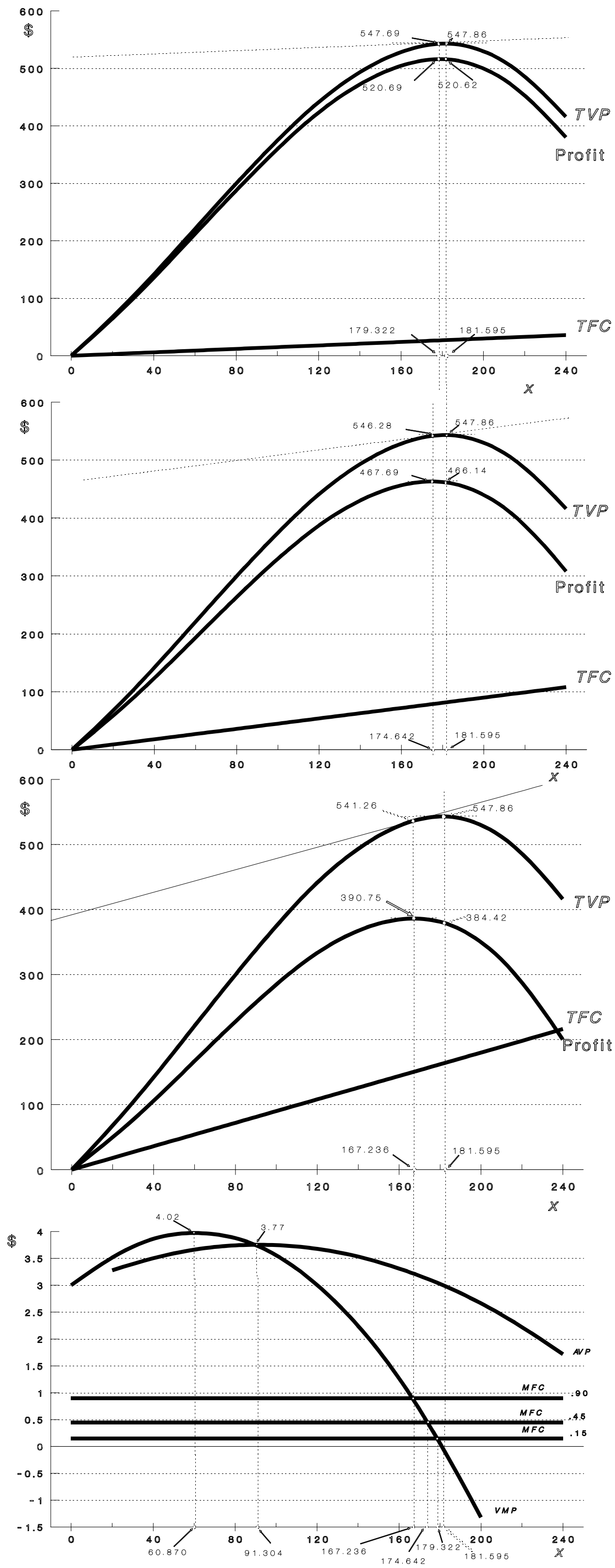

Figure 3.3 Profit Maximization under Varying Assumptions with Respect to Input Prices

In the third panel, the price of the input is further increased to \$0.90 per pound. As a result, the profit-maximizing level of input use is decreased to 167.236 pounds, and the maximum profit is also reduced to \$390.75.

The fourth panel illustrates the equivalent marginal conditions, and illustrates the profit-maximizing points for the three different prices at the intersection of the *VMP* curve and the three marginal factor costs (\$0.15, \$0.45, and \$0.90) The input level where *VMP* is maximum (60.870) and where *AVP* is maximum (91.304) is also illustrated.

3.6 General Conditions for Profit Maximization

The following are a set of rules for profit maximization. The total value of the product function is given as

†3.54 $r = h(x)$

or $r = TVP$

The cost function is given as

†3.55 $c = g(x)$

or $c = TFC$

Profits are defined by

†3.56 $\pi = r - c$

or $\pi = h(x) - g(x)$

or $\pi = TVP - TFC$

The first order conditions for profit maximization require that

†3.57 $d\pi/dx = h'(x) - g'(x) = 0$

†3.58 $= dr/dx - dc/dx = 0$

†3.59 $= dTVP/dx - dTFC/dx = 0$

†3.60 $= VMP - MFC = 0$

†3.61 $VMP = MFC$

†3.62 $VMP/MFC = 1$

The second derivative test is often used to ensure that profits are maximum, not minimum at this point. The second- derivative test requires that

†3.63 $d^2\pi/dx^2 = h''(x) - g''(x) < 0$

†3.64 $h''(x) < g''(x)$

†3.65 $d^2TVP/dx^2 < d^2TFC/dx^2$

†3.66 $dVMP/dx < dMFC/dx$

The slope of the *VMP* function must be less than the slope of *MFC*. This condition is met if *VMP* slopes downward and *MFC* is constant.

3.7 Necessary and Sufficient Conditions

The terms *necessary* and *sufficient* are used to describe conditions relating to the maximization or minimization of a function. These terms have a very special meaning. The term *necessary* means that the condition must hold for the event to occur. In this case, the slope of the profit function must be equal to zero if the function is to be maximized. In equations †3.57 to †3.66 , the condition is the slope of the function, and the event is the maximization of the function.

The necessary condition for the maximization of the function is that the slope be equal to zero. However, if the slope of the profit function is equal to zero, the profit function might also be at a minimum value. A necessary condition does not ensure that the event will occur but only describes a circumstance under which the event could take place. A necessary condition is required for profit maximization, but taken alone, a necessary condition does not ensure that profits will be maximum, only that profits could be maximum.

If the sufficient condition is present, the event will always occur. Thus a sufficient condition for profit maximization means that if the condition holds, profits will always be maximum. The term *sufficient* does not rule out the possibility that there may be other conditions under which the event will take place, but only states that if a particular condition is present, the event will always take place.

The terms *necessary* and *sufficient* are regularly used together. The necessary condition for the maximization of a profit function for corn is that the slope of the function be equal to zero. The sufficient condition for the maximization of a profit function for corn is that the slope of the function be equal to zero, and that the rate of change in the slope, or the second derivative of the profit function, be negative.

Requirements with respect to signs on first derivatives are sometimes called the first-order conditions, or first derivative tests for a maximum or minimum. Requirements with respect to signs on second derivatives are sometimes called the second order conditions, or second derivative tests. A necessary condition is sometimes, but not always, the same as a first derivative test. A second derivative test is normally only part of the requirements for a sufficient condition.

It is not a sufficient condition for a maximum if only the second derivative is negative. There are many points on the profit function that have negative second derivatives which are neither a minimum nor a maximum. Only when the necessary and sufficient conditions are taken together is a maximum achieved. Finally, necessary and sufficient conditions taken together will ensure that the event will always occur and that no other set of conditions will result in the occurrence of the event.

3.8 The Three Stages of the Neoclassical Production Function

The neoclassical production function described in Chapter 2 can be divided into three stages or regions of production (Figure 3.4). These are designated by Roman numerals I, II, and III. Stages I and III have traditionally been described as irrational stages of production. The terminology suggests that a farm manager would never choose levels of input use within these regions unless the behavior were irrational. Irrational behavior describes a farmer who

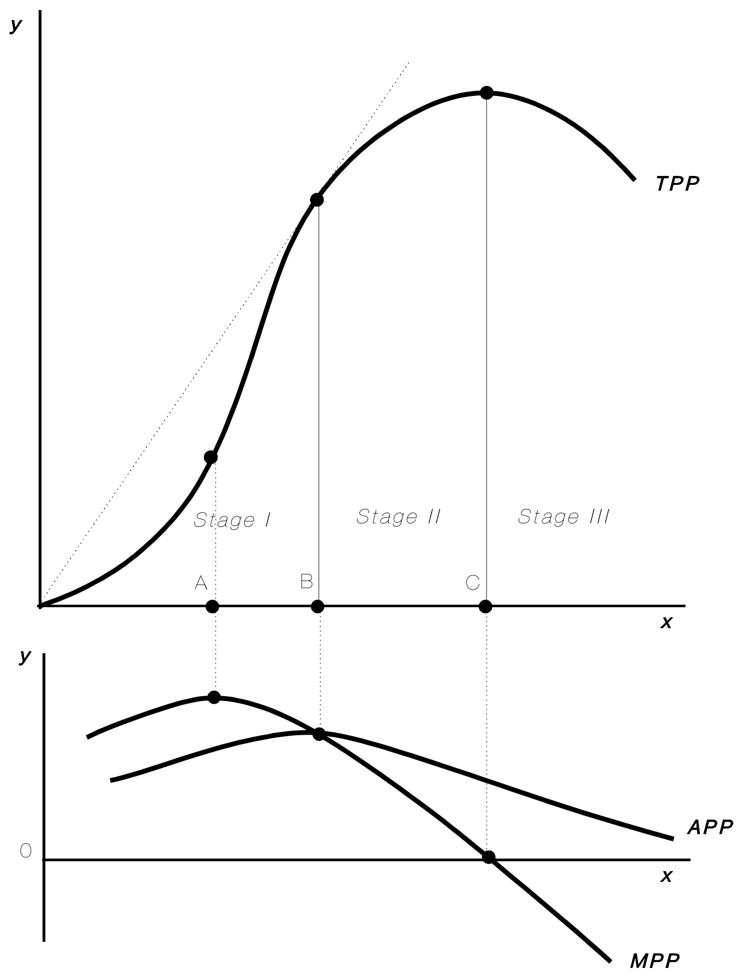

Figure 3.4 Stages of Production for a Neoclassical Production Function

chooses a goal inconsistent with the maximization of net returns, or profit. Stage II is sometimes called the rational stage, or economic region of production. This terminology suggests that rational farmers who have as their goal profit maximization will be found operating within this region. However, in certain instances, such as when dollars available for the purchase of inputs are limited, a rational farmer may not always operate in stage II of the production function.

Stage I of the neoclassical production function includes input levels from zero units up to the level of use where *MPP* = *APP*. *Stage II* includes the region from the point where *MPP* = *APP* to the point where the production function reaches its maximum and *MPP* is zero. *Stage III* includes the region where the production function is declining and *MPP* is negative.

The stages of production can also be described in terms of the elasticity of production. For the neoclassical production function, as the level of input use increases, the elasticity of production (E_p) also changes because the elasticity of production is equal to the ratio of *MPP* to *APP*. The value for the elasticity of production identifies the stage of production. If E_p is greater than 1, then *MPP* is greater than *APP* and we are in stage I. Stage I ends and stage II begins at the point where $E_p = 1$ and *MPP* = *APP*. Stage II ends and stage III begins at the point where E_p equals zero and *MPP* is also zero. Stage III exists anywhere that E_p is negative and hence *MPP* is also negative. Notice that the first stage of the neoclassical production function ends and the second stage begins at the point where the marginal product of the incremental or last unit of input *x* just equals the average product for all units of the input *x*.

It is easy to understand why a rational farmer interested in maximizing profits would never choose to operate in stage III (beyond point C, Figure 3.4). It would never make sense to apply inputs if, on so doing, output was reduced. Even if fertilizer were free, a farmer would never apply fertilizer beyond the point of output maximum.

Output could be increased and costs reduced by reducing the level of input use. The farmer would always make greater net returns by reducing the use of inputs such that he or she were operating instead in stage II.

It is also easy to see why a farmer would not choose to produce in the region where *MPP* is increasing (point A, Figure 3.4) in the first part of stage I, if output prices were constant and sufficient funds were available for the purchase of *x*. In this region, the marginal product of the input is increasing as more and more of the input is used. Diminishing marginal returns have not yet set in, and each additional unit of input used will produce a greater and greater additional net return. The additional return occurs despite the fact that for the first few units, the *MPP* for the incremental unit might still be below the cost of the incremental unit, as represented by the constant *MFC* function.

It is difficult to see why a farmer would not choose to operate in the second part of stage I, where *MPP* is declining but *APP* is increasing (line AB, Figure 3.4), if output prices were constant and sufficient funds were available to purchase additional units of *x*. However, using the definition

†3.67 $$AVP = p^{\circ}APP = p^{\circ}y/x.$$

the total value of the product (*TVP*) might then be defined as

†3.68 $$TVP = \mathrm{x}AVP = \mathrm{x}p^{\circ}y/x = p^{\circ}TPP$$

Look at Figure 3.5. Pick any level of input use and call that level x* corresponding with point A on Figure 3.5. Now draw a vertical line from the horizontal axis to the corresponding point on the *AVP* curve (point B). The value of the *AVP* curve at x* represents the average revenue obtained from the sale of output per unit of *x* used, assuming that the total amount of used was x*. With constant output prices, *AVP* might be thought of as the average revenue expressed per unit of *x* used. Now draw a horizontal line from the point on the *AVP* curve to the vertical axis. The length of the horizontal line represents the total amount of *x* used, or x^*. A rectangle has now been formed, with the lower sides being the axes of the graph. Thus, the rectangle OABE in Figure 3.5 represents the *TVP* for $x = x^*$. This is because the length of the rectangle is x* and its height is *AVP*.

Now draw a line from x^* to *MFC*. Another rectangle is formed by OACD. Input prices are assumed to be constant $v^{\circ} = MFC$. Since v° is constant, v° is equal to the average cost of a unit of *x*; or $TFC = v^{\circ}\mathrm{x}$ and $AFC = (v^{\circ}\mathrm{x})/x = v^{\circ} = dTFC/dx = MFC$. Then *TFC* at $x = x^*$ is equal to the area contained in the second rectangle as measured by OACD.

Profit equals returns less costs.

†3.69 $$A = TVP\ !\ TFC.$$

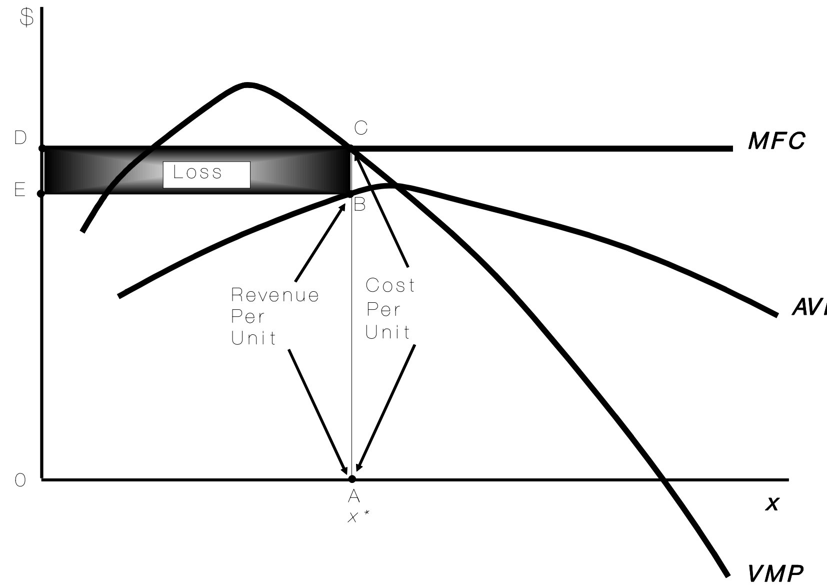

Figure 3.5 If *VMP* is Greater than *AVP*, the Farmer Will Not Operate

In Figure 3.5, the first rectangle is *TVP* and the second rectangle is *TFC*. Since *TVP* is less than *TFC*, the loss is represented by the rectangle EBCD. Suppose now that the input price is lower than the maximum value for *VMP* but higher than the maximum value for *AVP*. These conditions describe the second part of stage I. The farmer equates *VMP* and *MFC* and finds the resulting profit maximizing level of input use x^*. However, since *AVP* is less than *VMP*, the first rectangle representing *TVP* (OABE) would necessarily be less than the second rectangle representing *TFC* (OACD). This would imply that

†3.70 $$A = TVP - TFC < 0$$

Moreover, *TVP* < *TFC* occurs everywhere in stage I of the production function. The farmer would lose money if operation were continued in stage I. If the price of the input is higher than the maximum *AVP*, there is no way that the incremental unit of input can produce returns sufficient to cover its incremental cost. Under such circumstances a rational solution would be to use zero units of the input. This situation will be remedied if either of two events occurs: (1) the price of the input declines to a level below the maximum *AVP*, or (2) the price of the output increases such that *AVP* rises. New technology might also cause *APP* to increase, and the result would be an increase in *AVP*.

If *MFC* were below *AVP* in stage I, the farmer could always increase profit by increasing the use of the input. However, a farmer might not be able to always get the funds needed for the purchase of the input. In the special case, the farmer could operate in stage I if funds for the purchase of input *x* were restricted or limited. In this instance, the profit-maximizing level of input use would occur in stage II. Revenues exceed costs at many points within stage I, and the farmer may be better off to use available revenue for the purchase of *x* and to produce in stage I, even if the profit-maximizing point in stage II cannot be achieved. However, the farmer would never want to operate in stage III of the production function, or, for that matter, to the right of the point in stage II representing the profit-maximizing level of input use, assuming positive input and output prices. The profit-maximizing point is most desired, but other points to the left of the profit maximizing point may also generate a positive profit for the farmer.

3.9 Further Topics on Stages of Production

One of the reasons for the popularity of the neoclassical production function is that it includes all three stages of production. It is worthwhile to examine some features of other production functions in an effort to determine whether or not the various stages of production are accurately represented. As a starting point, a simple function might be

†3.71 $y = bx$

As indicated earlier, The $MPP(dy/dx)$ for this function is b and the $APP(y/x)$ equal to b. The elasticity of production (MPP/APP) is b/b, which is 1 everywhere. This implies that this function does not have any identifiable stages. The curious conclusion is that the function is at the dividing point between stages I and II throughout its range. No wonder this function has not proven popular with economists. If py [output (y) times its price (p)] were greater than bx [input (x) times its marginal product], profit maximization would entail obtaining as much x as one could possibly obtain, and producing as much y as possible. At some point, input prices would not hold constant, and hence the purely competitive assumptions would break down.

A production function with a constant slope produces *VMP* and *MFC* curves that are both horizontal lines, with *VMP* above *MFC*. For a given level of input use, the area under *VMP* represents returns, and the area under the *MFC* represents costs. The portion of the rectangles that do not overlap represents profits. If py were less than bx, returns would not cover costs and the farm would maximize profits by shutting down and producing no output. If py exactly equaled bx, the farmer would be indifferent toward producing or shutting down, since zero profit would result in either case.

Now consider the case where the production function is

†3.72 $y = \sqrt{x}$

As indicated earlier, the elasticity of production in this case is 0.5 throughout the range of the function, since the ratio MPP/APP is 0.5. This suggests that the farm is in stage II of the production function everywhere. Notice that this stage II is not a simple representation of stage II from the neoclassical production function. The elasticity of production for the neoclassical function decreases from 1 (at the start of stage II) to 0 (at the end of stage II) as the use of the input is increased.

For this production function, the elasticity of production remains constant. For any production function of the form $y = bx^{"}$, the elasticity of production is equal to the constant ". If " is greater than 1, the production function is in stage I everywhere. If " is less than zero the function is in stage III everywhere. The function $y = bx$ is a special case of this function with " equal to 1.

3.10 The Imputed Value of an Additional Unit of an Input

For profits to be maximum, a necessary condition is that the slope of the *TVP* function be equal to the slope of the total factor cost function. This might also be expressed as

†3.73 $VMP = MFC = v°$

†3.74 $p°MPP = MFC = v°$

†3.75 $p°dTPP/dx = dTFC/dx$

or

†3.76 $p°dy/dx = v°$

Equations †3.73 to †3.76 all describe the necessary condition for profit maximization. (The sufficient condition requires that *VMP* equal *MFC* and that the *VMP* curve intersect the *MFC* curve from above).

Another way of expressing the relationship *VMP* = *MFC* is

†3.77 $VMP/MFC = 1$

VMP is the return obtained from the incremental unit of *x*, or the value to the manager of the incremental unit of *x*. *MFC* is the cost of the incremental unit of *x*. The equation *VMP* = *MFC* is a decision rule that tells the farmer how much input should be used in order to maximize profits. This decision rule states that the use of the input should be increased until the point is reached whereby the last dollar spent on the input returns exactly its incremental cost. This is one of the fundamental marginal rules of economics. Many if not most of the previous incremental dollars spent on the input paid back more than the cost of the input. These units, taken together, generate the profit for the farm (Figure 3.6).

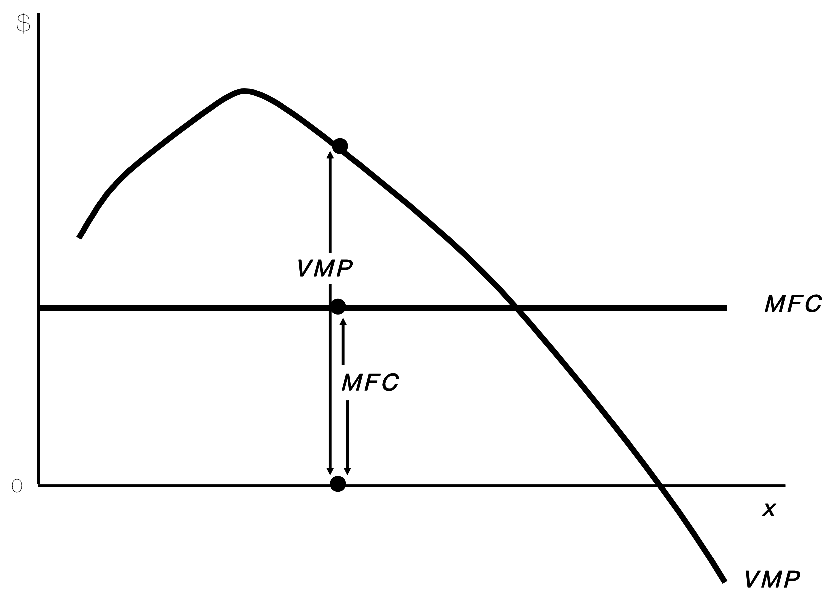

Figure 3.6 The Relationship Between *VMP* and *MFC* Illustrating the Imputed Value of an Input

Now suppose that

†3.78 $VMP/MFC = 3$

Equation †3.78 states that the value of the last dollar spent on the input in terms of its contribution to revenue to for the farm is three times its cost. Moreover, the last dollar spent on the input returns \$3 to the farm. This number is sometimes referred to as the *imputed value* or *implicit worth* of the incremental dollar spent on the input. Both terms refer to the same concept.

There is no particular reason to believe that the imputed value, or implicit worth of the last dollar spent on an input should necessarily be a dollar. The implicit worth of the last dollar spent on an input may be greater than, equal to, or less than a dollar. However, a necessary condition for profit maximization is for *VMP* to equal *MFC*. Profit maximization requires that the value of the last dollar spent on the input be a dollar. If profits are max-imized, the imputed value of an input will be 1 since its contribution to revenue exactly covers its cost. If the imputed value is 3, as in this instance, profits could be further increased by increasing the use of the input until the imputed value is reduced to 1.

Now suppose that

†3.79 *VMP*/*MFC* = 0.5

Equation †3.79 states that the value of the last dollar spent on the input in terms of its contribution to revenue for the farm is only one-half its cost. This is a point to the right of the profit-maximizing point, although it is still in stage II. Revenue from the sale of the output produced by the last unit of input only covers 50 percent of the cost or price of the input. The last dollar spent returns only 50 cents. The other 50 cents is loss. In this case, profits to the farm could probably be increased by reducing the use of the input. Since the *MPP* of the input usually increases as its use decreases, this has the effect of raising *MPP* and thus increasing *VMP* for the input.

Now suppose that

†3.80 *VMP*/*MFC* = 0

Assuming constant positive prices for both the input and output the only way this could happen is if *MPP* were zero. In this instance, the last dollar contributes nothing to revenue. The only point where this could happen is at the maximum of the production (*TPP* or *TVP*) function, the dividing point between stages II and III.

Finally, suppose that

†3.81 *VMP*/*MFC* = ! 5

Assuming constant positive prices for both the input and the output the only way this could happen is for *MPP* to be negative. This implies stage III of the production function. In this case, the last dollar spent on the input results in a loss in revenue of $5. This is a point in stage III where the farmer would never produce.

The implicit worth or imputed value of an input or factor of production has also sometimes been called the *shadow price* for the input. It is called a shadow price because it is not the price that the farmer might pay for the input, but rather the value of a dollar spent on the input to the farmer in his or her operation. A farmer might be willing to purchase an input at prices up to but not exceeding the imputed value or shadow price of the input on the farm.

Diagrammatically, the shadow price or imputed value of an input can easily be seen (Figure 3.6). The *VMP* represents the value of the input: the *MFC*, its price or cost per unit. The shadow price is the ratio of value to price. If *MFC* and product prices are constant, the shadow price usually increases until *MPP* reaches its maximum and then decreases. The shadow price is 1 where *MPP* (and *VMP*) intersects *MFC*, and zero where *MPP* intersects the horizontal axis of the graph.

3.11 Concluding Comments

Profit-maximization conditions for the factor-product model have been introduced. Profits are maximum when the necessary and sufficient conditions for a maximum have been met. The necessary conditions for profit maximization require that the profit function have a slope of zero. The necessary condition for profit maximization can be determined by finding the point on the profit function where the first derivative is zero. The sufficient condition, ensuring profit maximization, holds if the first derivative of the profit function is zero and the second derivative of the profit function is negative.

Alternatively, the level of input use that maximizes profits can be found by equating the *VMP* of the input with the *MFC*, which in pure competition is the price of the input. The slope of the total value of the product curve will be equal to the slope of the total factor cost curve. The slope of the total value of the product curve is its derivative, which if output prices are constant is the *VMP* curve. If the price of x is constant, the slope of the total factor cost curve is the *MFC*.

Under the assumptions of pure competition, with constant, positive input and output prices, a farmer interested in maximizing profits would never operate in stage III of the production function, where *MPP* and *VMP* are declining. A farmer would operate in stage I of the production function only if sufficient funds were not available for the purchase of inputs needed to reach stage II. A farmer would not produce at all if the price of x exceeded the maximum average value of the product.

Problems and Exercises

1. Suppose that the output sells for \$5 and the input sells for \$4. Fill in the blanks in the following table.

x (input)	y (output)	*VMP*	*AVP*
0	0		)))
		)))	
10	50		)))
		)))	
25	75		)))
		)))	
40	80		)))
		)))	
50	85		)))

2. In Problem 1, what appears to be the profit-maximizing level of input use? Verify this by calculating *TVP* and *TFC* for each level of input use as shown in the table.

3. Suppose that the production function is given by

$$y = 2x^{0.5}$$

The price of x is \$3 and the price of y is \$4. Derive the corresponding *VMP* and *AVP* functions. What is *MFC*? Solve for the profit-maximizing level for input use x.

4. When the input price is constant, the slope of the total factor cost function will also be constant. Is this statement true or false? Explain.

5. Whenever the total factor cost function and the total value of the product function are parallel to each other, profits will be maximized. Is this statement true or false? Explain.

6. Suppose that the production function is the one found in Problem 5, Chapter 2. Corn sells for $4.00 per bushel and nitrogen sells for $ 0.20 per pound. At what nitrogen application rate are profits maximized?

7. Explain the terms *necessary* and *sufficient*, in terms of a farmer seeking to maximize profits in the feeding of dairy cattle for milk production.

8. Is the shadow price of a dairy feed ration different from the price the farmer pays per pound of the ration? Explain. Of what importance is a shadow price to a farmer seeking to maximize profits from a dairy herd?

9. Explain the consequences to the farmer if the production function for milk were a linear function of the amount of feed fed to each cow.

10. Verify each of the numbers presented in Figure 3.3.