5

Production with Two Inputs

This chapter introduces the basics of the technical relationships underlying the factor-factor model, in which two inputs are used in the production of a single output. The concept of an isoquant is developed from a simple table containing data similar to that which might be available in a fertilizer response trial. The slope of the isoquant is defined as the marginal rate of substitution. Isoquants with varying shapes and slopes are illustrated. The shape of an isoquant is closely linked to the characteristics of the production function that transforms the two inputs into the output. The linkages between the marginal rate of substitution and the marginal products of each input are derived.

Key terms and definitions:

Isoquant
Marginal Rate of Substitution (*MRS*)
Diminishing Marginal Rate of Substitution
Constant Marginal Rate of Substitution
Increasing Marginal Rate of Substitution
Convex to the Origin
$)x_2/)x_1$
Asymptotic to the Axes
Concentric Rings
Synergistic Effect
Tangency
Infinite Slope
Zero Slope
Ridge Line
Family of Production Functions
Change in Output
Change in Input
Limit
Infinitesimally Small
Partial Derivative
Total Derivative
Total Differential

5.1 Introduction

The discussion in Chapters 2 to 4 centered on the problems faced by a farmer who wishes to determine how much of a single input should be used or how much of a single output should be produced to maximize profits or net returns to the farm. The basic assumption of this chapter is that two inputs, not one inputs, are allowed to vary. As a result, some modifications need to be made in the basic production function. The production function used in Chapters 1 to 4 was

†5.1 $y = f(x).$

Suppose instead that two inputs called x_1 and x_2 are allowed to vary. The resulting production function is

†5.2 $y = f(x_1, x_2)$

if there are no more inputs to the production process. If there are more than two, or n different inputs, the production function might be written as

†5.3 $y = f(x_1, x_2{}^* x_3, ..., x_n)$

The inputs $x_3, ..., x_n$ will be treated as fixed and given, with only the first two inputs allowed to vary.

In the single-input case, each level of input used produced a different level of output, as long as inputs were being used below the level resulting in maximum output. In the two-input case, there may be many different combinations of inputs that produce exactly the same amount of output. Table 5.1 illustrates some hypothetical relationships that might exist between phosphate (P_2O_5) application levels, potash (K_2O) application levels, and corn yields. The nitrogen application rate was assumed to be 180 pounds per acre.

The production function from which these data were generated is

†5.4 $y = f(x_1, x_2{}^* x_3)$

where y = corn yield in bushels per acre

x_1 = potash in pounds per acre
x_2 = phosphate in pounds per acre
x_3 = nitrogen in pounds per acre assumed constant at 180

Notice from Table 5.1 that potash is not very productive without an adequate availability of phosphate, The maximum yield with no phosphate is but 99 bushels per acre and that occurs at comparatively low levels of potash application of 20 to 30 pounds per acre. The production function for potash in the absence of any phosphate is actually decreasing at potash application rates of over 30 pounds per acre. In the absence of phosphate fertilizer, stage III for potash begins quite early.

Table 5.1 Hypothetical Corn Response to Phosphate and Potash Fertilizer

Phosphate (lb/acre)	Potash (lb/acre) 0	10	20	30	40	50	60	70	80
0	96	98	99	99	98	97	95	92	88
10	98	101	103	104	105	104	103	101	99
20	101	104	106	108	109	110	110	109	106
30	103	107	111	114	117	119	120	121	121
40	104	109	113	117	121	123	126	128	129
50	104	111	116	121	125	127	129	131	133
60	103	112	118	123	126	128	130	131	134
70	102	111	117	123	126	127	131	136	135
80	101	108	114	119	119	125	129	131	134

Phosphate in the absence of potash is more productive, but only slightly so. The maximum yield without any potash is 104 bushels per acre at between 40 and 50 pounds of phosphate. Stage III for phosphate begins at beyond 50 pounds per acre if no potash is applied.

Each of the rows of Table 5.1 represents a production function for potash fertilizer with the assumption that the level of phosphate applied is fixed at the level given by the application rate, which is the first number of the row. As the level of phosphate is increased, the productivity of the potash increases. The marginal product of an additional 10 pounds of potash is usually larger for rows near the bottom of the table than for rows near the top of the table. Moreover, production functions for potash with the larger quantities of phosphate typically achieve their maximum at higher levels of potash use.

Each of the columns of Table 5.1 represents a production function for phosphate fertilizer with the assumption that the level of potash remains constant as defined by the first number in the column. Again the same phenomenon is present. The productivity of phosphate is usually improved with the increased use of potash, and as the assumed fixed level of potash use increases, the maximum of each function with respect to phosphate occurs at larger levels of phosphate use.

These relationships are based on a basic agronomic or biological characteristic of crops. A crop would not be expected to produce high yields if an ample supply of all nutrients were not available. To a degree, phosphate can be substituted for potash, or potash for phosphate. In this example, there are several different combinations of phosphate and potash that will all produce the same yield.

But if the crops are to grow, some of both nutrients must be present, and the highest yields are obtained when both nutrients are in ample supply. This concept in economics is closely linked to Von Liebig's "Law of the Minimum," which states that plant growth is constrained by the most limiting nutrient.

Notice also that it is possible to use too much of both potash and phosphate. Yields using 70 pounds of each are greater than when 80 pounds of each are used. The law of diminishing returns applies to units of phosphate and potash fertilizer taken together when other inputs are held constant, just as it applies to each individual kind of fertilizer.

Table 5.1 contains data from nine production functions for phosphate, under nine different assumptions with regard to potash use. Table 5.1 also contains data from nine production functions for potash, each obtained from a different assumption with regard to the level of phosphate use.

Due to the biology of crop growth, a synergistic effect is present. This means that the presence of ample amounts of phosphate makes the productivity of potash greater. Ample amounts of potash makes the productivity of phosphate greater. The two fertilizers, taken together, result in productivity gains in terms of increased yields greater than would be expected by looking at yields resulting from the application of only one type of fertilizer.

This effect is not limited to crop production. The same phenomenon may be observed if data were collected on the use of the inputs grain (concentrate) and forage used in the production of milk. A cow that is fed all grain and no forage would not be a good milk producer. Similarly, a cow fed all forage and no grain would not produce much milk. Greatest milk production would be achieved with a ration containing a combination of grain and forage.

Each possible ration represents a particular combination or mix of inputs grain and forage. Some of these rations would be better than others in that they would produce more milk. The particular ration chosen by the farmer would depend not only on the amount of milk produced, but also on the relative prices of grain and forage. These ideas are fully developed in Chapter 7.

Figure 5.1 illustrates the production surface arising from the use of phosphate and potash. The x_1 and x_2 axes form a grid (series of agronomic test plots) with the vertical axis measuring corn yield response to the two fertilizers. The largest corn yields are produced from input combinations that include both potash and phosphate.

Data for yet another production function are contained in Table 5.1. From Table 5.1 it is possible to determine what will happen to corn yields if fertilizer application rates for potash and phosphate are increased by the same proportion. Suppose that 1 unit of fertilizer were to consist of 1 pound of phosphate and 1 pound of potash and that this proportion did not change. A table was constructed using numbers found on the diagonal of Table 5.1. These data points are illustrated on the production surface in Figure 5.1.

These data appear to be very similar to the data in the earlier chapters for single input production functions, and they are. The only difference here is that two types of fertilizer are assumed be used in fixed proportion to each other. Under this assumption, the amount of fertilizer needed to maximize profits could be found in a manner similar to that used in earlier chapters, but there is uncertainty as to whether or not the 1:1 ratio in the use of phosphate and potash is the correct ratio.

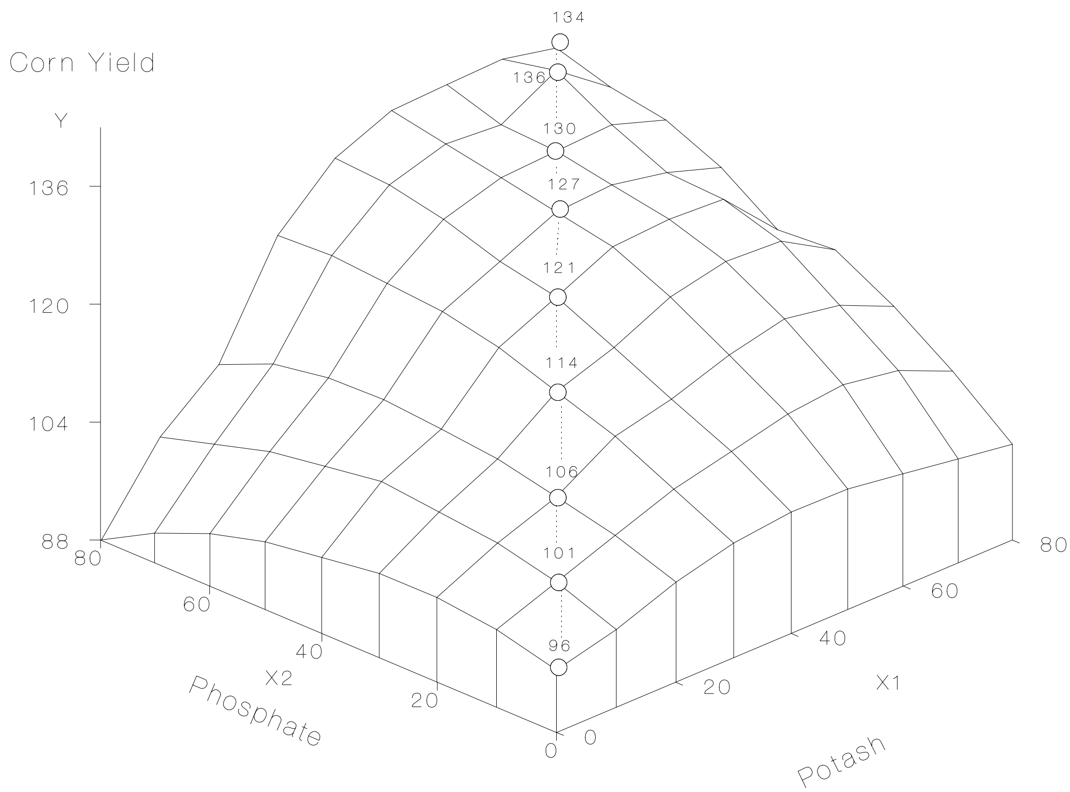


Figure 5.1 Production Response Surface Based on Data Contained in Table 5.1

Table 5.2 Corn Yield Response to 1:1 Proportionate Changes in Phosphate and Potash

Units of Fertilizer (1 Unit = 1 lb of Phosphate and 1 lb of Potash)	Corn Yield (bu/acre)
0	96
10	101
20	106
30	114
40	121
50	127
60	130
70	136
80	134

What would happen, for example, if phosphate were very expensive and potash were very cheap? Perhaps the 1:1 ratio should be changed to 1 unit of phosphate and 2 units of potash to represent a unit of fertilizer. Data for a production function with a 1:2 ratio could also be derived in part from Table 5.1. These data are presented in Table 5.3.

Table 5.3 Corn Yield Response to 1 : 2 Proportionate Changes in Phosphate and Potash

Units of Fertilizer (1 unit = 1 lb phosphate and 2 lb. potash)	Corn Yield (bu/acre)
10 !! 20	103
20 !! 40	109
30 !! 60	120
40 !! 80	129

Much of the next several chapters is devoted to the basic principles used for determining the combination of two inputs (such as phosphate and potash fertilizer) that represents maximum profit for the producer. Here the proper proportions are closely linked to the relative prices for the two types of fertilizer.

5.2 An Isoquant and the Marginal Rate of Substitution

Many combinations of phosphate and potash all result in exactly the same level of corn production. Despite the fact that Table 5.1 includes only discrete values, a bit of interpolation will result in additional combinations that produce the same corn yield. Take, for example, a corn yield of 121 bushels per acre (Table 5.1). This yield can be produced with the following input combinations:30 pounds of phosphate and 70 pounds of potash; 30 pounds of phosphate and 80 pounds of potash; 40 pounds of phosphate and 40 pounds of potash; and 50 pounds of phosphate and 30 pounds of potash.

Moreover, there are many more points that might also achieve approximately 121 bushels per acre-60 pounds of phosphate and approximately 27 pounds of potash; 70 pounds of phosphate and approximately 27 pounds of potash; and 80 pounds of phosphate and approximately 43 pounds of potash to name a few. All these combinations share a common characteristic in that they produce the same yield.

A line can be drawn that connects all points on Table 5.1 representing the same yield. This line is called an *isoquant*. The prefix *iso* comes from the Greek *isos* meaning equal. *Quant* is short for quantity. An isoquant is literally a line representing equal quantities. Every point on the line represents the same yield or output level, but each point on the line also represents a different combination of the two inputs. As one moves along an isoquant, the proportions of the two inputs vary, but output (yield) remains constant.

An isoquant could be drawn for any output or yield that one might choose. If it is possible to draw an isoquant for a yield of 121 bushels per acre it is also possible to draw one for a yield of 125.891 bushels per acre, if the data were sufficiently detailed, or an isoquant could be drawn for a yield of 120.999 bushels per acre, or any other plausible yield.

If isoquants are drawn on graph paper, the graph is usually drawn with the origin (0y, 0x) in the lower left-hand corner. The isoquants are therefore bowed toward the origin of the graph.

Figure 5.2 illustrates the isoquants based on the data contained in Table 5.1. These are the "contour lines" for the production surface illustrated in Figure 5.1. Notice that the isoquants are convex to the lower left hand corner, or origin, of Figure 5.2.

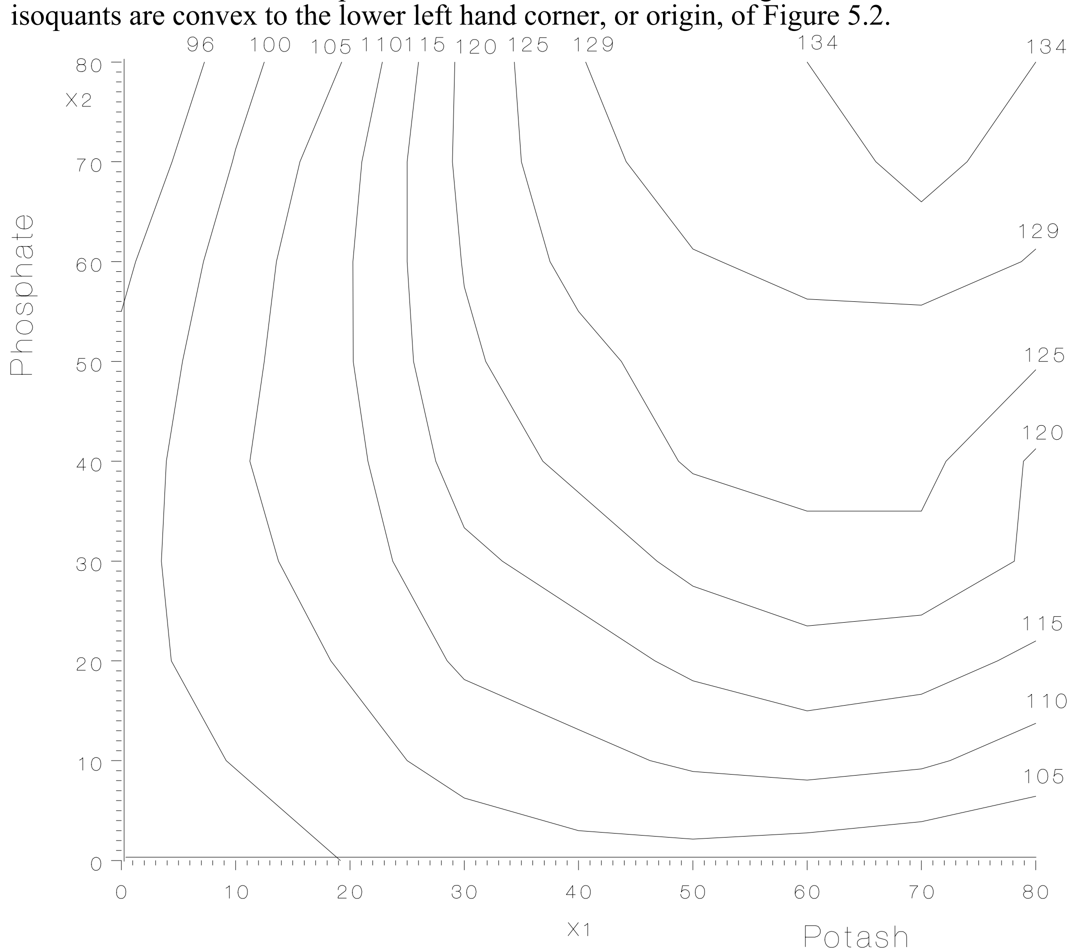


Figure 5.2 Isoquants for the Production Surface in Figure 5.1 Based on Data Contained in Table 5.1

The slope of an isoquant is referred to by some economists as the *marginal rate of substitution* (*MRS*).[1] Other authors refer to it as the rate of technical substitution (RTS) or the marginal rate of technical substitution (*MRTS*). This text uses the terminology *MRS*.

The *MRS* is a measurement of how well one input substitutes for another as one moves along a given isoquant. Suppose that the horizontal axis is labeled x_1, and the vertical axis is labeled x_2. The terminology $MRS_{x_1x_2}$ is used to describe the slope of the isoquant assuming that input x_1 is increasing and x_2 is decreasing. In this example, x_1 is the replacing input and x_2 is the input being replaced, moving down and to the right along the isoquant.

Figure 5.3 illustrates an isoquant exhibiting a diminishing marginal rate of substitution. As one moves farther and farther downward and to the right along the isoquant representing constant output, each incremental unit of x_1 ()x_1) replaces less and less x_2 ()x_2). The diminishing marginal rate of substitution between inputs accounts for the usual shape of an isoquant bowed inward, or convex to the origin. The shape is also linked to the synergistic effect of inputs used in combination with each other. An input is normally more productive when used with ample quantities of other inputs.

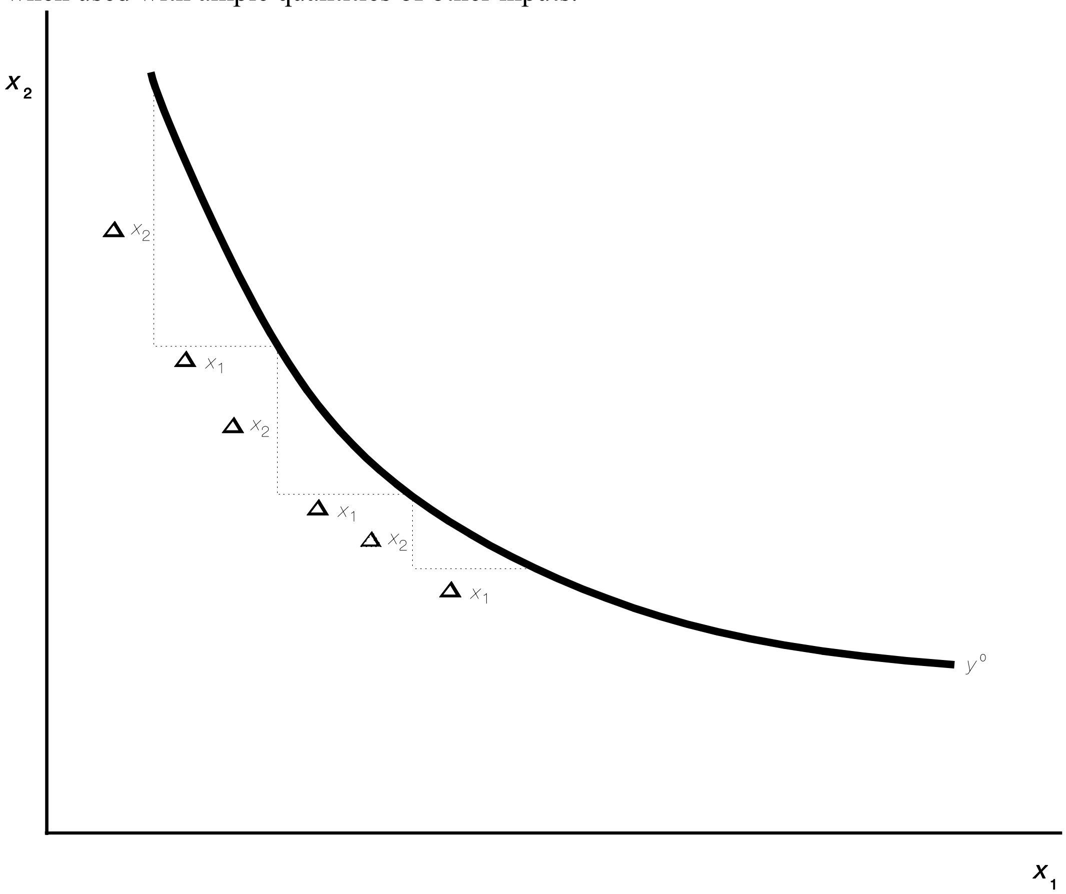


Figure 5.3 Illustration of Diminishing $MRS_{x_1x_2}$

The *MRS* might also measure the inverse slope of the isoquant. Suppose that the use of x_2 is being increased, while the use of x_1 is decreased. The terminology $MRS_{x_2x_1}$ is used to describe the inverse slope of the isoquant. In this example, x_2 is the replacing input, and x_1 is the input being replaced, as one moves up and to the left along the isoquant. The $MRS_{x_2x_1}$ is equal to $1/MRS_{x_1x_2}$.

The slope of an isoquant can also be defined as $) x_2/) x_1$.

Then[2]

†5.5 $$MRS_{x_1x_2} =) x_2/) x_1$$

and

†5.6 $$MRS_{x_2x_1} =) x_1/) x_2 = 1/MRS_{x_1x_2}$$

Isoquants are usually downward sloping, but not always. If the marginal product of both inputs is positive, isoquants will be downward sloping. It is possible for isoquants to slope upward if the marginal product of one of the inputs is negative.

Isoquants are usually bowed inward, convex to the origin, or exhibit diminishing marginal rates of substitution, but not always. The diminishing marginal rate of substitution is normally a direct result of the diminishing marginal product of each input. There are some instances, however, in which the *MPP* for both inputs can be increasing and yet the isoquant remains convex to the origin (see specific cases in Chapter 10).

Figure 5.4 illustrates the isoquants for a three-dimensional production surface derived from a polynomial production function that produces a three-dimensional surface illustrating all three stages of production, the two-input analog to the neoclassical production function employed in Chapter 2. To illustrate, horizontal cuts are made at varying output l

evels. In panel A, the entire three-dimensional production surface is illustrated. Panels C, D, E and F represent cuts at successively lower output levels. Note that in panel E, the isoquant is concave, rather than convex to the origin. Panel F illustrates an example isoquant beneath the production surface.

Figure 5.5 illustrates some possible patterns for isoquant maps and their corresponding production surfaces. Diagrams A and B illustrate isoquants as a series of concentric rings. The center of the series of rings corresponds to the input combination that results in maximum output or product. In Table 5.1, this would correspond with an input combination of 70 pounds of phosphate and 70 pounds of potash, for a yield of 136 bushels per acre. This pattern results when output is actually reduced because too much of both inputs have been used.

Diagrams C and D illustrate another common isoquant map and its corresponding production surface. The isoquants are not rings; rather they approach both axes but never reach them. These isoquants are called asymptotic to the x_1 and x_2 axes, since they approach but do not reach the axes. A diminishing marginal rate of substitution exists everywhere on these isoquants. These isoquants appear to be very similar to the average fixed-cost curve discussed in Chapter 4. However, depending on the relative productivity of the two inputs, these isoquants might be positioned nearer to or farther from one of the two axes. In this example, more of either input, or both inputs taken in combination, will always increase output. There are no maxima for the underlying production functions.

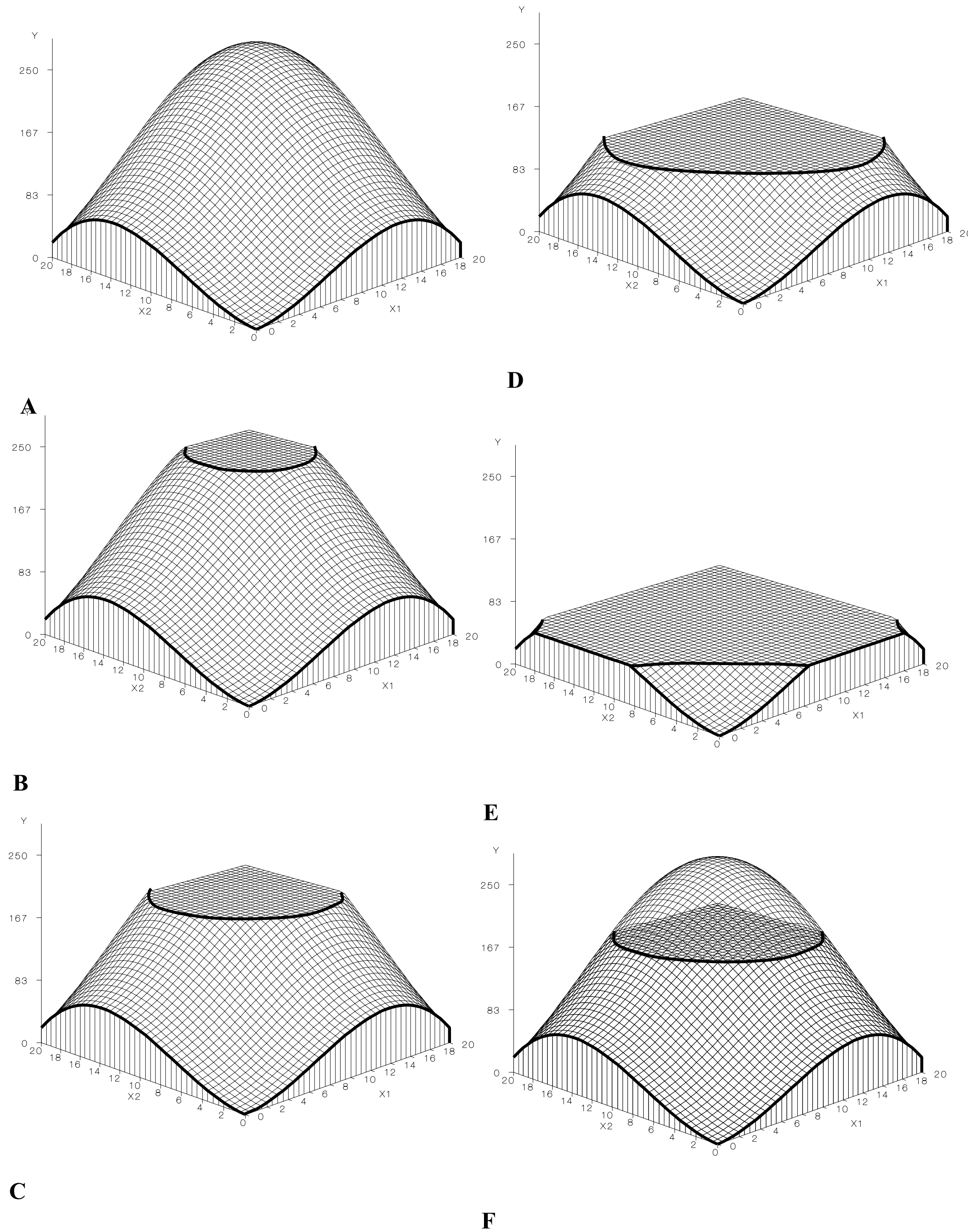
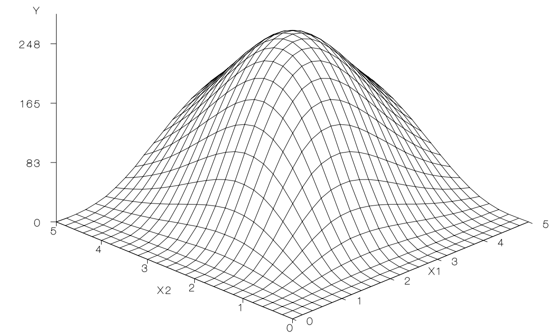
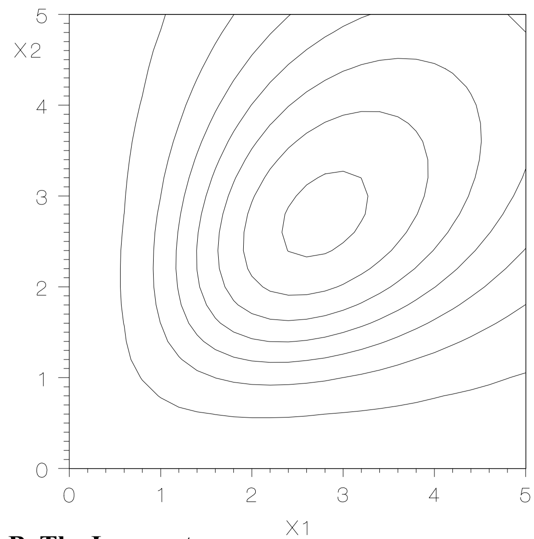


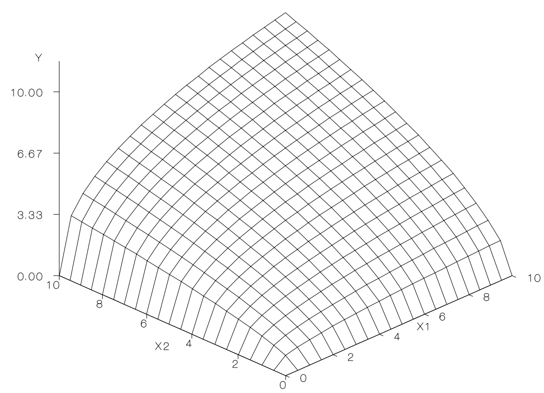
Figure 5.4 Isoquants and a Production Surface



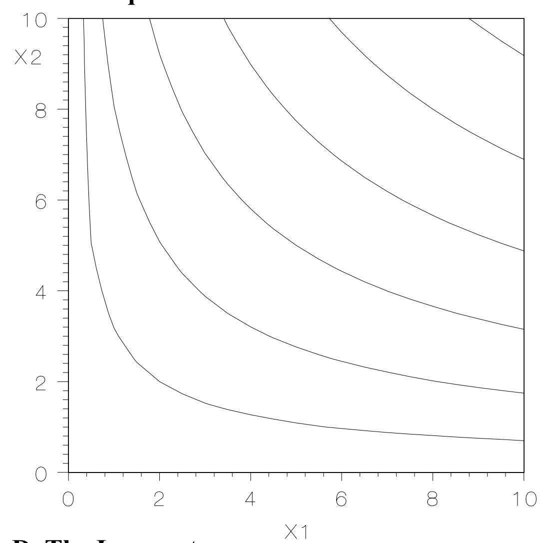
A The Production Surface



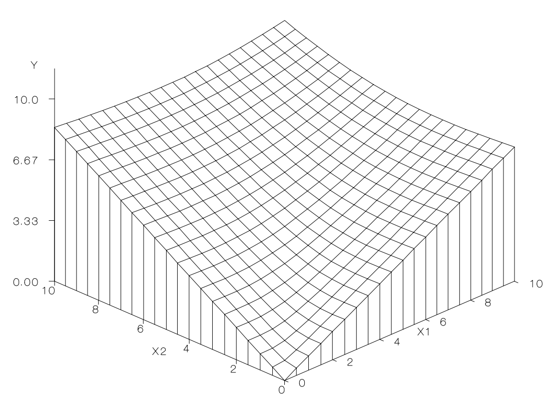
B The Isoquants



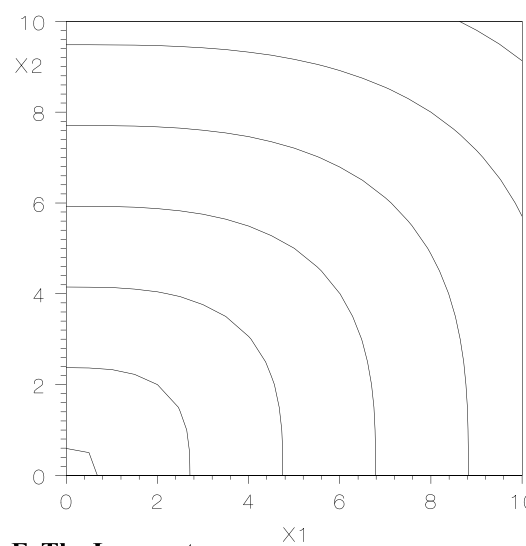
C The Production Surface



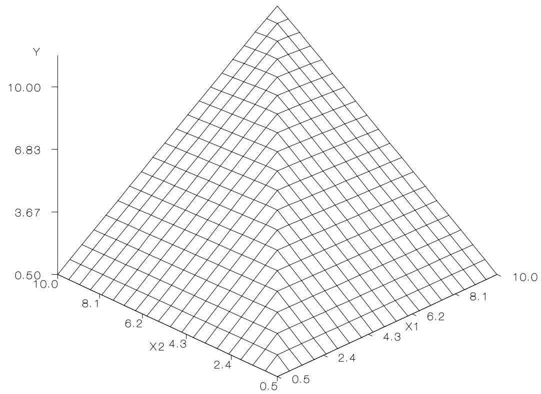
D The Isoquants



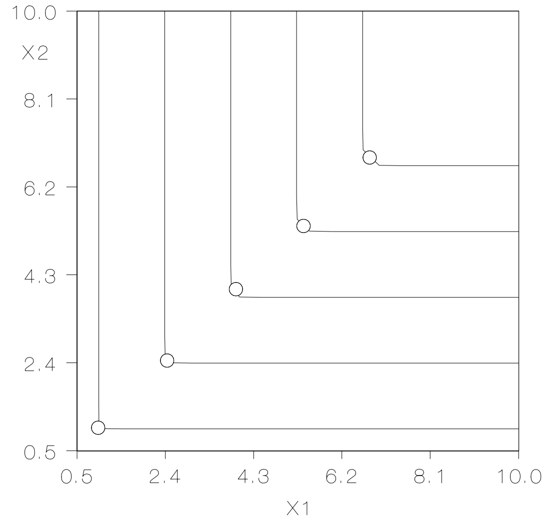
E The Production Surface



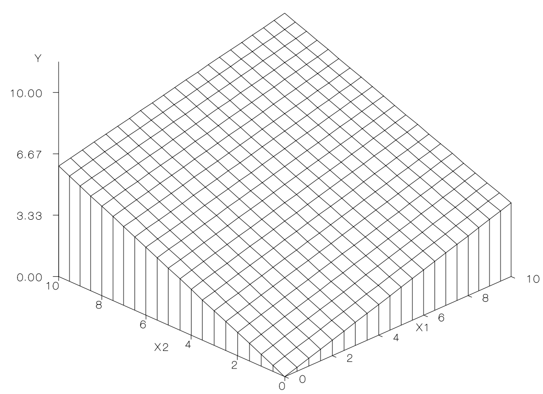
F The Isoquants



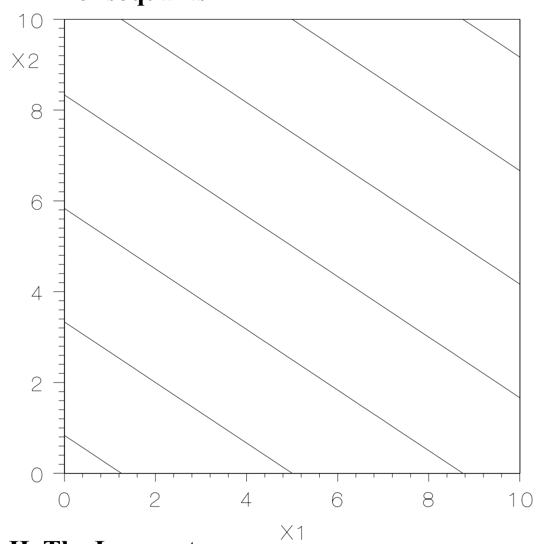
K The Production Surface



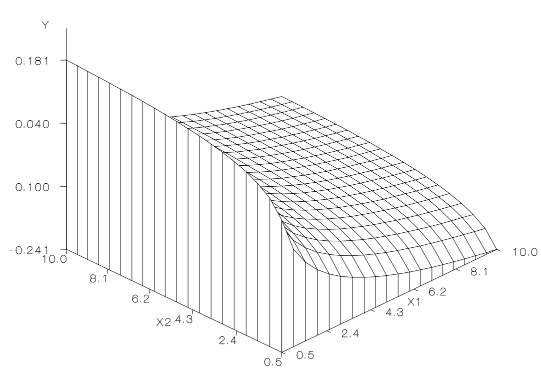
L The Isoquants



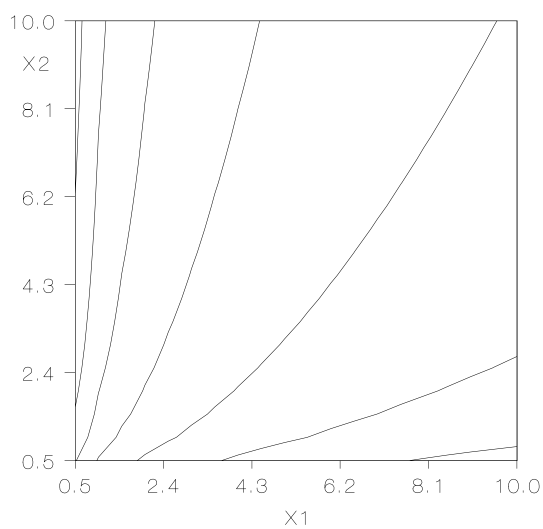
G The Production Surface



H The Isoquants



I The Production Surface



J The Isoquants

Figure 5.5 Some Possible Production Surfaces and Isoquant Maps

Another possibility is for a concave surface with isoquants bowed outward (concave to the origin (diagrams E and F). This pattern represents an increasing rather than diminishing marginal rate of substitution between input pairs. As the use of x_1 increases and the use of x_2 decreases along the isoquant, less and less additional x_1 is required to replace units of x_2 and maintain output. This shape is not very likely, because the pattern would suggest that the two inputs used in combination results in a decrease in relative productivity rather than the synergistic increase that was discussed earlier.

It is possible for isoquants to have a constant slope (diagrams G and H). The corresponding production surface is a hyperplane. In this instance, one input or factor of production substitutes for the other in a fixed proportion. Here, there is a constant, not a diminishing marginal rate of substitution. For example, if inputs substituted for each other in a fixed proportion of 1 unit x_1 to 2 units of x_2, the following input combinations would all result in exactly the same output ! $4x_1$, 0 x_2; $3x_1$, $2x_2$; $2x_1$, $4x_2$; $1x_1$, $6x_2$; $0x_1$, $8x_2$.

It is also possible for isoquants to have a positive slope (Diagrams I and J). This can occur in a situation where additional amounts of one of the inputs (in this instance, input x_1) *reduces* output. Diagram B also includes some points where the isoquants have a positive slope.

Finally, isoquants might be right angles, and the corresponding production surface is shaped like a pyramid (diagrams K and L). This can occur when two inputs must be used in fixed proportion with each other. The classic example here is tractors and tractor drivers. A tractor without a driver produces no output. A driver without a tractor produces no output. These inputs must be used in a constant fixed proportion to each other one tractor driver to one tractor.

5.3 Isoquants and Ridge Lines

Two families of production functions underlie every isoquant map. Figure 5.6 illustrates this relationship. Assume x_2 to be fixed at some predetermined level x_2^*. A horizontal line is drawn from x_2^* across the diagram. A production function for x_1 holding x_2 constant at x_2^* can then be drawn by putting x_1 on the horizontal axis, and noting the output obtained from the intersection of the line drawn at x_2^* with each isoquant.

Now choose another level of x_2. Call this level x_2^*. The process can be repeated over and over again for any level of x_2. Each alternative fixed level for x_2 generates a new production function for x_1 assuming that x_2 is held constant at the predetermined level.

Moreover, the same process can be repeated by holding x_1 constant and tracing out the production functions for x_2. Every time x_1 changes, a new production function is obtained for x_2. As one moves from one production function for x_2 to another, different quantities of output from x_2 are produced, despite the fact that neither the quality or quantity of x_2 has changed. This is because the varying assumptions about the quantity of x_1 either enhance or reduce the productivity of x_2. Another way of saying this is that the marginal productivity (or *MPP*) of x_2 is not independent of the assumption that was made about the availability of x_1, and the *MPP* of x_1 is not independent of the assumption that is made about the availability of x_2.

Now suppose that a level for x_2 is chosen of x_2^* that is just tangent to one of the isoquants. The point of tangency between the line drawn at x_2^* and the isoquant will represent the maximum possible output that can be produced from x_1 holding x_2 constant at x_2^*. The production function derived by holding x_2 constant at x_2^* will achieve its maximum at the point of tangency between the isoquant and the horizontal line drawn at x_2^*. The point of tangency is the point of zero slope on the isoquant and marks the dividing point between stages II and III for the production function

†5.7 $y = f(x_1 {}^* x_2 = x_2^*)$

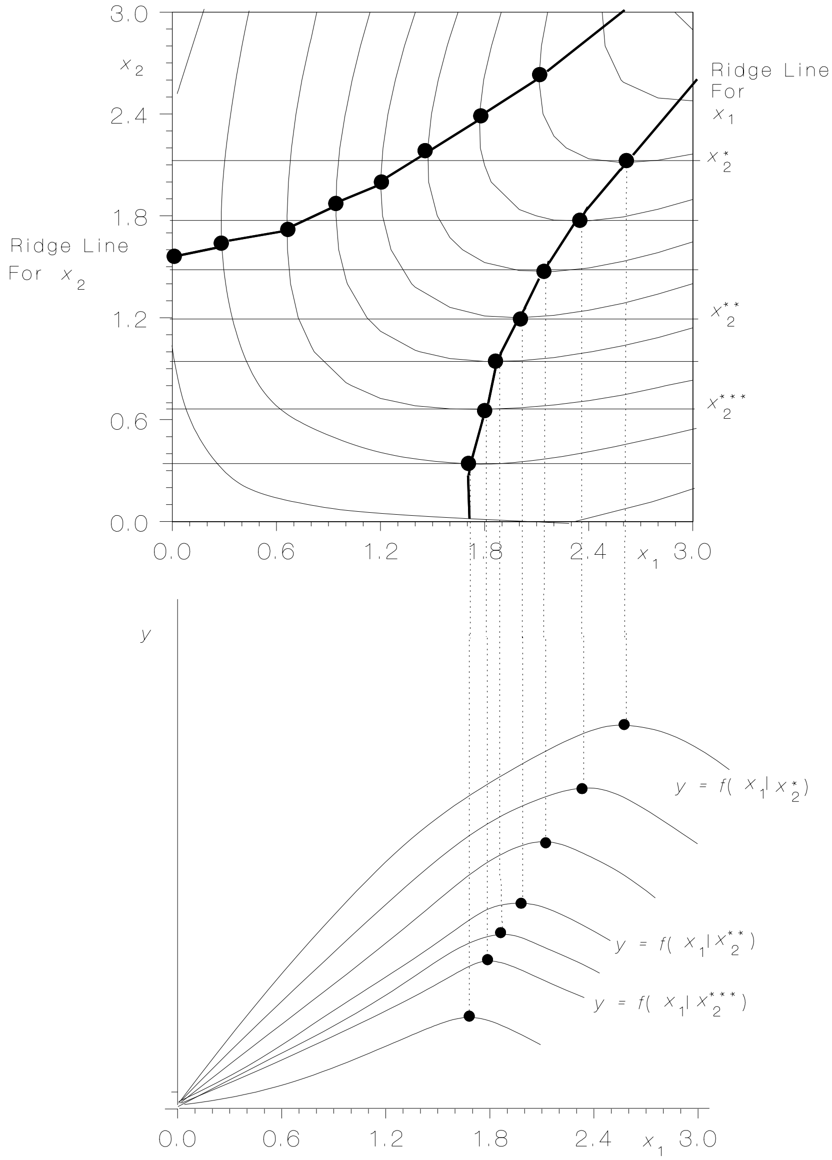


Figure 5.6 Ridge Lines and a Family of Production Functions for Input x_1

This process could be repeated over and over again by selecting alternative values for x_2 and drawing a horizontal line at the selected level for x_2. Each isoquant represents a different output level, just as each horizontal line represents a different assumption about the magnitude of x_2. An infinite number of isoquants could be drawn, each representing a slightly different output level. An infinite number of horizontal lines could be drawn across the isoquant map, each representing a slightly different assumption about the value for x_2. For each horizontal line, there would be a point of tangency on one (and only one!) of the isoquants. This point of tangency is a point of zero slope on the isoquant. Each isoquant would have a corresponding horizontal line tangent to it. The point of tangency represents the maximum for the underlying production function for x_1 under the predetermined assumption with regard to the fixed level of x_2.

The choice of the input to be labeled x_1 and x_2 is quite arbitrary. However, if x_2 remains on the vertical axis, the same process could be repeated by drawing vertical lines from the value chosen on the x_1 axis (the assumption with respect to the value for x_1) and finding the points of tangency between the vertical line and its corresponding isoquant. In this case however, the point of tangency will occur at the point where the isoquant assumes an infinite slope. Each point of tangency marks the division between stages II and III for the underlying production function for x_2 with x_1 set at some predetermined level x_1^*. The production function is

†5.8 $y = f(x_2;x_1{=}x_1^*)$

A line could be drawn that connects all points of zero slope on the isoquant map. This line is called a *ridge line* and marks the division between stages II and III for input x_1, under varying assumptions with regard to the quantity of x_2 that is used. This line is designated as ridge line 1 for x_1.

A second line could be drawn that connects all points of infinite slope on the isoquant map. This is also a ridge line, and marks the division between stages II and III for input x_2, under varying assumptions with regard to the quantity of x_1 that is used. This might be designated as ridge line 2 for x_2.

The two ridge lines intersect at the single point of maximum output. The neoclassical diagram, drawn from an isoquant map that consists of a series of concentric rings, appears not unlike a football. The ridge lines normally assume a positive slope. This is because the level of x_1 that results in maximum output increases as the assumption with regard to the fixed level for x_2 increases. Moreover, the level of x_2 that results in maximum output increases as the assumption with regard to the fixed level for x_1 is increased. The football appearance is the result of the underlying single-input production functions that assume the neoclassical three-stage appearance.

Notice that ridge line 1 connects points where the *MRS* is zero. Ridge line 2 connects points where the *MRS* is infinite. Finally, note that ridge lines can be drawn for only certain types of isoquant patterns or maps. For a ridge line to be drawn, isoquants must assume either a zero or an infinite slope. Look again at figure 5.5. Ridge lines can be drawn only for isoquants appearing in diagram B. For diagrams D, F, L, H and J , there are no points of zero or infinite slope. This suggests that the ridge lines do not exist. Moreover, this implies that the underlying families of production functions for x_1 and x_2 never achieve their respective maxima. Diagram L presents a unique problem. The right angle isoquants have either a zero or an infinite slope everywhere on either side of the angle. This would imply "thick" ridge lines. In this example, the underlying production functions for each input are but a series of points that represent the respective maximum output at each level of input use.

5.4 *MRS* and Marginal Product

The slope or *MRS* of an isoquant and the underlying productivity of the two families of production functions used to derive an isoquant map are closely intertwined. An algebraic relationship can be derived between the *MRS* and the marginal products of the underlying production functions.

Suppose that one wished to determine the change in output (called Δy) that would result if the use of x_1 were changed by some small amount (called Δx_1) and the use of x_2 were also changed by some small amount (called Δx_2). To determine the resulting change in output (Δy), two pieces of information would be needed. First, the exact magnitude of the changes in the use of each of the inputs x_1 and x_2. It is not possible to determine the change in output by merely summing the respective change in the use of the two inputs. An additional piece of information would also be needed. That information is the rate at which each input can be transformed into output. This rate is the marginal physical product of each input x_1 and x_2 (MPP_{x_1} and MPP_{x_2}).

The total change in output can be expressed as

†5.9 $\Delta y = MPP_{x_1}\,\Delta x_1 + MPP_{x_2}\,\Delta x_2$

The total change in output resulting from a given change in the use of two inputs is the change in each input multiplied by its respective *MPP*.

By definition, an isoquant is a line connecting points of equal output. Output does not change along an isoquant. The only way that output can change is to move on the isoquant map from one isoquant to another. Along any isoquant, Δy is exactly equal to zero. The equation for an isoquant can then be written as

†5.10 $$\Delta y = 0 = MPP_{x_1}\, \Delta x_1 + MPP_{x_2}\, \Delta x_2$$

Equation †5.10 can be rearranged such that

†5.11 $$MPP_{x_1}\, \Delta x_1 + MPP_{x_2}\, \Delta x_2 = 0$$

†5.12 $$MPP_{x_2}\, \Delta x_2 = -MPP_{x_1}\, \Delta x_1$$

Dividing both sides of equation †5.12 by Δx_1 gives us:

†5.13 $$MPP_{x_2}\, \Delta x_2/\Delta x_1 = -MPP_{x_1}$$

Dividing both sides by MPP_{x_2} yields:

†5.14 $$\Delta x_2/\Delta x_1 = -MPP_{x_1}/MPP_{x_2}$$

or[3]

†5.15 $$MRS_{x_1x_2} = -MPP_{x_1}/MPP_{x_2}$$

The marginal rate of substitution between a pair of inputs is equal to the negative ratio of the marginal products. Thus the slope of an isoquant at any point is equal to the negative ratio of the marginal products at that point, and if the marginal products for both inputs are positive at a point, the slope of the isoquant will be negative at that point. The replacing input (in this example, x_1) is the *MPP* on the top of the ratio. The replaced input (in this example, x_2) is the *MPP* on the bottom of the ratio. By again rearranging, we have

†5.16 $$MRS_{x_2x_1} = -MPP_{x_2}/MPP_{x_1}$$

The inverse slope of the isoquant is equal to the negative inverse ratio of the marginal products. Thus the slope (or inverse slope) of an isoquant is totally dependent on the *MPP* of each input.

In Section 5.3, a ridge line was defined as a line that connected points of zero or infinite slope on an isoquant map. Consider first a ridge line that connects points of zero slope on an isoquant map. This implies that $MRS_{x_1x_2} = 0$. But $MRS_{x_1x_2} = -MPP_{x_1}/MPP_{x_2}$. The only way for $MRS_{x_1x_2}$ to equal 0 is for MPP_{x_1} to equal zero. If MPP_{x_1} is zero, then the $TPPx_1$ (assuming a given value for x_2 again of x_2^*) must be maximum, and thus the underlying production function for x_1 holding x_2 constant at x_2^* must be at its maximum.

Now consider a ridge line that connects points of infinite slope on an isoquant map. This implies that $MRS_{x_1x_2}$ is infinite. Again $MRS_{x_1x_2} = -MPP_{x_1}/MPP_{x_2}$. $MRS_{x_1x_2}$ will become more and more negative as MPP_{x_2} comes closer and closer to zero. When MPP_{x_2} is exactly equal to zero, the $MRS_{x_1x_2}$ is actually undefined, since any number divided by a zero is undefined. However, note that when $MPP_{x_2} = 0$, then $MRS_{x_2x_1} = 0$, since MPP_{x_2} appears on the top, not the bottom of the ratio. A ridge line connecting points of infinite slope on an isoquant map connects points of zero inverse slope where the inverse slope is defined as $\Delta x_1/\Delta x_2$.

5.5 Partial and Total Derivatives and the Marginal Rate of Substitution

Consider again the Production function

†5.17 $$y = f(x_1, x_2)$$

For many production functions, the marginal product of x_1 (MPP_{x_1}) can be obtained only by making an assumption about the level of x_2. Similarly, the marginal product of x_2 cannot be obtained without making an assumption about the level of x_1. The MPP_{x_1} is defined as

†5.18 $$MPP_{x_1} = \partial f/\partial x_1 \mid x_2 = x_2^*$$

The expression $\partial y/\partial x_1$ is the partial derivative of the production function $y = f(x_1, x_2)$, assuming x_2 to be constant at x_2^*. It is the *MPP* function for the member of the family of production functions for x_1, assuming that x_2 is held constant at some predetermined level x_2^*.

Similarly, the MPP_{x_2}, under the assumption that x_1 is fixed at some predetermined level x_1^*, can be obtained from the expression

†5.19 $$MPP_{x_2} = \partial f/\partial x_2 \mid x_1 = x_1^*$$

In both examples the f refers to output or y.

The big difference between dy/dx_1 and $\partial y/\partial x_1$ is that the dy/dx_1 requires that no assumption be made about the quantity of x_2 that is used. dy/dx_1 might be thought of as the total derivative of the production function with respect to x_1, with no assumptions being made about the value of x_2. The expression $\partial y/\partial x_1$ is the partial derivative of the production function, holding x_2 constant at some predetermined level called x_2^*.

A few examples better illustrate these differences. Suppose that the production function is

†5.20 $$y = x_1^{0.5}x_2^{0.5}$$

Then

†5.21 $$MPP_{x_1} = \partial y/\partial x_1 = 0.5x_1^{-0.5}x_2^{0.5}$$

Since differentiation takes place with respect to x_1, x_2 is treated simply as if it were a constant in the differentiation process, and

†5.22 $$MPP_{x_2} = \partial y/\partial x_2 = 0.5x_2^{-0.5}x_1^{0.5}$$

Since differentiation takes place with respect to x_2, x_1 is treated as if it were a constant in the differentiation process.

Note that in this example, each marginal product contains the other input. An assumption needs to be made with respect to the amount of the other input that is used in order to calculate the respective *MPP* for the input under consideration. Again, the *MPP* of x_1 is conditional on the assumed level of use of x_2. The *MPP* of x_2 is conditional on the assumed level of use of x_1.

Now consider a slightly different production function

†5.23 $$y = x_1^{0.5} + x_2^{0.5}$$

In this production function, inputs are additive rather than multiplicative. The corresponding *MPP* for each input is

†5.24 $$MPP_{x_1} = \partial y/\partial x_1 = 0.5x_1^{-0.5}$$

†5.25 $$MPP_{x_2} = \partial y/\partial x_2 = 0.5x_2^{-0.5}$$

For this production function, MPP_{x_1} does not contain x_2, and MPP_{x_2} does not contain x_1. No assumption needs to be made with respect to the level of use of the other input in order to calculate the respective *MPP* for each input. Since this is true, this is an example where

†5.26 $$\partial y/\partial x_1 = dy/dx_1$$

and

†5.27 $$\partial y/\partial x_2 = dy/dx_2$$

The partial and the total derivatives are exactly the same for this particular production function.

Consider again the expression representing the total change in output

†5.28 $$\Delta y = MPP_{x_1}\,\Delta x_1 + MPP_{x_2}\,\Delta x_2$$

A Δ denotes a finite change, and the respective *MPP*'s for x_1 and x_2 are not exact but rather, merely approximations over the finite range.

Suppose that Δx_1 and Δx_2 become smaller and smaller. At the limit, the changes in x_1 and x_2 become infinitesimally small. If the changes in x_1 and x_2 are no longer assumed to be finite, at the limit, equation †5.28 can be rewritten as

†5.29 $$dy = MPP_{x_1}\,dx_1 + MPP_{x_2}\,dx_2$$

or

†5.30 $$dy = \partial y/\partial x_1\,dx_1 + \partial y/\partial x_2\,dx_2.$$

Equation †5.30 is the total differential for the production function $y = f(x_1,x_2)$.

Along an isoquant, there is no change in y, so $dy = 0$. An isoquant by definition connects points representing the exact same level of output. The total differential is equal to zero. The exact $MRS_{x_1x_2}$ at $x_1 = x_1^*$ and $x_2 = x_2^*$ is

†5.31 $$MRS_{x_1x_2} = dx_2/dx_1 = -MPP_{x_1}/MPP_{x_2} = -(\partial y/\partial x_1)/(\partial y/\partial x_2)$$

Similarly, the exact $MRS_{x_2x_1}$ is defined as

†5.32 $$MRS_{x_2x_1} = dx_1/dx_2 = -MPP_{x_2}/MPP_{x_1} = -(\partial y/\partial x_2)/(\partial y/\partial x_1)$$

The total change in the *MPP* for x_1 can be obtained by dividing the total differential of the production function by dx_1. The result is

†5.33 $$dy/dx_1 = \partial y/\partial x_1 + (\partial y/\partial x_2)(dx_2/dx_1)$$

Equation †5.33 is the total derivative of the production function $y = f(x_1, x_2)$. It recognizes specifically that the productivity of x_1 is not independent of the level of x_2 that is used.

The total change in output as a result of a change in the use of x_1 is the sum of two effects. The direct effect ($\partial y/\partial x_1$) measures the direct impact of the change in the use of x_1 on output. The indirect effect measures the impact of a change in the use of x_1 on the use of $x_2(dx_2/dx_1)$, which in turn affects y (through $\partial y/\partial x_2$).

The shape of the isoquant is closely linked to the production functions that underlie it. In fact, if the underlying production functions are known, it is possible to determine with certainty the exact shape of the isoquant and its slope and curvature at any particular point. The marginal rate of substitution, or slope of the isoquant at any particular point, is equal to the negative ratio of the marginal products of each input at that particular point. If the marginal product of each input is positive but declining, the isoquant normally will be bowed inward or convex to the origin.

The curvature of an isoquant can be determined by again differentiating the marginal rate of substitution with respect to x_1.[4] If the sign on the derivative is positive, the isoquant is bowed inward and exhibits a diminishing marginal rate of substitution. It is also possible for isoquants to be bowed inward in certain instances where the marginal product of both inputs is positive but not declining. Examples of this exception are contained in Chapter 10.

Diagrams B to D of Figure 5.2 all represent isoquants that are downward sloping, and hence dx_2/dx_1 is negative in each case. In diagram B, $d(dx_2/dx_1)/dx_1$ is positive, which is consistent with a a diminishing marginal rate of substitution. Diagram C illustrates a case in which $d(dx_2/dx_1)/dx_1$ is negative, resulting in isoquants concave to the origin, while for diagram D, $d(dx_2/dx_1)/dx_1$ is zero, and the isoquants have a constant slope with no diminishing or increasing marginal rates of substitution.

The derivative dx_2/dx_1 is positive in diagram *E* and undefined in diagram F. In diagram A, the isoquants have both positive and negative slopes, and the sign on dx_2/dx_1 depends on the particular point being evaluated.

Thus the concept of an isoquant with a particular marginal rate of substitution at any particular point and the concept of a production function with marginal products for each input are not separate and unrelated. Rather the slope, curvature and other characteristics of an isoquant are uniquely determined by the marginal productivity of each input in the underlying production function.

5.6 Concluding Comments

This chapter has been concerned with the physical and technical relationships underlying production in a setting in which two inputs are used in the production of a single output. An isoquant is a line connecting points of equal output on a graph with the axes represented by the two inputs. The slope of an isoquant is referred to as a marginal rate of substitution (*MRS*). The *MRS* indicates the extent to which one input substitutes for another as one moves from one point to another along an isoquant representing constant output. The marginal rate of substitution is usually diminishing. In other words, when output is maintained at the constant level represented by the isoquant, as units of input x_1 used in the production process are added, each additional unit of x_1 that is added replaces a smaller and smaller quantity of x_2.

A diminishing marginal rate of substitution between two inputs normally occurs if the production function exhibits positive but decreasing marginal product with respect to incremental increases in the use of each input, a condition normally found in stage II of production. Thus the marginal rate of substitution is closely linked to the marginal product functions for the inputs. This chapter has illustrated how the marginal rate of substitution can be calculated if the marginal products for the inputs are known.

Notes

[1] Not all textbooks define the marginal rate of substitution as the slope of the isoquant. A number of economics texts define the marginal rate of substitution as the *negative* of the slope of the isoquant. That is, $MRS_{x_1x_2} = -\Delta x_2/\Delta x_1$ (or $-dx_2/dx_1$). Following this definition, a downward-sloping isoquant exibits a positive marginal rate of substitution.

[2] or $-\Delta x_2/\Delta x_1$.

[3] If the marginal rate of substitution is defined as the negative of the slope of the isoquant, it is equal to the ratio of the marginal products, not the negative ratio of the marginal products.

[4] Let the Marginal rate of Substitution (*MRS*) of x_1 for x_2 be defined as dx_2/dx_1. Then the total differential of the *MRS* is defined as

$$dMRS = (\partial MRS/\partial x_1)dx1 + (\partial MRS/\partial x_2)dx_2$$

The total derivative with respect to x_1 is

$$dMRS/dx_1 = (\partial MRS/\partial x_1) + (\partial MRS/\partial x_2)(dx_2/dx_1)$$

or

$$dMRS/dx_1 = (\partial MRS/\partial x_1) + (\partial MRS/\partial x_2)\cdot MRS$$

As units of x_1 are increased, the total change in the marginal rate of substitution ($dMRS/dx_1$) is the sum of the direct effect of the change in the use of x_1 on the *MRS* $[(\partial MRS/\partial x_1)]$ plus the indirect effect [$(\partial MRS/\partial x_2)MRS$]. The indirect effect occurs because if output is to remain constant on the isoquant, an increase in x_1 must be compensated with a decrease in x_2.

Problems and Exercises

1. The following combinations of x_1 and x_2 all produce 100 bushels of corn. Calculate the $MRS_{x_1x_2}$ and the $MRS_{x_2x_1}$ at each midpoint.

Combination	Units of x_1	Units of x_2	$MRS_{x_1x_2}$	$MRS_{x_2x_1}$
A	10	1		
B	5	2		
C	3	3		
D	2	4		
E	1.5	5		

2. For the production function

$$y = 3x_1 + 2x_2$$

find

a. The MPP of x_1.
b. The MPP of x_2.
c. The marginal rate of substitution of x_1 for x_2.

3. Draw the isoquants for the production function given in Problem 1.

4. Find those items listed in Problem 2 for a production function given by

$$y = ax_1 + bx_2$$

where a and b are any constants. Is it possible for such a production function to produce isoquants with a positive slope? Explain.

5. Suppose that the production function is given by

$$y = x_1^{0.5}x_2^{0.333}$$

find

a. The MPP of x_1.
b. The MPP of x_2.
c. The Marginal rate of substitution of x_1 for x_2.

d. Draw the isoquants for this production function. Do they lie closer to the x_1 or the x_2 axis? Explain. What relationship does the position of the isoquants have relative to the productivity of each input?

6. Suppose that the production function is instead

$$y = 2x_1^{0.5}x_2^{0.333}$$

find

a. The *MPP* of x_1.
b. The *MPP* of x_2.
c. The Marginal rate of substitution of x_1 for x_2.

d. What happens to the position of the isoquants relative to those drawn for Problem 5? Compare your findings with those found for problem 5.