4

Costs, Returns and Profits on the Output Side

In this chapter the concept of a cost function defined in terms of units of output is introduced. Total, variable and marginal cost curves are illustrated using graphics and derived using mathematics. The necessary conditions for determining the level of output that maximizes profits are derived. The cost functions are shown to be closely linked to the parameters of the underlying production function. The supply function for the firm is derived.

Key terms and definitions:

Total Cost (*TC*)
Total Variable Cost (*VC*)
Marginal Cost (*MC*)
Total Fixed Cost (*FC*)
Average Cost (*AC*)
Average Fixed Cost (*AFC*)
Average Variable Cost (*AVC*)
Inverse Production Function
Duality of Cost and Production

4.1 Some Basic Definitions

In Chapter 3, a very simple cost equation was defined. This cost equation was

†4.1 $$TFC = v°x$$

Equation †4.1 states that the total cost for an input or factor of production is the constant price of the input ($v°$) multiplied by the quantity that is used.

However, the costs of production might also be defined not in terms of the use of the input, but in terms of the output. To do this, some basic terms need to be explained.

Variable costs (*VC*) are the costs of production that vary with the level of output produced by the farmer. For example, in the production of corn, with the time period being a single production season, variable costs might be thought of as the costs associated with the purchase of the variable inputs used to produce the corn. Examples of variable costs include the costs associated with the purchase of inputs such as seed, fertilizer, herbicides, insecticides, and so on. In the case of livestock production within a single production season, a major variable cost item is feed.

Fixed costs (*FC*) are the costs that must be incurred by the farmer whether or not production takes place. Examples of fixed-cost items include payments for land purchases, and depreciation on farm machinery, buildings, and equipment.

The categorization of a cost item as fixed or variable is often not entirely clear. The fertilizer and seed a farmer uses can only be treated as a variable cost item prior to the time in which it is placed in the ground. Once the item has been used, it is sometimes called a sunk, or unrecoverable, cost, in that a farmer cannot decide to sell seed and fertilizer already used and recover the purchase price.

Although depreciation on farm machinery is normally treated as a fixed cost, given sufficient time, the farmer does have the option of selling the machinery so that the depreciation would no longer be incurred. Payments for the purchase of land would not be made if the farmer elected to sell the land. The categorization of farm labor is very difficult. A farm laborer on an annual salary might be treated as a fixed cost which the farmer incurs whether or not production takes place. But if the laborer is laid off, the cost is no longer fixed. Temporary workers hired on an hourly basis might be more easily categorized as a variable cost.

The categorization of a particular input as a fixed cost or variable cost item is thus closely intertwined with the particular period involved. Over very long periods, a farmer is able to buy and sell land, machinery, and other inputs into the production process that would normally be considered fixed. Thus, over very long periods, all costs are normally treated as variable.

Over a very short period of time, perhaps during a few weeks within a single production season, a farmer might not be able to make any adjustment in the amounts of any of the inputs being used. For this length of time, all costs could be treated as fixed. Thus the categorization of each input as a fixed- or variable-cost item cannot be made without explicit reference to the particular period involved. A distinction between fixed and variable costs has thus been made on the basis of the period involved, with the proportion of fixed to variable costs increasing as the length of time is shortened, and declining as the length of time increases.

Some economists define the long run as a period of time of sufficient length such that the size of plant (in the case of farming, the farm) can be altered. Production takes place on a short-run average cost curve (*SRAC*) that is U shaped, with the manager equating marginal revenue (the price of the output in the purely competitive model) with short-run marginal cost (*SRMC*). There exists a series of short-run marginal and average cost curves corresponding to the size of the particular plant (farm). Given sufficient time, the size of the plant can be altered. Farmers can buy and sell land, machinery, and equipment. Long ! run average cost (*LRAC*) can be derived by drawing an envelope curve which comes tangent to each short run average cost curve (Figure 4.1).

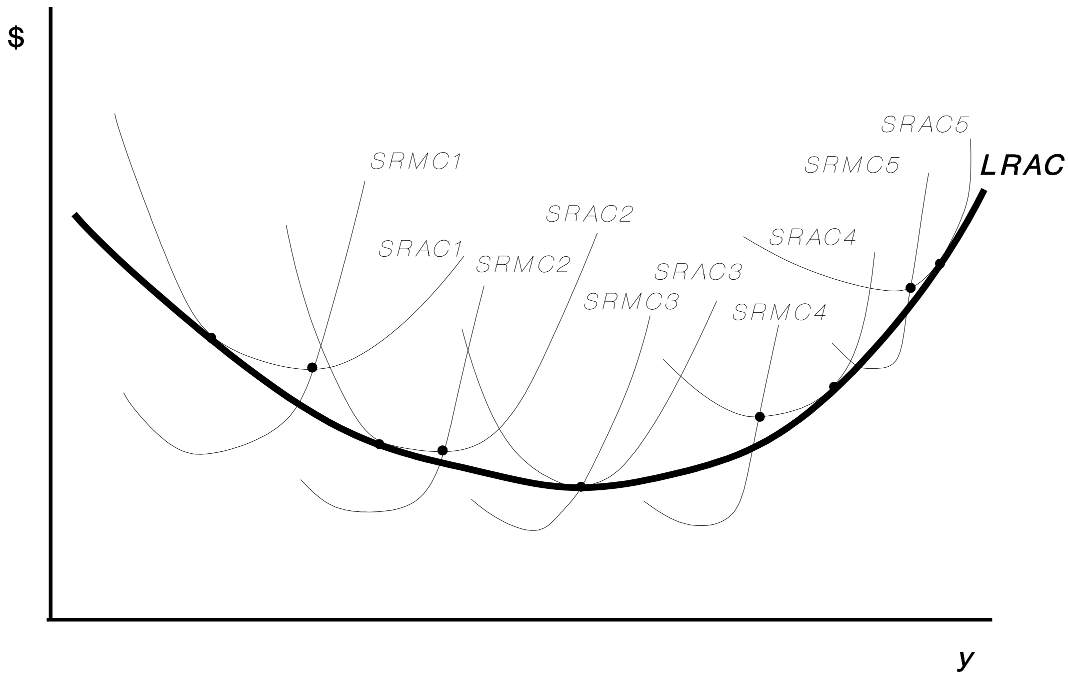

Figure 4.1 Short- and Long-Run Average and Marginal Cost with Envelope Long-Run Average Cost

A classic argument in economics was that between the economist Jacob Viner and his draftsman. Viner insisted that such a long-run average cost curve must necessarily come tangent to the minimum points on each short run average cost curve. The draftsman argued that this was impossible ! that plants operating with less capacity than that represented by the minimum point on the *LRAC* curve must necessarily be tangent to a point on the *LRAC* at higher than minimum *SRAC*. Plants operating at greater than the capacity suggested by the minimum *LRAC* would have a *SRAC* tangent to *LRAC* at a point at greater than minimum *SRAC*. Only for the plant operating with its *SRAC* curve at the point of minimum *LRAC* would the *LRAC* be tangent to the minimum point on the *SRAC*. The draftsman was, of course, correct (Figure 4.1).

In long-run equilibrium, producers discover and select a plant size at the minimum point on *LRAC*. Hence *MR* equals *LRMC* and there is no profit. In the short run, however, *MR* can exceed *MC*. Each producer would equate *MR* to his own *SRMC*. For the producers operating in the short run, this would entail using the plant beyond its point of minimum *SRAC*. No producer would ever be observed operating at the minimum *SRAC* and *LRAC*, save the firms in long run equilibrium.

Variable costs are normally expressed per unit of output (y) rather than per unit of input (x). This is because there is usually more than one variable cost item involved in the production of agricultural commodities. A general expression for a variable cost function is

†4.2 $$VC = g(y)$$

Since fixed costs do not vary with output, fixed costs are equal to some constant dollar value k; that is

†4.3 $$FC = k$$

Total costs (TC) are the sum of fixed plus variable costs.

†4.4 $$TC = VC + FC, \text{ or}$$

†4.5 $$TC = g(y) + k$$

Average variable cost (AVC) is the variable cost per unit of output

†4.6 $$AVC = VC/y = g(y)/y$$

Average fixed cost is equal to fixed cost per unit of output

†4.7 $$AFC = FC/y = k/y$$

The output level y is divided into the constant k, where k represents total fixed costs (FC).

There are two ways to obtain average cost (AC), sometimes also called average total cost (ATC). One way is to divide total cost (TC) by output (y)

†4.8 $$AC = ATC = TC/y$$

Another way is to sum average variable cost (AVC) and average fixed cost (AFC)

†4.9 $$AC = AVC + AFC, \text{ or}$$

†4.10 $$TC/y = VC/y + FC/y$$

Marginal cost is defined as the change in total cost, or total variable cost, resulting from an incremental change in output.

†4.11 $$MC = \Delta TC/\Delta y = \Delta VC/\Delta y$$

Since the value for fixed costs (FC) is a constant k, MC will be the same irrespective of whether it is based on total costs or total variable cost.

Marginal cost (MC) at a particular point is the slope of the total cost function. Marginal cost can be defined in terms of derivatives. In this instance

†4.12 $$MC = dTC/dy = dVC/dy$$

The marginal cost function is a function representing the slope of the total cost function. For example, a value for MC of \$5.00 indicates that the last or incremental unit of output cost an additional \$5.00 to produce.

Figure 4.2 illustrates the cost functions that have been defined. The illustration of *VC* looks like a production function that has been inverted. Output, rather than input, is on the horizontal axis. The vertical axis is dollars, not units of input. Moreover, the slope of the *VC* function appears to be exactly the inverse of the slope on a production function. The production function increased initially at an increasing rate until the inflection point was reached, then it increased at a decreasing rate. The cost function first increases at a decreasing rate until the inflection point is reached. Then the cost function increases at an increasing rate.

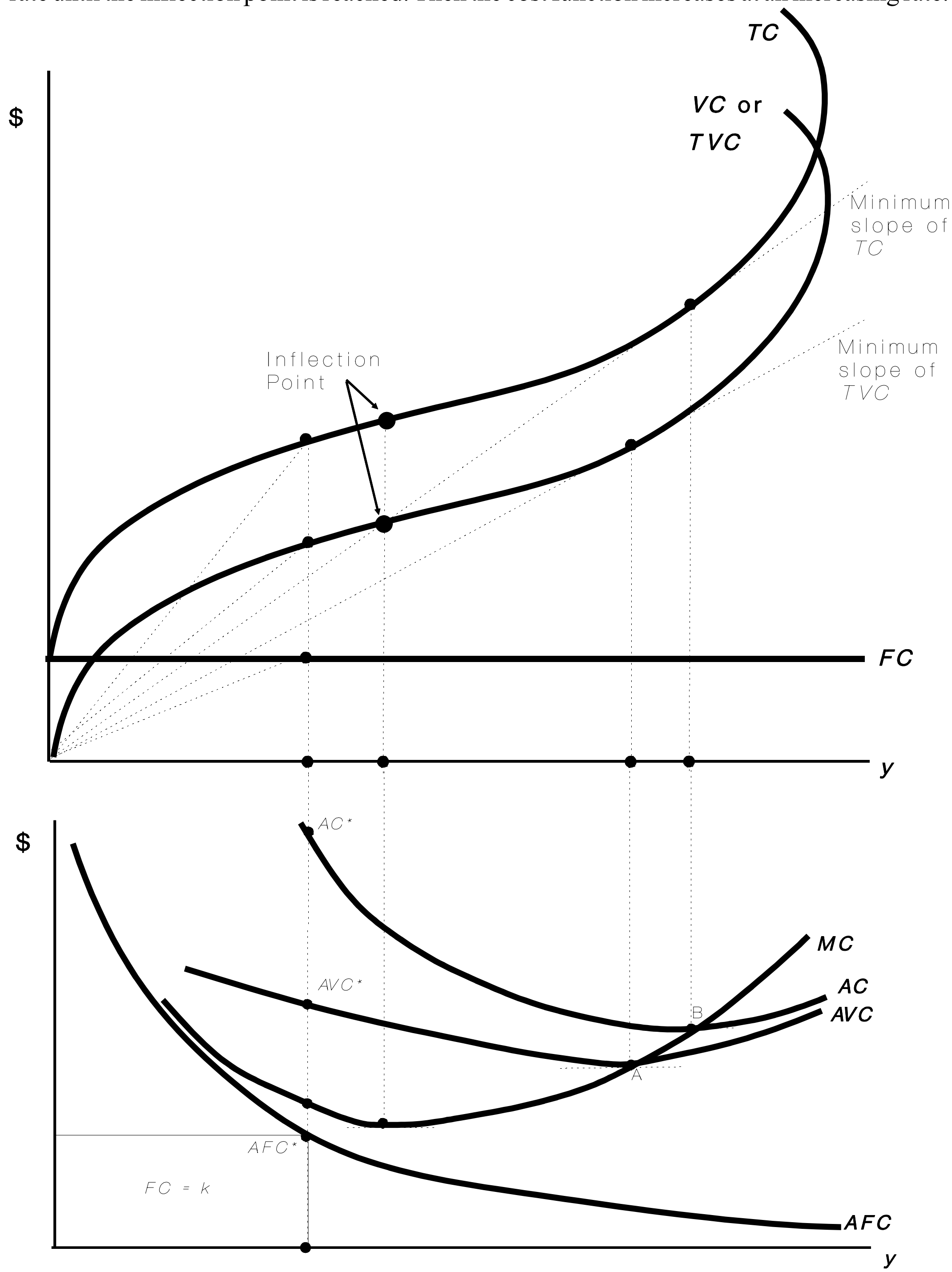

Figure 4.2 Cost Functions on the Output Side

The cost curves look rather strange when output reaches its technical maximum. Suppose that the maximum yield a farmer can achieve in the production of corn is 140 bushels per acre. Suppose that despite the farmer's best efforts to increase yields further by applying more seed, fertilizer, and pesticides, the additional yield is just not there. The additional seed results in more plants that become overly crowded in the field, and the additional plants become so crowded that yield is reduced. The additional fertilizer starts to do damage to the crop. The additional herbicides kill the corn plants. As more and more variable inputs are used, yield starts to drop off to 130, 120 or even 110 bushels per acre. Costs for the additional variable inputs are incurred even at yield levels that could have been achieved with a much lower level of input use and a corresponding reduction in the cost for seed, fertilizer, and pesticide. The variable-cost function must turn back on itself once the maximum yield is achieved. This is actually stage III of variable cost.

Once variable cost turns back on itself, it is no longer technically a function. This is because for some yield levels, two rather than one value for variable cost is assigned. Thus *VC* might be thought of in this case as a cost correspondence rather than a cost function.

Fixed cost (*FC*), being constant, is a horizontal line positioned at the corresponding dollar value on the vertical axis. Total cost (*TC*) appears nearly the same as variable cost (*VC*). Total cost has been shifted vertically by the fixed-cost amount. The difference between *TC* and *VC* at any point is *FC*. *TC* and *VC* are not parallel to each other, because *FC* is represented by the vertical distance between *TC* and *VC*. At each level of output, however, the slope of *TC* equals the slope of *VC*.

Any point on the average cost curves (*AC*, *AVC*, and *AFC*) can be represented by the slope of a line drawn from the origin of the graph to the corresponding point on *TC*, *VC* or *FC*. Suppose that the value for *AC*, *AVC* and *AFC* at some output level called y* is to be determined. Draw a vertical line from y* to the corresponding point on *TC VC* and *FC*. Call these points *TC**, *VC**, and *FC**. Now draw a line from each of these points to the origin of the graph. Three triangles will result. The slope of each of these triangles represents the corresponding *AC**, *AVC** and *AFC** for the output level y* (Figure 4.2).

Marginal cost (*MC*) at any point is represented by the slope of a line drawn tangent to either *TC* or *VC*. The minimum slope for both *TC* and *VC* occurs at the respective inflection points of *TC* and *VC*. The inflection points for both *TC* and *VC* correspond to the same level of output. Thus *MC* is minimum at the inflection point of either the *TC* or the *VC* curve, and there is but one *MC* curve that can be derived from either the *TC* or the *VC* curve.

Minimum *AVC* occurs where a line drawn from the origin comes tangent to *VC*. Minimum *AC* occurs where a line drawn from the origin comes tangent to *TC*. The point of tangency on *TC* occurs to the right of the point of tangency on *VC*. Thus the minimum *AC* will occur to the right of the minimum *AVC*. Since these lines are tangent to *TC* and *VC*, they also represent the slopes of the curves at the two points. Hence they also represent *MC* at the two points. Therefore, *MC* must be equal to and cut *AVC* and *AC* at each respective minimum (points A and B, Figure 4.2).

The relationship that must exist between *AC* and *MC* can be proven

†4.13 $$TC = (AC)y$$

†4.14 $$dTC/dy = AC \cdot 1 + y(dAC/dy)$$

†4.15 $$MC = AC + y(\text{the slope of } AC)$$

If the slope of *AC* is positive, *MC* must be greater than *AC*. If the slope of *AC* is negative, *MC* must be less than *AC*. If the slope of *AC* is zero, *AC* is at its minimum and *MC* must equal *AC*. The reader can verify that the same relationship must hold between *MC* and *AVC*.

AFC is a rectangular hyperbola. Draw a straight line from any point on the *AFC* curve to the corresponding vertical ($) and horizontal ($y$) axis. The area of the enclosed rectangle is equal to *FC* which is the constant k (Figure 4.2). To the point of maximum output, as y becomes larger and larger, *AFC* comes closer and closer to the horizontal axis but does not reach it. Similarly, as y becomes smaller and smaller, *AFC* becomes larger and larger and gets closer and closer to the vertical axis. Again *AFC* never reaches the vertical axis.

Since *AC* is the sum of *AVC* + *AFC*, and *AFC* becomes smaller and smaller to the point of maximum output, as output increases, *AC* should be drawn closer and closer to *AVC*. The minimum slope of a line drawn from the origin of the graph to the *TC* curve occurs at an output level larger than the output level associated with the minimum slope of a line drawn from the origin to the *VC* curve. Therefore, minimum *AVC* occurs at an output level smaller than the level at which minimum *AC* occurs.

The behavior of average and marginal cost curves beyond the point of output maximization is somewhat complicated. Beyond the point of output maximization, y is reduced. Since *FC* remains constant, *AFC* returns along the exact same curve. *AVC* and *AC* are increasing even as y is reduced, when inputs are used beyond the point of output maximization. Moreover, if there are any fixed costs, *AC* must remain above *AVC*. Both *AVC* and *AC* must turn back on themselves to represent the new higher average costs associated with the reduction in output when inputs are used beyond the point of output maximization. If this is to occur, *AC* must cross over *AVC*. At the point of output maximum, both *AC* and *AVC* have a perfectly vertical or infinite slope (Figure 4.3). In stage III, *MC* goes into the negative quadrant when *MPP* is negative and forms a mirror image of its appearance in the positive quadrant.

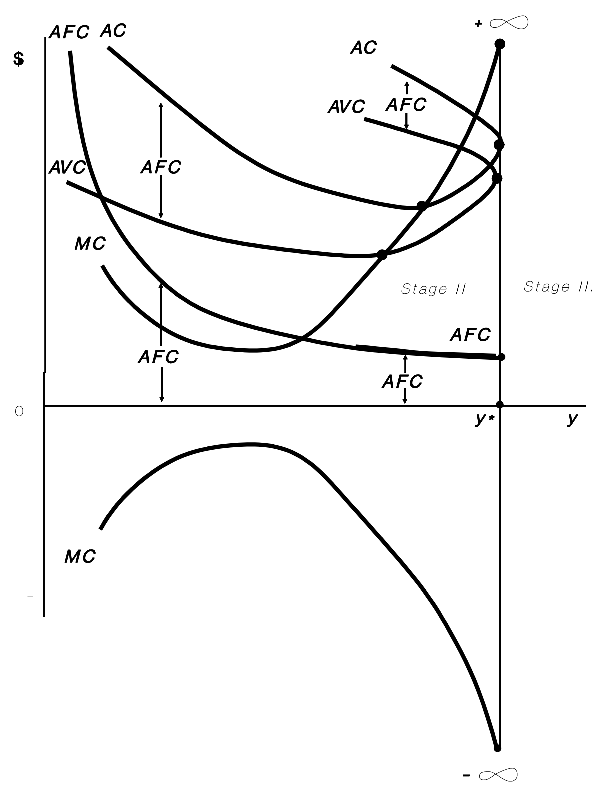

Figure 4.3 Behavior of Cost Curves as Output Approaches a Technical Maximum y^*

4.2 Simple Profit Maximization from the Output Side

Perhaps no criterion is more famous in economics than the expression "marginal cost equals marginal revenue." This simple rule is the basic requirement for selecting the level of output that maximizes profit.

If a farmer can sell all the output that he or she produces at the going market price, the resulting total revenue (TR) function is a line with a constant positive slope of $p°$

†4.16 $TR = p°y$

where $p°$ is some constant market price and y is the output.

The farmer's profit is equal to total revenue (TR) minus total cost (TC)

†4.17 $\pi = TR - TC$

The greatest profit will be achieved when the difference between TR and TC is greatest (Figure 4.4). Superimpose the TR function on the previously defined TC. The greatest vertical distance between TR and TC occurs at points where the slope of TR and TC are the same. There are two points where this occurs. At the first point, TC is above TR, so this point represents the minimum profit. The second point represents maximum profit, which is the desired point.

Maximum (or minimum) profit is achieved at the points where the slope of the profit function is equal to zero. Thus

†4.18 $d\pi/dy = dTR/dy - dTC/dy = 0$

Notice that dTR/dy represents the slope of TR, and dTC/dy is the slope of TC. The slope of TR is referred to as marginal revenue (MR). The slope of TC has already been defined as marginal cost (MC) Hence equation †4.18 can be rewritten as

†4.19 $MR - MC = 0$

or, the famous

†4.20 $MR = MC$

Under the assumptions of pure competition, the output price is constant. Incremental units of the output can be sold at the going market price $p°$. Hence MR must be $p°$.

†4.21 $dTR/dy = p° = MR$

Figure 4.4 illustrates the average and marginal cost curves with marginal revenue included. Marginal cost equals marginal revenue at two points. The first point corresponds to the point of profit minimization, the second to the point of profit maximization. The second derivative test can be used to confirm this.

Differentiate the equation

†4.22 $MR - MC = 0$

which results in

†4.23 $dMR/dy - dMC/dy = +$ or $-$?

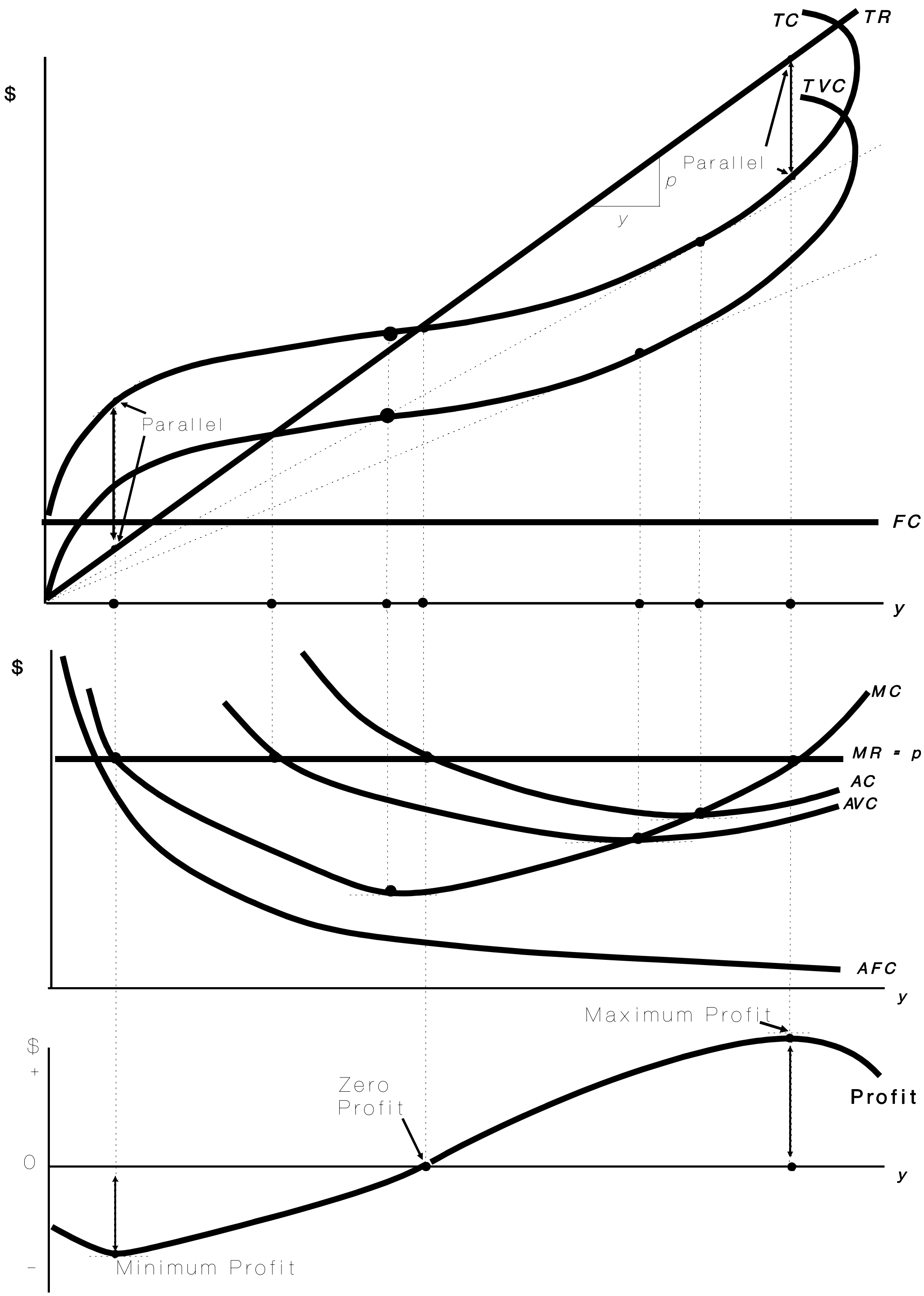

Figure 4.4 Cost Functions and Profit Functions

The sign on equation †4.23 tells if the point is a maximum or a minimum on the profit function. A negative sign indicates a maximum, and a positive sign is a minimum. Another way of looking at equation †4.23 is that the slope of *MC* must be greater than the slope of *MR* for profits to be maximized.

The term *dMR*/*dy* represents the slope of the the marginal revenue curve. In this case, marginal revenue is a constant with a zero slope. The sign on equation †4.23 is thus determined by the slope of *MC*, which is *dMC*/*dy*. If the slope of *MC* is negative, equation †4.23 will be positive. This condition corresponds to the first point of intersection between *MC* and *MR* in Figure 4.4. If the slope of *MC* is positive, then equation †4.23 is negative, and a point of profit maximization is found corresponding to the second point of intersection between *MC* and *MR* in Figure 4.4. The minimum point on the profit function represents the maximum loss for the farmer.

The farmer has an option not recognized by the mathematics. Suppose that *MC* has a positive slope, but *MR* = *MC* at a price level so low that it is below *AVC*. In this instance, the farmer would be better off not to produce, because he or she would lose only his fixed costs (*FC*). This would be less than the loss incurred at the point where *MR* = *MC*. If, however, *MC* = *MR* at a level between *AVC* and *AC*, the farmer would be better off to produce. In this instance, the farmer, by producing, would cover all the variable costs plus a portion of the fixed costs. The total loss would be less than if production ceased and all the fixed costs had to be paid. This explains why farmers might continue to produce corn even though the market price is less than the total costs of production. With a high ratio of fixed to variable costs (as would often be the case in grain production), the farmer is better off to produce and incur only the partial loss, at least in the short run.

Of course, in the long run, the farmer can make major adjustments, and all costs should be treated as variable. Farmers can buy and sell land and machinery in the long run, making these costs variable. If the length of run is sufficiently long, a farmer will continue to produce only insofar as all costs are covered. A farmer cannot continue to lose money indefinitely without going bankrupt.

Table 4.1 illustrates some hypothetical total cost data for corn production and shows the corresponding average and marginal costs. Corn is assumed to sell for $4.00 per bushel.The relationships represented in the data contained in Table 4.1 are the same as those illustrated in Figure 4.4. Marginal cost (*MC*) is the change in cost over the 10-bushel increment obtained by calculating the change in *TC* (or *VC*) and dividing by the change in output. Marginal cost equals marginal revenue at between 110 and 120 bushels of corn per acre. Profits are maximum at that output level. It is not possible to determine the exact output level without first knowing the exact mathematical function underlying the data contained in Table 4.1.

Figure 4.5 illustrates the data contained in Table 4.1, and confirms the profit-maximizing output level at approximately 115 bushels of corn per acre. Thic corresponds withthe point where the slope of *TR* equals the slope of *TC*, or *MR* = *MC*. Notice also that the *TR* curve intersects *TC* at exactly the output level at which *AC* equals *MR* equals *Average Revenue* (*AR*) per unit of output equals the price (p) of the output, which in this example is $4.00 per bushel. An increase in the price of the output would increase the profit-maximizing output level beyond 115 bushels per acre; a decrease would reduce the profit-maximizing output level below the 115 bushel level. An increase in the variable input price(s) would reduce the profit-maximizing output level, whereas adecrease inthe input price(s) would increase the profit-maximizing output level.

Table 4.1 Hypothetical Cost Data for Corn Production

Yield, (y)	*TVC*	*FC*	*TC*	*AVC*	*AFC*	*AC*	*MC*	*MR*
40	89	75	164	2.23	1.88	4.11		
							2.10	4.00
50	110	75	185	2.20	1.50	3.70		
							2.00	4.00
60	130	75	205	2.17	1.25	3.42		
							1.00	4.00
70	140	75	215	2.00	1.07	3.07		
							1.50	4.00
80	155	75	230	1.94	0.94	2.88		
							2.00	4.00
90	175	75	250	1.94	0.83	2.78		
							2.50	4.00
100	200	75	275	2.00	0.75	2.75		
							3.00	4.00
110	230	75	305	2.09	0.68	2.77		
							4.00	4.00
120	270	75	345	2.25	0.63	2.88		
							5.00	4.00
130	320	75	395	2.46	0.58	3.04		
							6.00	4.00
140	380	75	455	2.71	0.54	3.25		

4.3 The Duality of Cost and Production

The shape of the total variable cost function is closely linked to the shape of the production function that underlies it. If input prices are constant, all the information about the shape of the *VC* function is contained in the equation for the underlying production function. Moreover, if the *VC* function and the prices for the inputs are known, so is the shape of the underlying production function. If input prices are constant, then all of the needed information for determining the shape of the *VC* is given by the production function, and all the information for determining the shape of the production function is given by the *VC* function.

In Chapter 2, the law of diminishing returns was stated "As units of a variable input are added to units of a fixed input, after a point, each additional unit of variable input produces less and less additional output." Another way of stating this law is that after a point, incremental or additional units of input each produce less and less additional output.

The law of diminishing returns might also be interpreted from the output side. From the output side, the law states that as output is increased by 1 unit at a time, after a point, each incremental or additional unit of output requires more and more additional units of one or more variable inputs. Another way of saying this is that if output is increased incrementally, after a point, each incremental or additional unit of output becomes more and more costly with respect to the use of inputs. Another unit of output is produced but only at the expense of using more and more additional input.

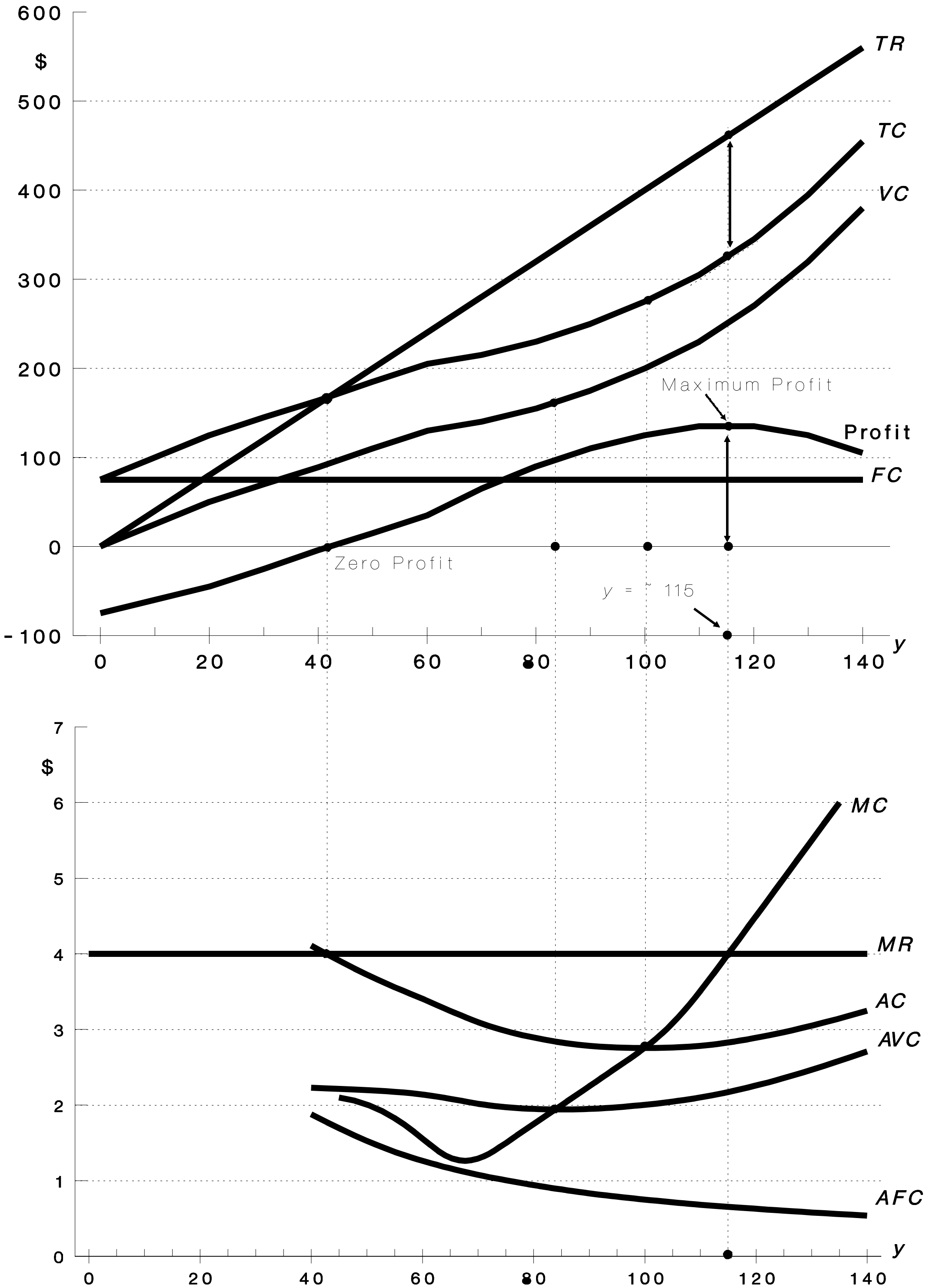

Figure 4.5 The Profit-Maximizing Output Level Based on Data Contained in Table 4.1

The reason the variable cost function appears to be a mirror- image production function with its axes reversed now becomes clear. The production function reflects the fact that each incremental unit of input produces less and less additional output. The corresponding variable cost function reflects the fact that incremental units of output become more and more costly in terms of input requirements.

The fertilizer response data contained in table 2.5 in chapter 2 is presented in a manner in which this dual relationship can be readily observed (Table 4.2).

Table 4.2 Corn Response to Nitrogen Fertilizer

Nitrogen x	Corn y	Exact MPP	$1/MPP$	$v°$	$v°/MPP$ (MC)	Exact APP	$1/APP$	$v°/APP$ (AVC)
0	0.0	0.7500	1.33	0.15	0.200	a	a	a
20	16.496	0.8904	1.12	0.15	0.168	0.8248	1.21	0.182
40	35.248	0.9756	1.03	0.15	0.154	0.8812	1.13	0.170
60	55.152	1.0056	0.99	0.15	0.149	0.9192	1.09	0.163
80	75.104	0.9804	1.02	0.15	0.153	0.9388	1.07	0.160
100	94.000	0.9000	1.11	0.15	0.167	0.9400	1.06	0.160
120	110.736	0.7644	1.31	0.15	0.196	0.9228	1.08	0.163
140	124.208	0.5736	1.74	0.15	0.262	0.8872	1.13	0.169
160	133.312	0.3276	3.05	0.15	0.458	0.8332	1.20	0.180
180	136.944	0.0264	37.88	0.15	5.682	0.7608	1.31	0.197
200	134.000	-0.3300	-3.03	0.15	-0.454	0.6700	1.49	0.224
220	123.376	-0.7416	-1.35	0.15	-0.202	0.5608	1.78	0.267
240	103.968	-1.2084	-0.83	0.15	-0.124	0.4332	2.31	0.346

[a]Undefined. Errors due to rounding.

Compared with Table 2.5, the data appear inverted. In Chapter 2, average physical product was defined as y/x, and marginal physical product was defined as $\Delta y/\Delta x$. Now x/y and $\Delta x/\Delta y$ have been calculated.

If $y/x = APP$, then x/y must be $1/APP$. The expression x/y represents the average cost for nitrogen to produce the incremental unit of output, but the cost is expressed in terms of physical units of the input, not in dollar terms. This cost is equal to $1/APP$. This cost can be converted to dollar units by multiplying by the price of nitrogen, earlier called $v°$. The result is the average variable cost for nitrogen per unit of output $AVC_n = v°/APP$.

If $\Delta y/\Delta x = MPP$, then $\Delta x/\Delta y$ must be $1/MPP$. The expression $\Delta x/\Delta y$ represents the marginal cost for nitrogen to produce the incremental unit of output, but again the cost is represented in physical terms, not in dollar terms. This cost is equal to $1/MPP$. This cost can again be converted to dollar units by multiplying by the price of nitrogen or $v°$. The result is the marginal cost for nitrogen per unit of output $MC_n = v°/MPP$.

At a nitrogen application rate of 180 pounds per acre, marginal cost is \$5.68 per bushel of corn produced. If corn is selling for \$4.00 per bushel, the incremental bushel of corn costs \$5.68 but returns only \$4.00. However, at a nitrogen application rate of 160 pounds per acre, the marginal cost of the incremental bushel of corn is but \$0.458. If corn is selling for \$4.00 per bushel, the difference of \$3.54 is profit to the farmer. The farmer could increase profits by increasing the use of nitrogen fertilizer until the marginal cost associated with the production of the incremental bushel of corn just equals marginal revenue. This should be at a nitrogen application level of slightly less than 180 pounds per acre- 179.322 pounds to be exact. That is exactly the solution found in Chapter 3. It makes no difference whether *VMP* is equated to *MFC* or *MR* is equated to *MC*. The solution provides the farmer with exactly the same conclusion with regard to how much input should be used. The solution to the profit-maximization problem is the same regardless of whether it is done on the output or input side.

4.4 The Inverse of a Production Function

Any production function has an underlying dual cost function or correspondence (Figure 4.6). The production function has input (nitrogen or x) on the horizontal axis and output (corn or y) on the vertical axis. The corresponding cost function expressed in physical terms is the production function with the axes reversed. The result is the inverse production function, or cost function expressed in physical terms. This cost function is dual to the production function.

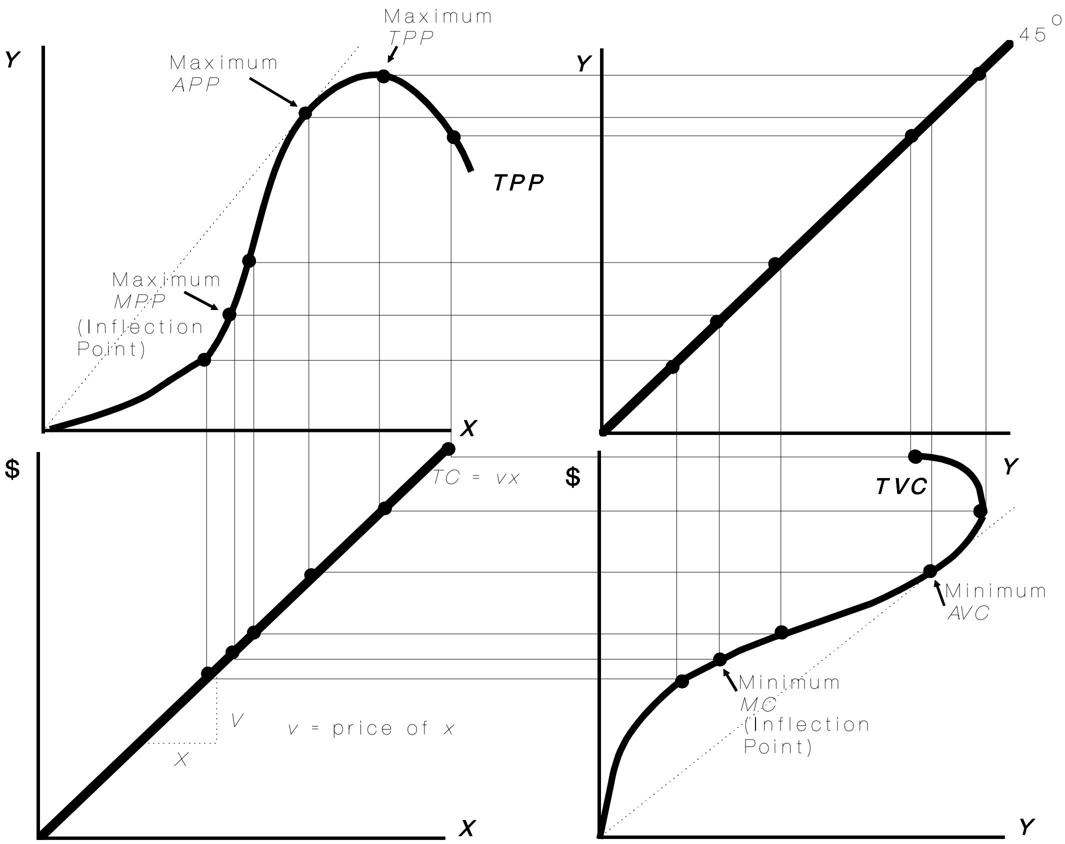

Figure 4.6 A Cost Function as an Inverse Production Function

Note that this function is in many ways the mirror image of the underlying production function. If the production function is increasing at an increasing rate, the inverse production function increases at a decreasing rate. If the production function is increasing at a decreasing rate, the inverse production function increases at an increasing rate.

Inverses to production functions for some simple functions might readily be calculated. All that is required is to solve the function in terms of the *x* instead of y. For example, suppose that the production function is

†4.24 $y = 2x$

The corresponding inverse production function is

†4.25 $x = y/2 = 0.5y$

If the production function is

†4.26 $y = bx$ where b is any number

the corresponding inverse production function is

†4.27 $x = y/b$.

Suppose that the production function is

†4.28 $y = x^{0.5}$

The corresponding inverse production function is

†4.29 $x = y^{1/0.5} = y^2$

And if the production function is

†4.30 $x = y^{1/0.5} = y^2$

The inverse production function is

†4.31 $x = y^{1/2} = y^{0.5}$

For the production function

†4.32 $y = ax^b$

The corresponding inverse function is

†4.33 $x = (y/a)^{1/b}$

In each of these examples, the inverse function contains all the coefficients contained in the original production function and can be converted into true variable cost functions by multiplying by the constant price ($v°$) of the input x. If these functions were drawn, the vertical axis would then be in terms of dollars rather than physical units of the input x.

It is therefore not necessary to know the physical quantities of the inputs that are used in the production process in order to determine the coefficients of the production function. If the cost function is known, it is frequently possible to determine the underlying production function.

A general rule is that if the production function is

†4.34 $y = f(x)$

then the corresponding inverse production function is

†4.35 $x = f^{-1}(y)$

Not all production functions can be inverted into another function to obtain the corresponding dual cost function. Any production function that includes both increasing and decreasing *TPP* will not have a inverse function, but only an inverse correspondence. The neoclassical production function is an example. The inverse in Figure 4.6 is actually a correspondence, but not a function.

The total cost for the input expressed in terms of units of output is obtained by multiplying the inverse function times the input price. Suppose that

†4.36 $$y = f(x)$$

Then

†4.37 $$x = f^{-1}(y).$$

Multiplying by $v°$ results in the total cost (TC_x) for the input (x or nitrogen) from the production function for corn [$y = f(x)$]

†4.38 $$v°x = TFC = TC_x = v°f^{-1}(y)$$

4.5 Linkages between Cost and Production Functions

Suppose that the price of the input is $v°$ and the production function is

†4.39 $$y = 2x$$

Then $MPP = APP = 2$, and $MC_x = AVC_x = v°/2$.

If the production function is

†4.40 $$y = bx$$

Then $MPP = APP = b$, and $MC_x = AVC_x = v°/b$.

If the production function is

†4.41 $$y = x^{0.5}$$

then $MPP = 0.5/x^{0.5}$, $APP = x^{0.5}/x = x^{0.5}x^{-1} = x^{-0.5} = 1/x^{0.5}$,

$MC_x = (v°x^{0.5})/0.5 = 2v°x^{0.5}$, and $AVC_x = v°x^{0.5}$.

If *MPP* is precisely one half of *APP*, then MC_x will be precisely twice AVC_x. If the elasticity of production (E_p) is defined as the ratio *MPP*/*APP*, then $1/E_p$ is the ratio of MC_x/AVC_x.

If the production function is

†4.42 $$y = ax^b$$

then the inverse production function is

†4.43 $$x = (y/a)^{1/b}$$

†4.44 $$MPP = abx^{b-1}$$

†4.45 $APP = ax^{b!1}$

†4.46 $E_p = b$

†4.47 $MC_x = v°/abx^{b!1}$

†4.48 $AVC_x = v°/ax^{b!1}$

†4.49 ratio of $MC_x/AVC_x = 1/b$

Some important relationships between APP, MPP, MC, and AVC become clear. In stage I, MPP is greater than APP and E_p is greater than 1. As a result, MC_x must be less than AVC_x in stage I. The exact proportion is defined by $1/E_p$. In stages II and III, MPP is less than APP, and as a result, E_p is less than 1. Therefore, MC_x must be greater than AVC_x. The exact proportion is again defined by $1/E_p$. At the dividing point between stages I and II, $MPP = APP$ and $E_p = 1$. $1/E_p = 1$ and $MC_x = AC_x$, and at the dividing point between stages II and III, $MPP = 0$, $E_p = 0$, $1/E_p$ is undefined, and MC_x is undefined.

4.6 The Supply Function for the Firm

The profit-maximizing firm will equate marginal cost with marginal revenue. If the firm operates under conditions of pure competition, marginal revenue will be the same as the constant price of the output. If the farmer produces but one output, the marginal cost curve that lies above average variable cost will be the supply curve for the farm. Each point on the marginal cost curve above average variable cost consists of a point of profit maximization if the output sells for the price associated with the point. The supply curve or function for the farm will consist of the series of profit maximizing points under alternative assumptions with respect to marginal revenue or the price of the product.

Consider, for example, the production function

†4.50 $y = ax^b$

The inverse production function is

†4.51 $x = (y/a)^{1/b}$

Variable cost is defined as

†4.52 $VC = vx = v(y/a)^{1/b}$

Marginal cost can be found by differentiating equation †4.52 with respect to y

†4.53 $MC = d(vx)/dy = (1/b)vy^{(1/b)!1}a^{!1/b}$

†4.54 $MC = (1/b)vy^{(1!b)/b}a^{!1/b}$

Equating marginal cost with marginal revenue or the price (p) of the product yields

†4.55 $p = (1/b)vy^{(1!b)/b}a^{!1/b}$

$MR = MC$

Solving equation †4.55 for y yields the supply function for the firm

†4.56 $y = (bp)^{b/(1-b)}v^{-b/(1-b)}a^{(1/b)(b/(1-b))}$

The elasticity of supply with respect to the product price is

†4.57 $(dy/dp)(p/y) = b/(1 - b)$

The elasticity of supply is positive when b is less than 1.

The elasticity of supply with respect to the input price is

†4.58 $(dy/dv)(v/y) = -b/(1 - b)$

The elasticity of supply with respect to the input price is negative if b is less than 1.

Average (variable) cost is

†4.59 $AC = vx/y = [v(y/a)^{1/b}]/y = vy^{(1-b)/b}\ a^{-1/b}$

Since marginal cost is

†4.60 $MC = (1/b)vy^{(1-b)/b}a^{-1/b}$

The ratio of marginal to average cost is

†4.61 $MC/AC = 1/b = 1/E_p$

In this example, the marginal and average cost functions must remain in fixed proportion to each other. The proportion is equal to 1 over the elasticity of production for the production function. Figure 4.7 illustrates the aggregate supply function derived for a production function in which b is less than 1, and the product price is set at alternative levels. The supply function is the portion of the marginal cost function above average variable cost. However, in this example marginal cost lies above average variable cost everywhere and is at the fixed ratio to average variable cost of $1/b$.

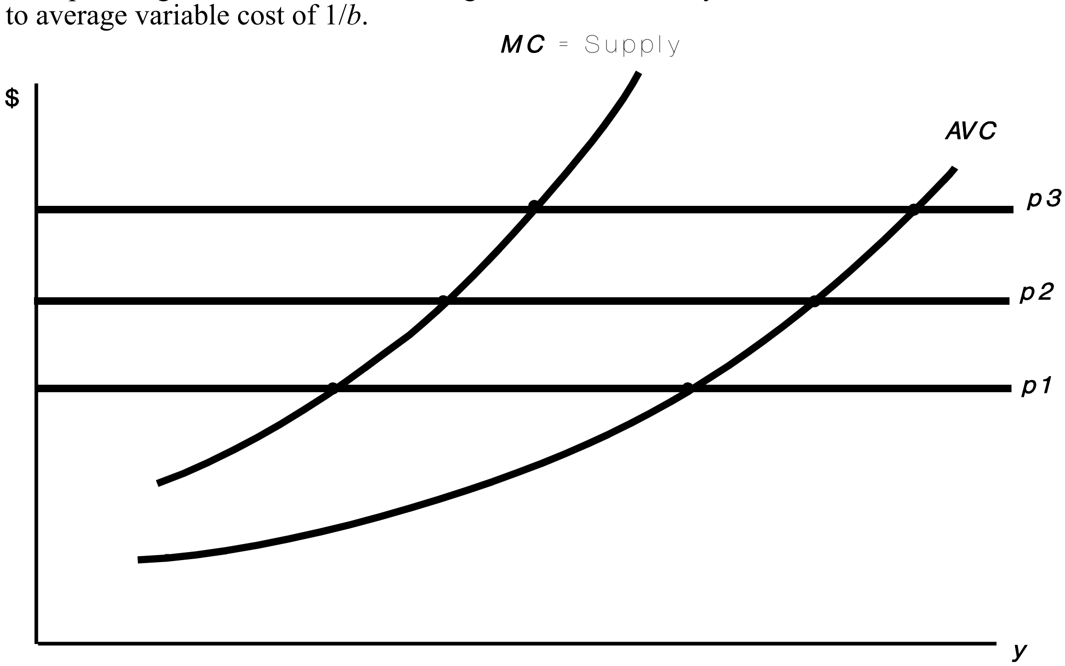

Figure 4.7 Aggregate Supply When the Ratio *MC*/*AC* = 1/*b* and *b* is Less Than 1

4.7 Concluding Comments

Profit maximizing conditions for the firm have been derived. Profits are maximum when the level of output chosen is where marginal cost equals marginal revenue. The cost function is the inverse of the production function that underlies it multiplied by the price of the input. A close linkage thus exists between the coefficients of the production function and those of the underlying cost function. The firm's supply curve can be derived from the equilibrium *MC* = *MR* conditions and is represented by the marginal cost curve above average variable cost. Expressions for elasticities of supply with respect to product and input prices can be obtained from the equilibrium conditions.

Problems and Exercises

1. Explain the difference between total value of the product (*TVP*) and total revenue (*TR*).

2. Explain the difference between total cost (*TC*) and total factor cost (*TFC*).

3. Suppose that the price of the input x is \$3. Total fixed costs are \$200. Fill in the blanks.

x (Input)	y (Output)	*TVC*	*TC*	*MC*	*AVC*	*AC*
0	0	)))	)))		)))	)))
				)))		
10	50	)))	)))		)))	)))
				)))		
25	75	)))	)))		)))	)))
				)))		
40	80	)))	)))		)))	)))
				)))		
50	85	)))	)))		)))	)))

4. Suppose that the production function is

$$y = 3x^{0.5}$$

The price of the input is \$3. per unit, and total fixed costs are \$50. Find and graph the functions that represent.

a. *MPP*
b. *APP*
c. *AVC*
d. *AC* (or *ATC*)
e. *MC*

Suppose that the output price is \$5. Find:

f. *AVP*
g. *VMP*
h. *MFC*

5. Using the data contained in Problem 4, find the profit- maximizing level of input use by equating *VMP* and *MFC*.

6. Using the data contained in Problem 4, find the profit-maximizing output level by equating *MR* and *MC*. What is the relationship between the profit- maximizing output level and the profit-maximizing input level?

7. Draw a three-stage production function on a sheet of paper. Now turn the paper so that the input x is on the vertical axis and output y is on the horizontal axis. Now turn the sheet of paper over and hold the sheet of paper up to a light. Look at the production function through the back side. What you see is the cost function that underlies the production function, with costs expressed in physical units of input use rather than dollars. If input prices are constant, the vertical axis can be converted to dollars by multiplying the physical units of input by the corresponding input price.

8. Draw a graph of the corresponding total cost correspondence when fixed costs are zero, the input costs \$2 per unit, and the production function is given by

$$y = 0.4x + 0.09x^2 - 0.003x^3$$

Reference

Viner, Jacob, "Cost Curves and Supply Curves," *Zeitschrift fur Nationalokonomie* III (1931) pp. 23-46. Also in American Economics Association, *Readings in Price Theory*, K. E. Boulding and G. J. Stigler eds. Homewood, Ill.: Richard D. Irwin, 1952.