6

Maximization in the Two-Input Case

This chapter develops the fundamental mathematics for the maximization or minimization of a function with two or more inputs and a single output. The necessary and sufficient conditions for the maximization or minimization of a function are derived in detail. Illustrations are used to show why certain conditions are required if a function is to be maximized or minimized. Examples of functions that fulfill and violate the rules are illustrated. An application of the rules is made using the yield maximization problem.

Key terms and definitions:

Maximization
Minimization
First-Order Conditions
Second-Order Conditions
Young's Theorem
Necessary Conditions
Sufficient Conditions
Matrix
Matrix of Partial Derivatives
Principal Minors
Local Maximum
Global Maximum
Saddle Point
Determinant
Critical Value
Unconstrained Maximization and Minimization
Constrained Maximization and Minimization

6.1 An Introduction to Maximization

An isoquant map might be thought of as a contour map of a hill. The height of the hill at any point is measured by the amount of output that is produced. An *isoquant* connects all points producing the same quantity of output, or having the same elevation on the hill. In general, isoquants consist of concentric rings, just as there are points on all sides of the hill that have the same elevation. Similarly, there are many different combinations of two inputs that would all produce exactly the same amount of output.

An infinite number of isoquants can be drawn. Each isoquant represents a slightly different output level or elevation on the hill. Isoquants never intersect or cross each other, for this would imply that the same combination of two inputs could produce two different levels of output. The quantity of output produced from each combination of the two inputs is unique. If one is standing at a particular point on a side of a hill, that particular point has one and only one elevation.

If the isoquants are concentric rings, any isoquant drawn inside another isoquant will always represent a slightly greater output level than the one on the outside (Figure 5.1, diagram A). If the isoquants are not rings, the greatest output is normally associated with the isoquant at the greatest distance from the origin of the graph. No two isoquants can represent exactly the same level of output. Each isoquant by definition represents a slightly different quantity of output from any other isoquant.

If an isoquant map is drawn as a series of concentric rings, these rings become smaller and smaller as one moves toward the center of the diagram. At comparatively low levels of output, the possible combinations of the two inputs x_1 and x_2 suggest a wide range of options: a large quantity of x_2 and a small quantity of x_1: a small quantity of x_2 and a large quantity of x_1, or something in between. At higher levels of output, the isoquant rings become smaller and smaller, suggesting that the range of options becomes more restricted, but there remains an infinite number of possible combinations on a particular isoquant within the restricted range, each representing a slightly different combination of x_1 and x_2.

The concentric rings finally become a single point. This is the global point of maximum output and would be the position where the farm manager would prefer to operate a farm if inputs were free and there were no other restrictions on the use of the inputs. This single point is the point where the two ridge lines intersect. The *MRS* for an isoquant consisting of a single point is undefined, but this point represents the maximum amount of output that can be produced from any combination of the two inputs x_1 and x_2.

If one were standing on the top of a hill, at the very top, the place where one would be standing would be level. Moreover, regardless of the direction that one looked from the top of a hill, the hill would slope downward from its level top. If one were standing on the hilltop, no other point on the hill would slope upward. If it did, one would not be on the top of the hill. Every other point on the hill would be at a somewhat lower elevation.

The top of the highest hill represents the greatest possible elevation, or global maximum. However, hills that are not as high are also level at the top. The tops of these hills represent local, but not global maxima.

Minimum points can be defined similarly. The bottom of a valley is also level. The bottom of the deepest valley represents a global minimum, while the bottom of other valleys not as deep represent local but not global minima. If one were to draw contour lines for a valley, they would be indistinguishable from the contour lines for a hill.

The slope at both the bottom of a valley and at the top of the hill is zero in all directions. It is not possible to distinguish the bottom of a valley from the top of a hill simply by looking at the slope at that point, because the slope for both is zero. Much of the mathematics of maximization and minimization is concerned with the problem of distinguishing bottoms of valleys from tops of hills based on second derivative tests or second order conditions.

6.2 The Maximum of a Function

The problem of finding the combination of inputs x_1 and x_2 that results in the true maximum output from a two-input production function is the mathematical equivalent of finding the top of the hill, or the point on a hill with the greatest elevation. Two conditions need to be checked. First, the point under consideration must be level, or have a zero slope, which is a necessary condition, but level points are found not only at the top of hills but at the bottom of valleys.

The saddle for a horse provides another example and problem for the mathematician. The saddle is level in the middle, but it slopes upward at both ends and downward at both sides. A saddle looks like neither a hill nor a valley, but is a combination of both. So an approach needs to be taken that will separate the true hill from the valley and the saddle point.

Suppose again the general production function

†6.1 $y = f(x_1, x_2)$

The first-order or necessary conditions for the maximization of output are

†6.2 $\partial y/\partial x_1 = 0$, or $f_1 = 0$

and

†6.3 $\partial y/\partial x_2 = 0$ or $f_2 = 0$

Equations †6.2 and †6.3 ensure that the point is level relative to both the x_1 and the x_2 axes.

The second order conditions for the maximization of output require that the partial derivatives be obtained from the first order conditions. There are four possible second derivatives obtained by differentiating the first equation with respect to x_1 and then with respect to x_2. The second equation can also be differentiated with respect to both x_1 and x_2.

These four second partial derivatives are

†6.4 $\partial(\partial y/\partial x_1)/\partial x_1 = \partial^2 y/\partial x_1^2 = f_{11}$

†6.5 $\partial(\partial y/\partial x_1)/\partial x_2 = \partial^2 y/\partial x_1 \partial x_2 = f_{12}$

†6.6 $\partial(\partial y/\partial x_2)/\partial x_1 = \partial^2 y/\partial x_2 \partial x_1 = f_{21}$

†6.7 $\partial(\partial y/\partial x_2)/\partial x_2 = \partial^2 y/\partial x_2^2 = f_{22}$

Young's theorem states that the order of the partial differentiation makes no difference and that $f_{12} = f_{21}$.[1]

The second order conditions for a maximum require that

†6.8 $f_{11} < 0$

and

†6.9 $f_{11}f_{22} > f_{12}f_{21}$.

Since $f_{12}f_{21}$ is non-negative, $f_{11}f_{22}$ must be positive for equation †6.9 to hold, and $f_{11}f_{22}$ can be positive only if f_{22} is also negative. Taken together, these first- and second-order conditions provide the necessary and sufficient conditions for the maximization of a two-input production function that has one maximum.

6.3 Some Illustrative Examples

Some specific examples will further illustrate these points. Suppose that the production function is

†6.10 $y = 10x_1 + 10x_2 - x_1^2 - x_2^2$

The first order or necessary conditions for a maximum are

†6.11 $f_1 = 10 - 2x_1 = 0$

†6.12 $x_1 = 5$

†6.13 $f_2 = 10 - 2x_2 = 0$

†6.14 $x_2 = 5$

The critical values for a function is a point where the slope of the function is equal to zero. The critical values for this function occur at the point where $x_1 = 5$, and $x_2 = 5$. This point could be a maximum, a minimum or a saddle point.

For a maximum, the second order conditions require that

†6.15 $f_{11} < 0$ and $f_{11}f_{22} > f_{12}f_{21}$

For equation †6.10

†6.16 $f_{11} = -2 < 0$

†6.17 $f_{22} = -2$

†6.18 $f_{12} = f_{21} = 0$, since x_2 does not appear in f_1, nor x_1 in f_2.

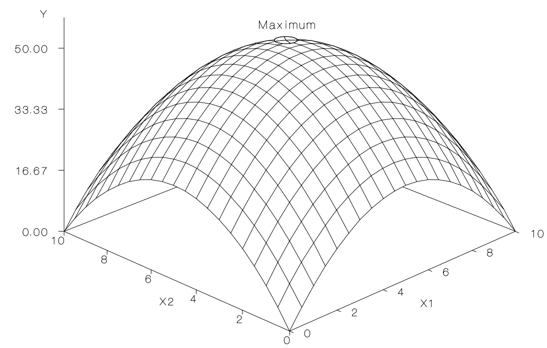
Hence

†6.19 $f_{11}f_{22} - f_{12}f_{21} = 4 > 0$

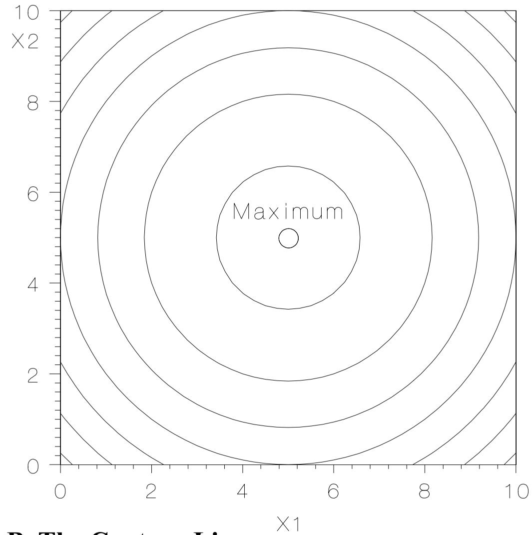
The necessary and sufficient conditions have been met for the maximization of equation †6.10 at $x_1 = 5$, $x_2 = 5$. This function and its contour lines are illustrated in panels A and B of Figure 6.1.

Now consider a production function

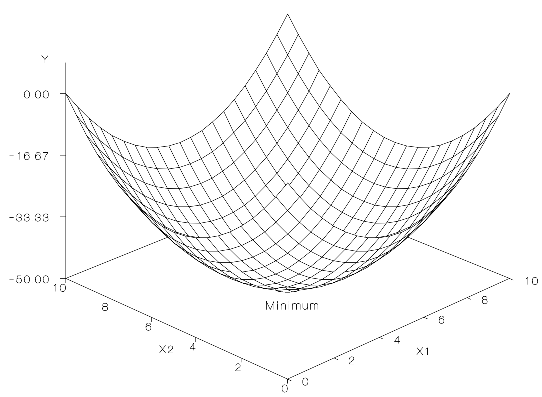
†6.20 $y = -10x_1 - 10x_2 + x_1^2 + x_2^2$



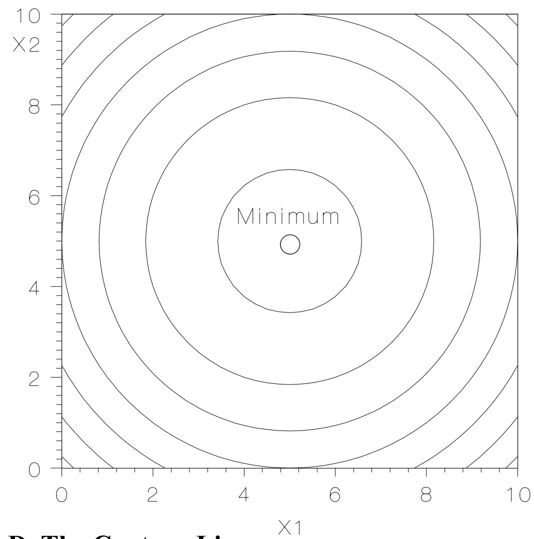
A The Surface



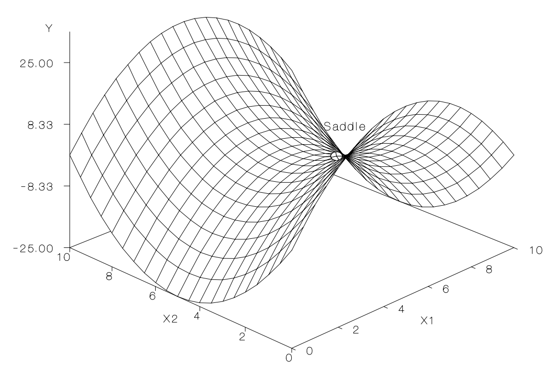
B The Contour Lines



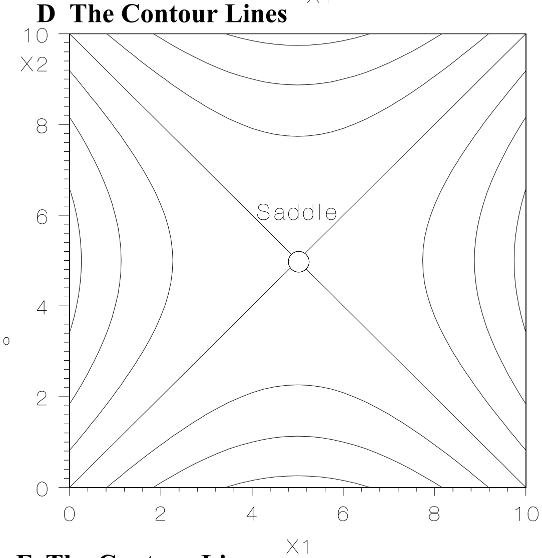
C The Surface



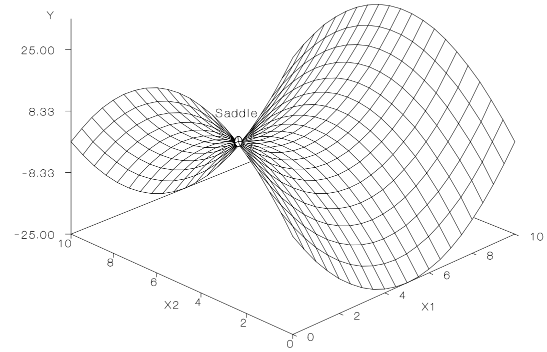
D The Contour Lines



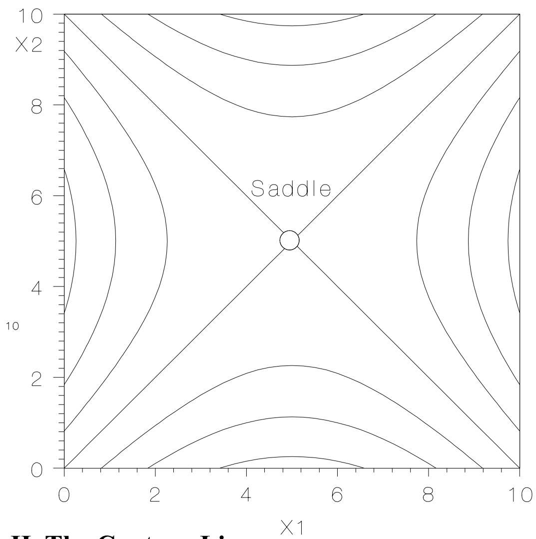
E The Surface



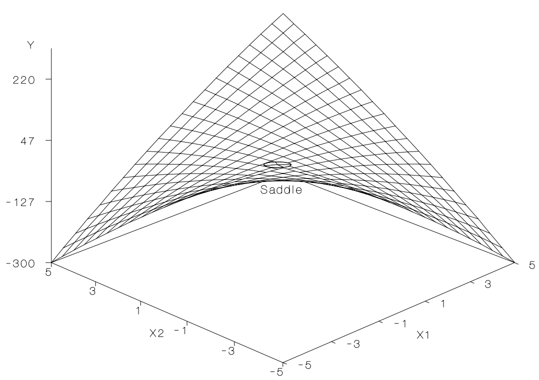
F The Contour Lines



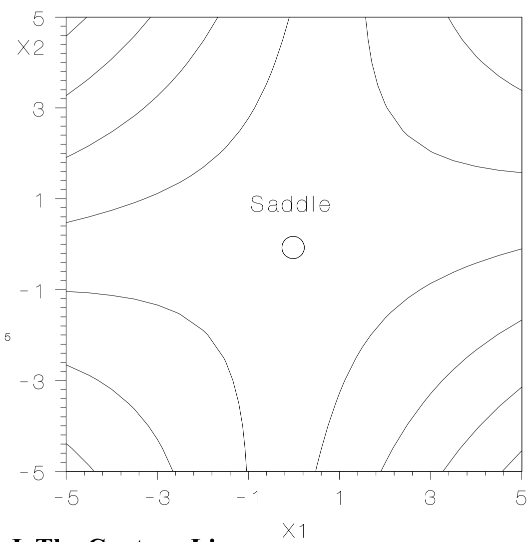
G The Surface



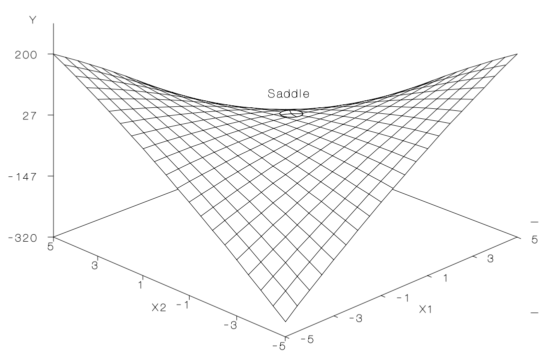
H The Contour Lines



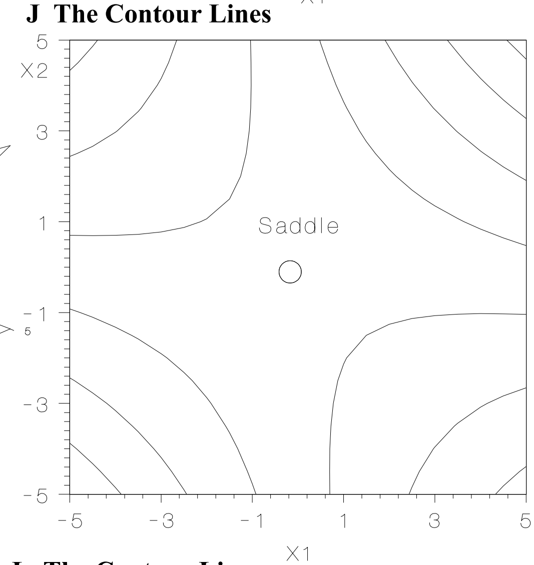
I The Surface



J The Contour Lines



K The Surface



L The Contour Lines

Figure 6.1 Alternative Surfaces and Contours Illustrating Second-Order Conditions

The first-order conditions are

†6.21 $f_1 = -10 + 2x_1 = 0$

†6.22 $x_1 = 5$

†6.23 $f_2 = -10 + 2x_2 = 0$

†6.24 $x_2 = 5$

The second order conditions for a minimum require that

†6.25 $f_{11} > 0$

†6.26 $f_{11} f_{22} > f_{12} f_{21}$

For equation †6.20 the second order conditions are

†6.27 $f_{11} = 2 > 0$

†6.28 $f_{22} = 2$

Moreover

†6.29 $f_{11}f_{22} - f_{12}f_{21} = 4 > 0$

The necessary and sufficient conditions have been met for the minimization of equation †6.20 at $x_1 = 5$, $x_2 = 5$. This function and its contour lines are illustrated in panels C and D of Figure 6.1.

Now consider a function

†6.30 $y = 10x_1 - 10x_2 - x_1^2 + x_2^2$

The first order conditions are

†6.31 $f_1 = 10 - 2x_1 = 0$

†6.32 $x_1 = 5$

†6.33 $f_2 = -10 + 2x_2 = 0$

†6.34 $x_2 = 5$

For equation †6.30 , the second order conditions are

†6.35 $f_{11} = -2 < 0$

†6.36 $f_{22} = 2$

Moreover

†6.37 $f_{11}f_{22} - f_{12}f_{21} = -4 < 0$

The necessary and sufficient conditions have not been met for the minimization or maximization of equation †6.30 at $x_1 = 5$, $x_2 = 5$. This function is the unique saddle point illustrated panels E and F of Figure 6.1 that represents a maximum in the direction parallel to the x_1 axis, but a minimum in the direction parallel to the x_2 axis.

The function

†6.38 $$y = -10x_1 + 10x_2 + x_1^2 - x_2^2$$

results in a very similar saddle point with the axes reversed. That is, a minimum occurs parallel to the x_1 axis, but a maximum occurs parallel to the x_2 axis. The surface of this function is illustrated in panels G and H of Figure 6.1.

Now consider a function

†6.39 $$y = -2x_1 - 2x_2 - x_1^2 - x_2^2 + 10x_1x_2$$

The first order conditions are

†6.40 $$f_1 = -2 - 2x_1 + 10x_2 = 0$$

†6.41 $$f_2 = -2 - 2x_2 + 10x_1 = 0$$

Solving for x_2 in equation †6.41 for f_2 gives us

†6.42 $$-2x_2 = 2 - 10x_1$$

†6.43 $$x_2 = 5x_1 - 1$$

Inserting equation †6.43 x_2 into equation †6.40 for f_1 results in

†6.44 $$x_1 = 0.25$$

Since $x_2 = 5x_1 - 5$, x_2 also equals 0.25.

In this instance the second order conditions are

†6.45 $$f_{11} = -2 < 0$$

†6.46 $$f_{22} = -2 < 0$$

However

†6.47 $$f_{12} = f_{21} = 10$$

Thus

†6.48 $$f_{11}f_{22} - f_{12}f_{21} = 4 - 100 = -96 < 0$$

Although these conditions may at first appear to be sufficient for a maximum at $x_1 = x_2 = 0.25$, the second order conditions have not been fully met. In this example, the product of the direct second partial derivatives $f_{11}f_{22}$ is less than the product of the second cross partial derivatives $f_{12}f_{21}$, and therefore $f_{11}f_{22} - f_{12}f_{21}$ is less than zero. In the earlier examples, the second cross partial derivatives were always zero, since an interaction term such as $10x_1x_2$ did not appear in the original production function.

As a result, another type of saddle point occurs, as illustrated in panels I and J of Figure 6.1, which appears somewhat like a bird with wings outstretched. Like the earlier saddle points, a minimum exists in one direction and a maximum in another direction at a value for x_1 and x_2 of 0.25, but the saddle no longer is parallel to one of the axes, but rather lies along a line running between the two axes. This is the result of the product of the second cross partials being greater than the second direct partials. By changing the function only slightly and making the coefficient 10 on the product of x_1 and x_2 a -10 results in the surface and contour lines illustrated in panels K and L of Figure 16.1. Compare these with panels I and J.

In the preceding examples, care was taken to develop polynomial functions that had potential maxima or minima at levels for x_1 and x_2 at positive but finite amounts. If a true maximum exists, the resultant isoquant map will consist of a series of concentric rings centered on the maximum with ridge lines intersecting at the maximum.

One is sometimes tempted to attempt the same approach for other types of functions. For example, consider a function such as

†6.49 $$y = 10x_1^{0.5}x_2^{0.5}$$

In this instance

†6.50 $$f_1 = 5x_1^{-0.5}x_2^{0.5}$$

And

†6.51 $$f_2 = 5x_1^{0.5}x_2^{-0.5}$$

These first partial derivatives of equation †6.49 could be set equal to zero, but they would each assume a value of zero only at $x_1 = 0$ and $x_2 = 0$. There is no possibility that f_1 and f_2 could be zero for any combination of positive values for x_1 and x_2. Hence the function never achieves a maximum.

6.4 Some Matrix Algebra Principles

Matrix algebra is a useful tool for determining if a function has achieved a maximum or minimum.A *matrix* consists of a series of numbers (also called *values* or *elements*) organized into rows and columns. The matrix

†6.52 $$\begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array}$$

is a square 3 *x* 3 matrix, since it has the same number of rows and columns. For each element, the first subscript indicates its row, the second subscript its column. For example a_{23} refers to the element or value located in the second row and third column.

Every square matrix has a number associated with it called its *determinant*. For a 1 x 1 matrix with only one value or element, its determinant is a_{11}. The determinant of a 2 x 2 matrix is $a_{11}a_{22} - a_{12}a_{21}$. The determinant of a 3 x 3 matrix is $a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{21}a_{32}a_{13}$

$- a_{31}a_{22}a_{13} - a_{11}a_{32}a_{23} - a_{33}a_{21}a_{12}$. Determinants for matrices larger than 3 x 3 are very difficult to calculate, and a computer routine is usually used to calculate them.

The *principal minors* of a matrix are obtained by deleting first all rows and columns of the matrix except the element located in the first row and column (a_{11}) and finding the resultant determinant. In this example, the first principal minor is a_{11}. Next, all rows and columns except the first two rows and columns are deleted, and the determinant for the remaining 2 x 2 matrix is calculated. In this example, the second principal minor is $a_{11}a_{22} - a_{12}a_{21}$. The third principal minor would be obtained by deleting all rows and columns with row or column subscripts larger than 3, and then again finding the resultant determinant.

The second order conditions can better be explained with the aid of matrix algebra. The second direct and cross partial derivatives of a two input production function could form the square 2 x 2 matrix

†6.53
$$\begin{matrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{matrix}$$

The principal minors of equation †6.53 are

†6.54
$$H_1 = f_{11}$$
$$H_2 = f_{11}f_{22} - f_{12}f_{21}$$

Assuming that the first-order conditions have been met, The second-order condition for a maximum requires that the principal minors H_1 and H_2 alternate in sign, starting with a negative sign. In other words, $H_1 < 0$; $H_2 > 0$.

For a minimum, all principal minors must be positive. That is, $H_1, H_2 > 0$.

A saddle point results for either of the remaining conditions

$$H_1 > 0; H_2 < 0$$

or,
$$H_1 < 0; H_2 < 0$$

6.5 A Further Illustration

A further illustration of second-order conditions is obtained from the two input polynomial

†6.55
$$y = 40x_1 - 12x_1^2 + 1.2x_1^3 - 0.035x_1^4 + 40x_2 - 12x_2^2 + 1.2x_2^3 - 0.035x_2^4$$

This function has nine values where the first derivatives are equal to zero. Each of these values, called *critical values*, represents a maximum, a minimum, or a saddle point. Figure 6.2 illustrates the function. Table 6.1 illustrates the corresponding second order conditions. In this example, H_1 is f_{11} and H_2 is $f_{11}f_{22} - f_{12}f_{21}$.

This function differs from the previous functions in that there are several combinations of x_1 and x_2 that generate critical values where the slope of the function is equal to zero. There is but one global maximum for the function, but several local maxima. A global maximum might be thought of as the top of the highest mountain, whereas a local maximum might be considered the top of a nearby hill. There are also numerous saddle points. The second-order conditions can be verified by carefully studying figure 6.2.

6.6 Maximizing a Profit Function with Two Inputs

The usefulness of the criteria for maximizing a function can be further illustrated with an agricultural example using a profit function for corn. Suppose that the production function for corn is given by

†6.56 $$y = f(x_1, x_2)$$

where y = corn yield in bushels per acre
x_1 = pounds of potash applied per acre
x_2 = pounds of phosphate applied per acre

Table 6.1 Critical Values for the Polynomial $y = 40x_1 - 12x_1^2 + 1.2x_1^3 - 0.035x_1^4 + 40x_2 - 12x_2^2 + 1.2x_2^3 - 0.035x_2^4$

	x_1		
x_2	2.54	6.93	16.24
16.24	local maximum $y = 232.3$ $H_1 < 0$ $H_2 > 0$	saddle point $y = 209.5$ $H_1 > 0$ $H_2 < 0$	global maximum $y = 378.8$ $H_1 < 0$ $H_2 > 0$
6.93	saddle point $y = 61.9$ $H_1 < 0$ $H_2 < 0$	local minimum $y = 39.1$ $H_1 > 0$ $H_2 > 0$	saddle point $y = 209.5$ $H_1 < 0$ $H_2 < 0$
2.54	local maximum $y = 84.8$ $H_1 < 0$ $H_2 > 0$	saddle point $y = 61.9$ $H_1 > 0$ $H_2 < 0$	local maximum $y = 232.3$ $H_1 < 0$ $H_2 > 0$

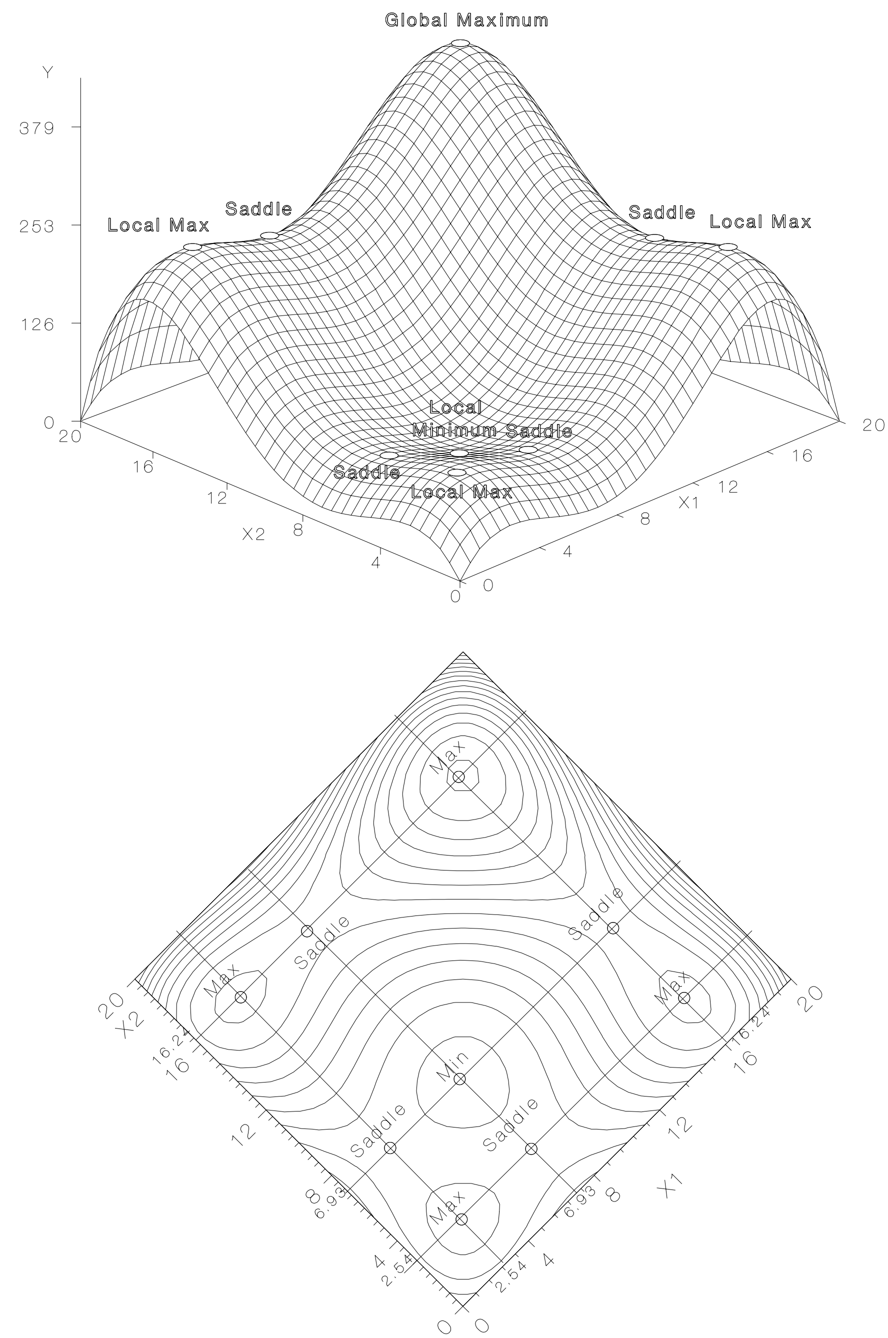


Figure 6.2 Critical Values for the Polynomial $y = 40x_1 ! 12x_1^2 + 1.2x_1^3 ! 0.035x_1^4 + 40x_2 ! 12x_2^2 + 1.2x_2^3 ! 0.035x_2^4$

All other inputs are presumed to be fixed and given, or already owned by the farm manager. The decision faced by the farm manager is how much of the two fertilizer inputs or factors of production to apply to maximize profits to the farm firm.

The total revenue or total value of the product from the sale of the corn from 1 acre of land is

†6.57 $TVP = py$

where
p = price of corn per bushel
y = corn yield in bushels per acre

The total input or factor cost is

†6.58 $TFC = v_1x_1 + v_2x_2$

where v_1 and v_2 are the prices on potash and phosphate respectively in cents per pound. The profit function is

†6.59 $\Pi = TVP - TFC$

Equation †6.59 can also be expressed as

†6.60 $\Pi = py - v_1x_1 - v_2x_2$

or

†6.61 $\Pi = pf(x_1,x_2) - v_1x_1 - v_2x_2$

The first order, or necessary conditions for a maximum are

†6.62 $\Pi_1 = pf_1 - v_1 = 0$

†6.63 $\Pi_2 = pf_2 - v_2 = 0$

Equations †6.62 and †6.63 require that the slope of the *TVP* function with respect to each input equal the slope of the *TFC* function for each input, or that the difference between the slopes of the two functions be zero for both inputs, or as

†6.64 $pf_1 = v_1$

†6.65 $pf_2 = v_2$

The value of the marginal product must equal the marginal factor cost for each input. If the farmer is able to purchase as much of each type of fertilizer as he or she wishes at the going market price, the marginal factor cost is the price of the input, v_1 or v_2. This also implies that at the point of profit maximization the ratio of *VMP* to *MFC* for each input is 1. In other words

†6.66 $pf_1/v_1 = pf_2/v_2 = 1$

The last dollar spent on each input must return exactly \$1, and most if not all previous units will have given back more than a dollar. The accumulation of the excess dollars in returns over costs represents the profits or net revenues accruing to the farm firm.

Moreover, the equations representing the first order conditions can be divided by each other:

†6.67 $pf_1/pf_2 = v_1/v_2$.

Note that the output price cancels in equation †6.67 such that

†6.68 $f_1/f_2 = v_1/v_2$

Recall from Chapter 5 that f_1 is the *MPP* of x_1 and f_2 is the *MPP* of x_2. The negative ratio of the respective marginal products is one definition of the marginal rate of substitution of x_1 for x_2 or $MRS_{x_1x_2}$. Then at the point of profit maximization

†6.69 $$MRS_{x_1x_2} = v_1/v_2$$

or

†6.70 $$dx_2/dx_1 = v_1/v_2$$

As will be seen later, equation †6.70 holds at other points on the isoquant map in addition to the point of profit maximization.

The second order conditions also play a role. Assuming fixed input prices (v_1 and v_2), the second order conditions for the profit function are

†6.71 $$A_{11} = pf_{11}$$

†6.72 $$A_{22} = pf_{22}$$

†6.73 $$A_{12} = A_{21} = pf_{12} = pf_{21} \text{ (by Young's theorem)}$$

Or in the form of a matrix

†6.74

$$\begin{matrix} pf_{11} & pf_{12} \\ pf_{21} & pf_{22} \end{matrix}$$

For a maximum

†6.75 $$pf_{11} < 0$$

and

†6.76 $$pf_{11}pf_{22} \; ! \; pf_{12}pf_{21} > 0$$

The principal minors must alternate in sign starting with a minus. Equations †6.75 and †6.76 require that the *VMP* functions for both x_1 and x_2 be downsloping. With fixed input prices, the input cost function will have a constant slope, or the slope of *MFC* will be zero.

The conditions that have been outlined determine a single point of global profit maximization, assuming that the underlying production function itself has but a single maximum. This single profit-maximization point will require less of both x_1 and x_2 than would be required to maximize output, unless one or both of the inputs were free.

6.7 A Comparison with Output- or Yield-Maximization Criteria

A comparison can be made of the criteria for profit maximization versus the criteria for yield maximization. If the production function is

†6.77 $$y = f(x_1, x_2)$$

Maximum yield occurs where

†6.78 $$f_1 = MPP_{x_1} = 0$$

†6.79 $$f_2 = MPP_{x_2} = 0$$

or

†6.80 $$f_1 = f_2 = 0$$

The second-order conditions for maximum output require that $f_{11} < 0$; and $f_{11}f_{22} > f_{12}f_{21}$. The *MPP* for both inputs must be downward sloping.

The first- and second-order conditions comprise the necessary and sufficient conditions for the maximization of output or yield and are the mathematical conditions that define the center of an isoquant map that consists of a series of concentric rings.

Since zero can be multiplied or divided by any number other than zero, and zero would still result, when *MPP* for x_1 and x_2 is zero,

†6.81 $$pf_1/v_1 = pf_2/v_2 = 0$$

To be at maximum output, the last dollar spent on each input must produce no additional output, yield, or revenue.

Recall that the first-order, or necessary conditions for maximum profit occur at the point where

†6.82 $$pf_1 - v_1 = 0$$

†6.83 $$pf_2 - v_2 = 0$$

†6.84 $$pf_1/v_1 = pf_2/v_2 = 1$$

and the corresponding second order conditions for maximum profit require that

†6.85 $$pf_{11} < 0$$

†6.86 $$pf_{11}pf_{22} - pf_{12}pf_{21} > 0$$

†6.87 $$p^2(f_{11}f_{22} - f_{12}f_{21}) > 0$$

Since p^2 is positive, the required signs on the second-order conditions are the same for both profit and yield maximization.

6.8 Concluding Comments

This chapter has developed some of the fundamental rules for determining if a function is at a maximum or a minimum. The rules developed here are useful in finding a solution to the unconstrained maximization problem. These rules also provide the basis for finding the solution to the problem of constrained maximization or minimization. The constrained maximization or minimization problem makes it possible to determine the combination of inputs that is required to produce a given level of output for the least cost, or to maximize the level of output for a given cost. The constrained maximization problem is presented in further detail in Chapters 7 and 8.

Notes

1. A simple example can be used to illustrate that Young's theorem does indeed hold in a specific case. Suppose that a production function

$$y = x_1^2 x_2^3$$

Then

$$f_1 = 2x_1 x_2^3$$

$$f_2 = 3x_1^2 x_2^2$$

$$f_{12} = 6x_1 x_2^2$$

$$f_{21} = 6x_1 x_2^2$$

A formal proof of Young's theorem in the general case can be found in most intermediate calculus texts.

Problems and Exercises

1. Does the function $y = x_1 x_2$ ever achieve a maximum? Explain.

2. Does the function $y = x_1^2 - 2x_2^2$ ever achieve a maximum? Explain.

3. Does the function $y = x_1 + 0.1x_1^2 - 0.05x_1^3 + x_2 + 0.1x_2^2 - 0.05x_2^3$ ever achieve a maximum? If so, at what level of input use is output maximized.

4. Suppose that price of the output is \$2. For the function given in Problem 3, what level of input use will maximize the total value of the product?

5. Assume that the following conditions exist

$$f_1 = 0$$

$$f_2 = 0$$

Does a maximum, minimum, or saddle point exist in each case?

a. $f_{11} > 0$

$f_{11} \cdot f_{22} - f_{12} \cdot f_{21} < 0$

b. $f_{11} < 0$

$f_{11} \cdot f_{22} - f_{12} \cdot f_{21} > 0$

c. $f_{11} > 0$

$f_{11} \cdot f_{22} - f_{12} \cdot f_{21} > 0$

d. $f_{11} < 0$

$f_{11} \cdot f_{22} - f_{12} \cdot f_{21} < 0$

6. Suppose that the price of the output is \$3, the price of the input x_1 is \$5, and the price of input x_2 is \$4. Is it possible to produce and achieve a profit? Explain. What are the necessary and sufficient conditions for profit maximization?