7

Maximization Subject to Budget Constraints

This chapter presents the factor-factor model by relying primarily on simple algebra and graphics. Here the concept of a constraint to the maximization process is introduced. Points of tangency between the budget constraint and the isoquant are defined. Conditions along the expansion path are outlined, and the least-cost combination of inputs is defined. Pseudo scale lines are developed, and the global point of profit maximization is identified. The chapter concludes with a summary of the fundamental marginal conditions for the factor-factor model. The algebraic and graphical presentation forms the basis for a better understanding of the mathematical presentation contained in Chapter 8.

Key terms and definitions:

Constraint
Budget Constraint
Iso-outlay Line
Isoquant Map
Points of Tangency
Isocline
Expansion Path
Least-Cost Combination
Equimarginal Return Principle
Input Bundle
Pseudo Scale Lines
Global Output Maximization
Global Profit Maximization
Marginal Conditions
Decision Rules

7.1 Introduction

Chapter 6 dealt with basic relationships governing the maximization of output or profit without regard for constraints or limitations on the maximization process. However, farmers do not normally operate in an environment where maximization of profit can take place without regard to constraints on the maximization process.

The consumer, seeking to purchase goods and services in such a manner as to maximize utility, must invariably face constraints or limitations imposed by the availability of money income. The consumer must operate within these constraints by choosing a mix of goods that requires a total outlay not to exceed income. While the consumer might borrow money to purchase goods and services, eventually loans need to be paid back. Ultimately, the bundle of goods and services purchased by the consumer must be in line with the consumer's money income.

The producer, too, faces constraints. The constraints or limitations imposed on the producer fall into two categories: (1) internal constraints occurring as a result of limitations in the amount of money available for the purchase of inputs, and (2) external constraints imposed by the federal government or other institutions. An example of such a constraint might be an acreage allotment within a government farm program.

This chapter is devoted to a discussion of how constraints internal to a farm firm might limit the farmer's ability to achieve profit maximization. The models developed in this chapter also provide a useful analytical tool for assessing the impact of certain external constraints on the behavior of the farm manager. The application of these models to situations where external constraints are imposed is developed fully in chapter 8.

7.2 The Budget Constraint

Suppose that a farmer again uses two inputs (x_1 and x_2) to produce an output (y). The farmer can no longer purchase as much of both inputs as is needed to maximize profits. The farmer faces a budget constraint that limits the amount total expenditures on the two inputs to some fixed number of dollars $C°$. The budget constraint faced by the farmer can be written as

†7.1 $$C° = v_1x_1 + v_2x_2$$

where v_1 and v_2 are prices on the inputs x_1 and x_2, respectively.

Another way of writing equation †7.1 is

†7.2 $$C° = \Sigma v_ix_i \text{ for i = 1, 2}$$

Now suppose that the farmer has \$100 available for the purchase of the two inputs x_1 and x_2. Suppose also that x_1 costs \$5.00 per unit and x_2 costs \$3.00 per unit. Table 7.1 illustrates possible combinations of x_1 and x_2 that could be purchased with the \$100.

Table 7.1 Alternative Combinations of x_1 and x_2 Purchased with \$100

Combination	Units of x_1	Units of x_2	Total Cost $C°$
A	20.00	0.00	\$100
B	15.00	8.33	\$100
C	10.00	16.67	\$100
D	8.00	20.00	\$100
E	5.00	25.00	\$100
F	0.00	33.33	\$100

Table 7.1 illustrates but a few of the possible combinations of x_1 and x_2 that could be purchased with a total budget outlay ($C°$) of exactly \$100. If inputs are assumed to be infinitely divisible, there are an infinite number of alternative combinations that could be purchased for exactly \$100. The assumption that inputs are infinitely divisible is not a bad one for certain classes of inputs such as fertilizer or livestock feed. For example, 186.202 pounds of fertilizer or 149.301 bushels of feed could be purchased. For other classes of inputs in agriculture, the assumption is silly. No farmer would purchase 2.09 tractors or 1.57 bulls. However, the basic model has as an underlying assumption that inputs are infinitely divisible.

Now suppose that the budget line or constraint indicated by the tabular data in Table 7.1 is plotted with input x_1 on the horizontal axis and x_2 on the vertical axis. It may seem surprising that a budget line that has dollars as its units can be plotted on a diagram in which the axes are physical quantities of inputs. However, the position of the budget constraint on both the horizontal and vertical axis can be determined. First, suppose that the farmer chooses to purchase with the \$100 dollars all x_1 and no x_2. The total amount of x_1 that would be purchased is \$100/\$5.00 ($C°/v_1$) or 20 units of x_1. The budget constraint therefore intersects the x_1 axis at 20 units.

Tabular data similar to that contained in Table 7.1 can be derived for any chosen budget outlay. The terms *iso-outlay* or *isocost* have frequently been used by economists to refer to the budget constraint or outlay line. The iso-outlay function can be thought of as a line of constant or equal budget outlay.

Suppose instead that the farmer chose to allocate the \$100 in such a way that no x_1 was purchased and all of the \$100 was used to purchase x_2. The total amount of x_2 that could be purchased is \$100/\$3.00 ($C°/v_2$) or approximately 33.33 units of x_2. The budget constraint therefore intersects the vertical axis at 33.33 units of x_2.

The final step is to determine the shape of the budget constraint between the points of intersection with the axes. If input prices are constant, the budget constraint will have a constant slope. A line with a constant slope might be drawn between the previously identified points on the two axes to form a triangle. The height of this triangle is \$100/\$3.00 (33.33 units of x_2). The length of the triangle is \$100/\$5.00 (20 units of x_2). The slope of the triangle is height divided by length, or

†7.3 ($100/$3.00) ÷ ($100/$5.00)

= ($100/$3.00) · ($5.00/$100)

= $5.00/$3.00 = 1.67

In equation †7.3 , the budget constraint has a constant slope of 5/3 or 1.67. Under the assumption of fixed input prices, the budget constraint will always have a constant slope of $(C°/v_2)(v_1/C°) = v_1/v_2$, sometimes called the *inverse input price ratio*. The term *inverse* is used because the price for the input appearing on the horizontal axis appears on the top of the fraction, the price for the input on the vertical axis at the bottom of the fraction.

By varying the total amount of the budget constraint or outlay ($C°$), a family of budget constraints can be developed, each representing a slightly different total outlay. Like isoquants, budget constraint lines are everywhere dense. That is, an infinite number of budget constraint lines can be drawn, each with the constant slope v_1/v_2.

The characteristics of an iso-outlay line can be summarized by making use of the total differential. The iso-outlay line is

$$C° = v_1x_1 + v_2x_2$$

The input prices are taken as fixed constants. The total differential of the iso-outlay line is

†7.4 $$\mathrm{d}C° = v_1dx_1 + v_2dx_2$$

The outlay ($C°$) along the iso-outlay line is assumed to be constant. Thus $dC° = 0$. Therefore

†7.5 $$0 = v_1dx_1 + v_2dx_2$$

†7.6 $$-v_1dx_1 = v_2dx_2$$

†7.7 $$dx_2/dx_1 = -v_1/v_2,$$

The term dx_2/dx_1 in equation †7.7 is the slope of the iso-outlay or budget constraint line in factor-factor (x_1 on the horizontal axis, x_2 on the vertical axis) space. The slope of the budget line is equal to the negative inverse ratio of input prices. The negative sign indicates that the iso-outlay line is downward sloping when both input prices are positive.

7.3 The Budget Constraint and the Isoquant Map

A diagram showing a series of isoquants is sometimes referred to as an isoquant map. The budget constraint or iso-outlay line developed in Section 7.2 is placed on a diagram with input x_1 on the horizontal axis and x_2 on the vertical axis. This factor-factor space is the same as that used to graph isoquants. Figure 7.1 illustrates an isoquant map superimposed on top of a series of budget constraints. In each case, only selected isoquants and selected iso-outlay lines are shown. An infinite number of either isoquants or iso-outlay lines could be drawn, each representing a slightly different level of output or a slightly different total outlay.

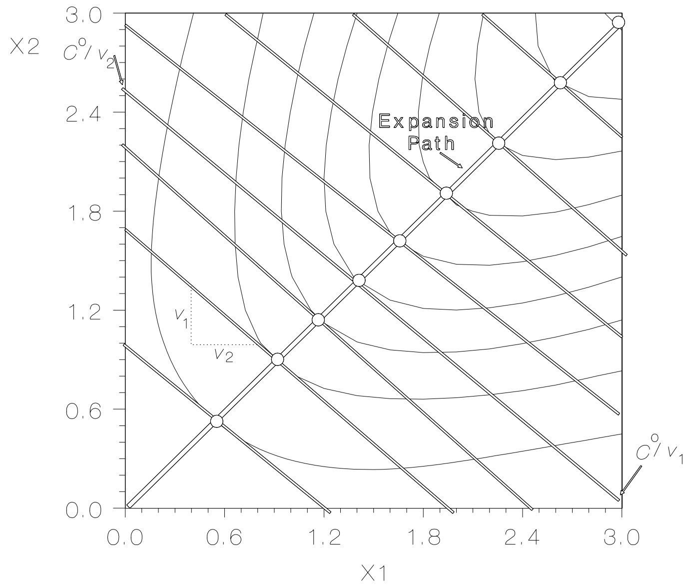

Figure 7.1 Iso-outlay Lines and an Isoquant Map

Eachisoquant has a corresponding iso-outlay line that comes just tangent to it. Moreover, for each iso-outlay line there is a corresponding isoquant that comes just tangent to it.

Assuming that the isoquant is bowed inward or convex to the origin of the graph, the point of tangency between the isoquant and the iso-outlay line represents the combination of inputs that will the produce the greatest quantity of output for the expenditure represented by the iso-outlay line. This is the maximum output given the budgeted dollars $C°$ or subject to the budget constraint.

Another approach is to think of the amount of output represented by a particular isoquant as being fixed. Then the point of tangency between the isoquant and the iso-outlay line represents the minimum ! cost, or least cost combination of input x_1 and x_2 that can be used to produce the fixed level of output represented by the isoquant.

Either rationale leads to the same important conclusion. If the farmer faces a budget constraint in the purchase of inputs x_1 and x_2, and as a result is unable to globally maximize profits, the next best alternative is to select a point of least-cost combination where the budget constraint faced by the farmer comes just tangent to the corresponding isoquant.

Any line drawn tangent to an isoquant represents the slope or $MRS_{x_1x_2}$ of the isoquant at that point. As indicated earlier, the slope of the isoquant can be represented by dx_2/dx_1. But the slope of the iso-outlay line was also found to be dx_2/dx_1, so the point of least cost combination is defined as the point where the slope of the iso-outlay line equals the slope of the corresponding isoquant. At the point of least-cost combination, both the isoquant and the iso-outlay line will be downward sloping.

One definition of the slope of the isoquant is ! $MRS_{x_1x_2}$. The slope of the iso-outlay line is ! v_1/v_2. Both the isoquant and the iso-outlay line are downward sloping, so the point of tangency between the isoquant and the iso-outlay line can be defined as

†7.8 $$dx_2/dx_1 = ! v_1/v_2$$

or

†7.9 $$dx_2/dx_1 = dx_2/dx_1 = v_1/v_2$$

At the point of least cost combination, the *MRS* of x_1 for x_2 (dx_2/dx_1) must equal the inverse price ratio (v_1/v_2).

7.4 Isoclines and the Expansion Path

The term *isocline* is used to refer to any line that connects points of the same slope on a series of isoquants. The ridge lines developed in Chapter 5 were examples of isoclines. Ridge line I connected all points of zero slope on the series of isoquants. Ridge line II connected all points of infinite slope on the same series of isoquants. Each are examples of isoclines because each connects points with the same slope.

As outlined in Section 7.3, the inverse ratio of input prices v_1/v_2 is very important in determining where within a series of isoquants a farm manager can operate. To produce a given amount of output at minimum cost for inputs, or to produce the maximum amount of output for a given level of expenditure on x_1 and x_2, the farmer must equate $MRS_{x_1x_2}$ with v_1/v_2. However, if input prices are constant, a key assumption of the model of pure competition outlined in Chapter 1, the slope of the iso-outlay line will be a constant v_1/v_2.

A line connecting all points of constant slope v_1/v_2 on an isoquant map is a very important isocline. This isocline has a special name, the *expansion path* (Figure 7.1). The expansion path is a specialized isocline that connects all points on an isoquant map where the slope of the isoquants is equal to the ratio v_1/v_2, where v_1 and v_2 refer to the prices on the inputs.

The term *expansion path* is used because the line refers to the path on which the farmer would expand or contract the size of the operation with respect to the purchases of x_1 and x_2. A farmer seeking to produce a given amount of output at minimum cost, or seeking to produce maximum output for a given expenditure on x_1 and x_2, would always use inputs x_1 and x_2 in the combinations indicated along the expansion path. The exact point on the expansion path where the farmer would operate would depend on the availability of dollars ($C°$) for the purchase of inputs.

The points of tangency between the iso-outlay lines and the corresponding isoquant on the expansion path thus represent the least cost combination of inputs that can be used to produce the output level associated with the isoquant. There is no combination of x_1 and x_2 that can produce that specific quantity of output at lower cost. If isoquants are convex to the origin, or bowed inward, all points of tangency represent points of least-cost combination for the output level associated with the particular tangent isoquant. While every point on the expansion path is a point of least cost combination, there is only one point on the expansion path that represents the global point of profit maximization for the farmer. This particular point is derived in Section 7.7.

The expansion path begins at the origin of the graph ($x_1 = 0, x_2 = 0$) and travels across isoquants until the global point of output maximization in reached, where the *MPP* of both x_1 and x_2 is zero. Points beyond the global point of output maximization, while having the same constant slope v_1/v_2, would never be chosen by the entrepreneur. Note that at points beyond global output maximization, isoquants are no longer convex to the origin of the graph. Points of tangency occur as a result of the isoquants curving upward from below, not downward from above. These points of tangency represent the maximum expenditure for a given level of output, not the desired minimum expenditure. So these points would never be considered economic for the farmer.

Some widely used agricultural production functions generate expansion paths with a constant slope. The class of production functions that generate linear expansion paths when input prices are constant are referred to as *homothetic production functions*.

The equation for an expansion path can be derived through the use of the general expansion path conditions

†7.10 $$MRS_{x_1x_2} = v_1/v_2$$

But

†7.11 $$MRS_{x_1x_2} = -MPP_{x_1}/MPP_{x_2}$$

The equation for the expansion path can be obtained by solving the expression $MPP_{x_1}/MPP_{x_2} = v_1/v_2$ for x_2 in terms of x_1. For example, suppose that the production function is

†7.12 $$y = ax_1^{0.5}x_2^{0.5}$$

The corresponding *MPP*'s are

†7.13 $$MPP_{x_1} = 0.5ax_1^{-0.5}x_2^{0.5}$$

†7.14 $$MPP_{x_2} = 0.5ax_1^{0.5}x_2^{-0.5}$$

The $MRS_{x_1x_2}$ is

†7.15 $$(0.5ax_1^{-0.5}x_2^{0.5})/(0.5ax_1^{0.5}x_2^{-0.5})$$

†7.16 $$x_2/x_1 = v_1/v_2$$

Thus, the equation for the expansion path is

†7.17 $$x_2 = (v_1/v_2) \cdot x_1$$

Since the ratio v_1/v_2 is a constant b, the expansion path [equation †7.17] in this example is linear

†7.18 $$x_2 = bx_1$$

7.5 General Expansion Path Conditions

In chapter 6 the general conditions for the maximization of profit were defined as

†7.19 $$VMP_{x_1}/v_1 = VMP_{x_2}/v_2 = 1$$

There are two parts to the rule in equation †7.19 . The first part requires that the ratio of *VMP* to the corresponding input price be the same for both (all) inputs. The second part requires that ratio to be equal to 1. The farmer should use inputs up to the point where the last dollar spent on the input returns back a dollar, and most if not all prior units of input returns more than a dollar.

What if the farmer faces a limitation or constraint on the availability of funds for the purchase of inputs x_1 and x_2? The farmer's next best alternative is to apply the *equimarginal return principle*. The equimarginal return principle ensures that if the farmer is not at the point of profit maximization, at least costs are being minimized for the level of output that can be produced. Alternatively, maximum output is being produced for a given budget outlay.

The equimarginal return principle requires the farmer to operate using combinations of inputs such that

†7.20 $$VMP_{x_1}/v_1 = VMP_{x_2}/v_2 = K$$

Equation †7.20 is only slightly different from the profit maximizing condition outlined in equation †7.19 . Instead of requiring that the ratio of the *VMP* to the corresponding input The most important characteristic of pseudo scale lines is that the two lines intersect at the global point of profit maximization. The intersection of the pseudo scale lines defines precisely the point on the expansion path where profits are greatest. There is no other point more profitable (Figure 7.4).
price to be equal to 1, now the ratio of *VMP* to the corresponding input price must be equal to some constant number K, where K can be any number. The ratios of the *VMP* to the input price must be the same for both inputs, and thus the ratio for both inputs must be equal to a number K.

Another way of looking at the expansion path is that it represents the series of points defined by equation †7.20 . Any point on the expansion path has a different value for K assigned to it. In general, as one moves outward along the expansion path, the value of K will decline. Points along the expansion path can be identified according to the value of K.

Suppose, for example, that $K = 3$. The last dollar spent on the input returns \$3. This is a point on the expansion path that represents a least cost combination of inputs (since the ratio of $VMP_{x_1}/v_1 = VMP_{x_2}/v_2 = 3$). This is not a point of profit maximization. The farmer is constrained by the availability of funds available for the purchase of inputs x_1 and x_2.

Suppose that $K = 1$. This is also a point of least-cost combination on the expansion path, but this is the same as the previously defined point of profit maximization. The point of global profit maximization is a special point along the expansion path where the value of K is equal to 1, indicating that the last dollar spent on each input returns exactly a dollar of revenue. This is probably a point on the expansion path farther out than the point where $K = 3$, where funds for the purchase of input were restricted.

Now suppose that $K = 0$. This is also a point of least cost combination on the expansion path, but $VMP = pMPP$ where p is the price of the output. If p is positive, the only way that K can be zero is for *MPP* to be zero. The last dollar spent on each input returns back absolutely nothing in terms of revenue. The point where $VMP_{x_1}/v_1 = VMP_{x_2}/v_2 = 0$ defines the global point of output maximization where the two ridge lines intersect. There is no other point where output is greater. This is a point that normally requires more of both x_1 and x_2 than the global point of profit maximization.

Whenever $0 < K < 1$, the last dollar spent on each input is returning less than its incremental cost. The section of the expansion path between the point of profit maximization and the point of output maximization represents a section where the farmer would never wish to operate. This is despite the fact that the isoquants in this section are curving downward toward the budget or iso-outlay line. For example, a value for K of 0.3 suggests that the last dollar spent on each input returns only 30 cents. The farmer would never wish to use an input at levels beyond the point of profit maximization, despite the fact that funds can be available for the purchase of additional units. Not only is stage III of the production function irrational, but any point that uses more of x_1 and x_2 than the profit maximizing point in stage II is also irrational.

Finally, suppose that $K < 0$. If input and output prices are positive, this suggests that *MPP* must be negative. The use of both inputs must exceed the level required to globally maximize output. The last dollar spent on an additional unit of input not only does not return its cost in terms of *VMP*, but revenues are declining as a result of the incremental use of inputs. A value for K of -0.2 suggests that the last dollar spent on the input results in a reduction in revenue of 20 cents. The total loss from the last dollar spent on the input is \$1.00 + \$0.20 = \$1.20. This is clearly not economic and is stage III for the use of both inputs, since *MPP* for both inputs is negative. Isoquants are tangent to the iso-outlay line, but are bowed outward (concave to the origin), not inward (convex to the origin). The entrepreneur could increase profit by a reduction in the use of both x_1 and x_2.

7.6 The Production Function for the Bundle

Envision a bundle of the two inputs x_1 and x_2. Suppose that the proportion of each input contained in the bundle is defined by the expansion path. If the expansion path has a constant slope, then as one moves up the expansion path, the proportion of x_1 and x_2 does not change. Suppose that a point on the expansion path requires 2 units of x_1 and 1 unit of x_2. If the expansion path has a constant slope, the point requiring 6 units of x_1 would require 3 units of x_2. The point requiring 8.8 units of x_1 would require 4.4 units of x_2, and so on. The size of the bundle varies, but if the expansion path has a constant slope, the proportion of each input contained in the bundle remains constant. In this example, that constant proportion is 2 units of x_1 to 1 unit of x_2.

Now suppose that a single input production function is drawn (Figure 7.2). The difference here is that instead of showing input x_1 on the horizontal axis, the horizontal axis is instead the bundle of x_1 and x_2. Each unit of the bundle consists of 2 units of x_1 and 1 unit of x_2. The production function for the bundle looks very similar to the traditional three stage single- input production function. This production function has a point of output maximization where the *MPP* of the bundle of x_1 and x_2 is equal to zero. It also has a point of profit maximization, where the *VMP* of the bundle is exactly equal to the price per unit of the bundle, or the cost of 2 units of x_1 and 1 unit of x_2 taken together.

Now consider a series of isoquants in three dimensions [Figure 7.2(b)]. The inputs x_1 and x_2 are on the horizontal plane. The third dimension is y or output. If one were to look along the expansion path of production surface such as that depicted in figure 7.2, the shape would correspond exactly to the shape of the production function for the bundle. The output maximization point on the production function for the bundle would correspond exactly to the global point of output maximization defined by the center of the series of concentric ring isoquants.

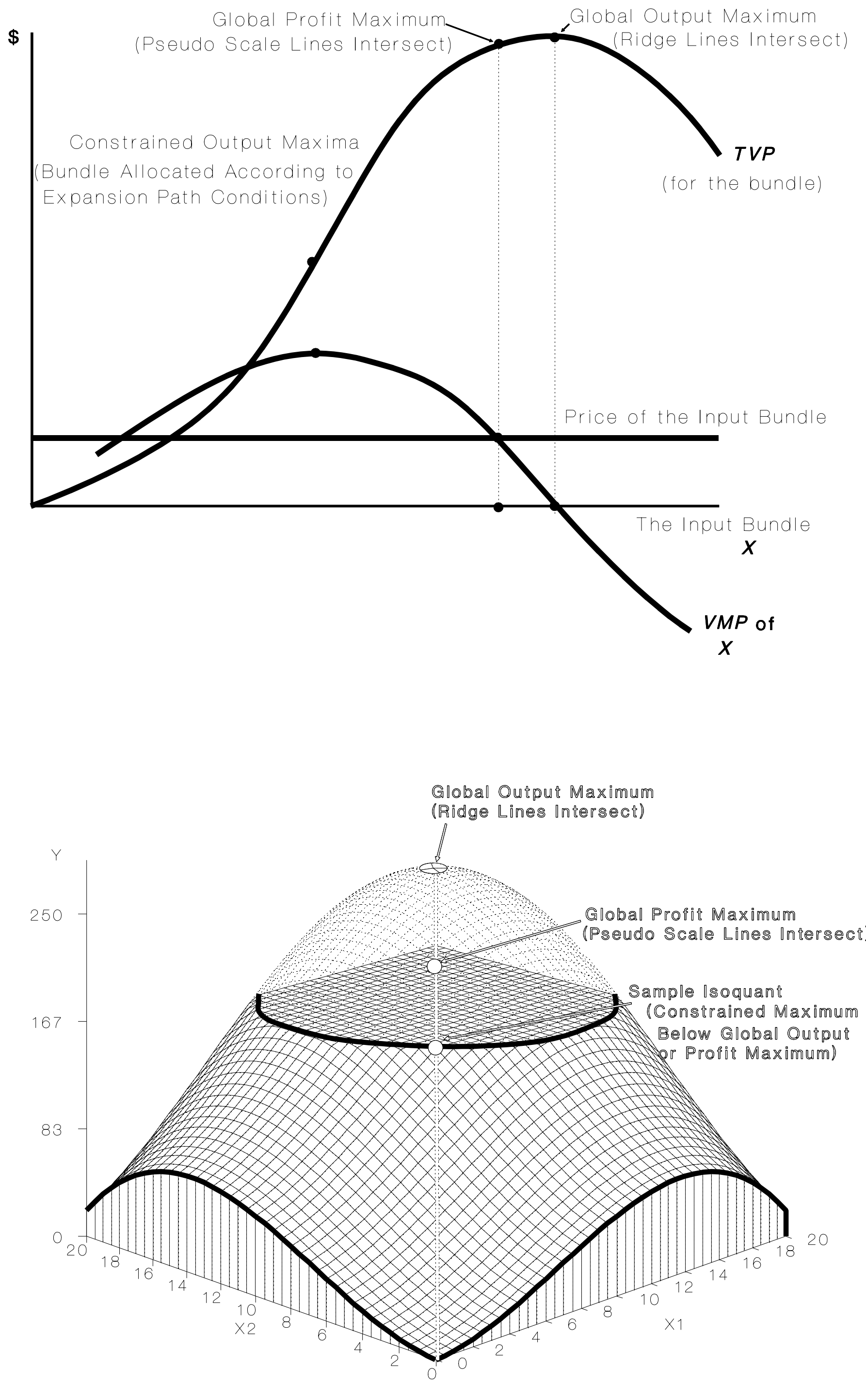

Figure 7.2 Global Output and Profit Maximization for the Bundle

The point of global profit maximization, where the *VMP* for the bundle equals its cost, would correspond to a point on an isoquant that is on the expansion path but below the point of global output maximization. This is where the farmer would most like to be, in that the point represents the greatest total profits of any possible point. The only reason that a farmer would not operate here would be as the result of a limitation in the availability of dollars needed to purchase such a globally optimal bundle of x_1 and x_2, or some institutional constraint, such as a government farm program, that would prohibit the use of the required amount of one or both of the inputs.

7.7 Pseudo Scale Lines

Recall from Chapter 5 that two families of production functions underlie any series of isoquants. A single-input production function can be obtained for one of the inputs by assuming that the other input is held constant at some fixed level. By making alternative assumptions about the level at which the second input is to be fixed, a series of production functions for the first input can be derived. The family of production functions thus derived each has a maximum. The maximum value for each production function for the first input (x_1) holding the second input (x_2) constant corresponds to a point on ridge line I (where the slope of the isoquant is zero). The maximum value for each production function for the second input (x_2) holding the first input (x_1) constant corresponds to a point on ridge line II (where the slope of the isoquant is infinite).

Now suppose that output has some positive price called p, and the prices for x_1 and x_2 are v_1 and v_2 respectively. Each member of the two families of underlying production functions will have a profit maximizing level of input use for one input, assuming that the second input is fixed. This is not the global point of profit maximization, since only one input is allowed to vary. For input x_1, this is where $pMPP_{x_1}/v_1 = 1$, assuming that x_2 is fixed at x_2^*. For input x_2, this is where $pMPP_{x_2}/v_2 = 1$, assuming that x_1 is fixed at x_1^*.

If input prices are positive, this profit maximizing level of input use for each member of the family will require less x_1 or x_2 than did the output maximizing level of input use. Figure 7.3 illustrates the relationship for input x_1, which is assumed to be on the horizontal axis. A vertical line drawn from the profit maximizing level of input use to the line that represents the assumed fixed level of the other input (x_2 on the vertical axis) defines also a point on an isoquant. This point will be on an isoquant that lies below the isoquant that defines the ridge line. This isoquant will intersect but not be tangent to the line representing the fixed level of x_2. This is because profit maximization results in less output than does yield maximization.

For input x_1, this point will lie to the left of the ridge line. The greater the price of x_1 (v_1), the farther to the left of the ridge line this point will lie, and the lower the profit maximizing level of input x_1 and the resulting output from the use of x_1. This procedure can be repeated for each member in the family of the production functions for x_1, by assuming alternative values for the input x_2, which is treated as fixed.

A similar approach can be used for the family of production functions for x_2, assuming that x_1 is held constant at alternative fixed levels. Here the points of profit maximization for x_2 (holding x_1 constant at alternative levels) will occur below the ridge line for the second input along the vertical line defined by the assumption with respect to the quantity of x_1 that is to be used. Again the process can be repeated over and over for varying assumptions this time with regard to the level at which x_1 is to be fixed.

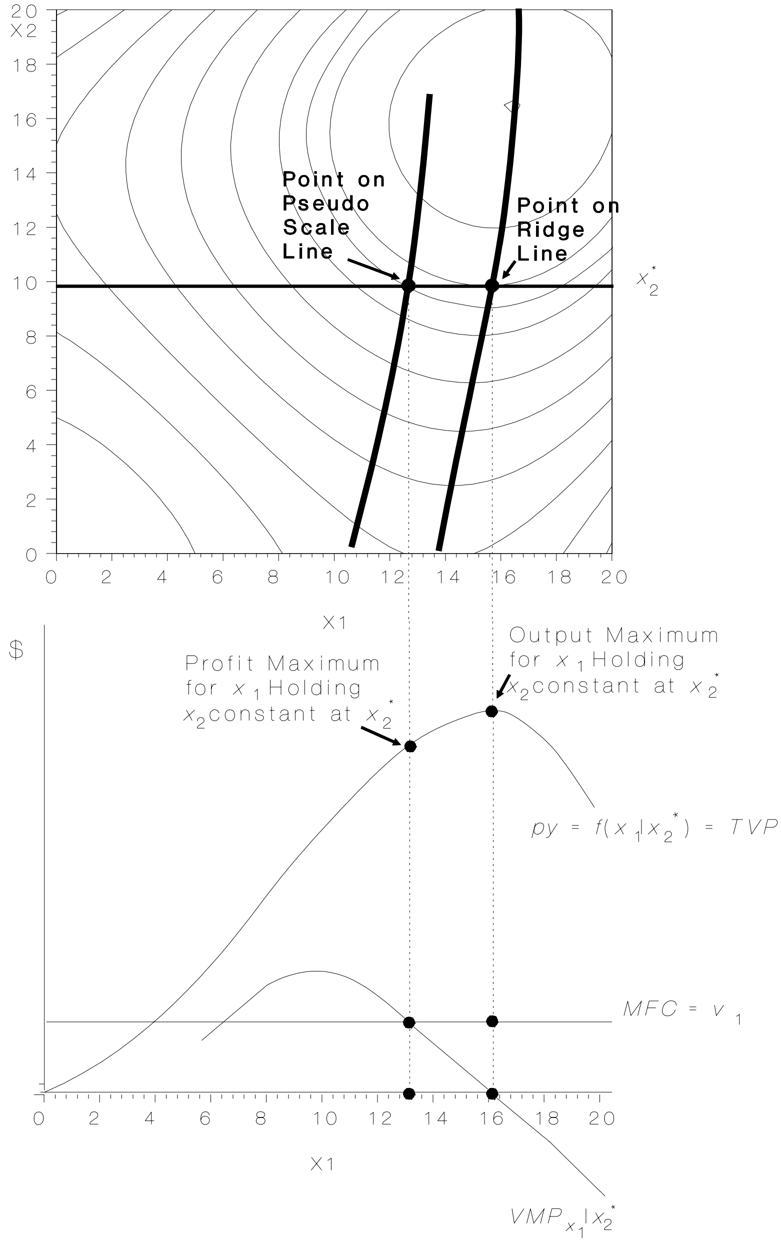

Figure 7.3 Deriving a Point on a Pseudo Scale Line

A line connecting all points of profit maximization for one input, assuming the other input to be fixed at some constant level, is called a *pseudo scale line*. Just as there are two ridge lines, so also are there two pseudo scale lines, one for each input. If input prices are positive, pseudo scale lines will lie interior to the ridge lines. If inputs were free, the pseudo scale lines would lie on top of the ridge lines, just as profits would be maximized by maximizing output. The greater the input price, the farther will be the pseudo scale line for that input from the ridge line for that input.

The most important characteristic of pseudo scale lines is that the two lines intersect at the global point of profit maximization. The intersection of the pseudo scale lines defines precisely the point on the expansion path where profits are greatest. There is no other point more profitable (Figure 7.4).

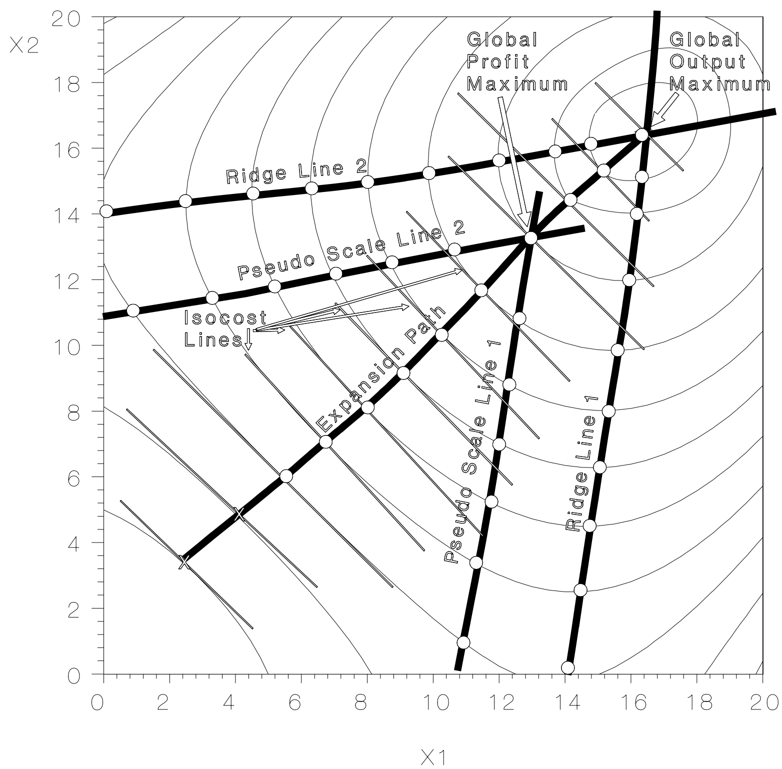

Figure 7.4 The Complete Factor-Factor Model

The global point of profit maximization, where profits are greatest when both inputs can be varied, is at once a point on the expansion path, a point of least cost combination, and a point where the pseudo scale lines intersect. There is no other point where these conditions are met. Any other point on a pseudo scale line is no longer on the expansion path, and the expansion path meets both pseudo scale lines only once.

Another way of looking at the concept of the pseudo scale line is in relation to the equimarginal returns equations. The global point of profit maximization is defined by

†7.21 $$VMP_{x_1}/v_1 = VMP_{x_2}/v_2 = 1$$

This is the point where the pseudo scale lines intersect.

Points on the pseudo scale line for input x_1 are defined by

†7.22 $VMP_{x_1}/v_1 = 1 \; VMP_{x_2}/v_2 \geq 1$

If

†7.23 $VMP_{x_1}/v_1 = 1$ and $VMP_{x_2}/v_2 > 1$

the farmer could increase profit by increasing the use of x_2. This could be accomplished either by increasing total outlay for x_2 until the global profit maximizing condition was met for both inputs, or by a reduction in the use of x_1 until the expansion path condition that

†7.24 $VMP_{x_1}/v_1 = VMP_{x_2}/v_2 = K$

is met. The closer K could be be brought to 1, the closer the farmer would be to maximum global profit.

Points on the pseudo scale line for input x_2 are defined by

†7.25 $VMP_{x_1}/v_1 \geq 1 \; VMP_{x_2}/v_2 = 1$

If $VMP_{x_1}/v_1 > 1$, profits could be increased by increasing the use of x_1 such that the expansion path condition is again met. Again, the closer K is brought to 1, the closer the farmer would be to maximum profit.

7.8 Summary of Marginal Conditions and Concluding Comments

Table 7.2 summarizes the marginal conditions associated with the ridge lines, expansion path, and pseudo scale lines. These marginal conditions comprise the decision rules for the farmer in choosing the amount and combination of inputs to be used in a two input, single-output, factor-factor setting.

Figure 7.5 summarizes this information. Notice that any point on the expansion path is at once, a minimum and a maximum, that is, maximum output for a specific expenditure level, or minimum expenditure for a specific output level. Although all points on the expansion path represent optimal input combinations for a specific expenditure level, not all points on the expansion path are equally preferred. In general, as the farmer expands the operation along the expansion path, the will increase profitability only to the point on the expansion path where the pseudo scale lines intersect, that is, to the global point of profit maximization. Points of tangency between isocost lines and isoquants beyond the point of profit maximization represent a reduction in profit, and are analogous to the input levels that lie between profit maximization and output maximization in the single input case.

This chapter has developed graphically and algebraically the fundamental conditions for the least cost combination of inputs with the factor-factor model. The expansion path along which a farmer would expand or contract the scale of his operation was derived. All points along the expansion path represent points of least cost combination for the farmer. Both the global point of output maximization and the global point of profit maximization are on the expansion path. All points on the expansion path are points of least cost combination of inputs as long as isoquants are convex to the origin. However, there is but a single point of global profit maximization for the farmer, as defined by the point where the pseudo scale lines intersect the expansion path.

Table 7.2 Marginal Conditions for Ridge Lines, the Expansion Path and Pseudo Scale Lines

Condition	Comment
On the expansion path:	
$VMP_{x_1}/v_1 = VMP_{x_2}/v_2 = 0$	Global output maximization
$VMP_{x_1}/v_1 = VMP_{x_2}/v_2 < 0$	Stage III for both inputs; the profit-maximizing farmer would not operate here
$VMP_{x_1}/v_1 = VMP_{x_2}/v_2 = 0 < ... < 1$	Between profit and output maximum; farmer would not operate here.
$VMP_{x_1}/v_1 = VMP_{x_2}/v_2 = 1$	Global profit maximization; point of least-cost combination
$VMP_{x_1}/v_1 = VMP_{x_2}/v_2 > 1$	Point of least-cost combination; not global profit maximization
On the pseudo scale lines:	
$VMP_{x_1}/v_1 = 1, \; VMP_{x_2}/v_2 > 1$	Point on pseudo scale line for input x_1; not global profit maximization
$VMP_{x_1}/v_1 > 1, \; VMP_{x_2}/v_2 = 1$	Point on pseudo scale line for input x_2; not global profit maximization
$VMP_{x_1}/v_1 = 1, \; VMP_{x_2}/v_2 = 1$	Global profit maximization and on the expansion path
On the ridge lines:	
$VMP_{x_1}/v_1 = 0, \; VMP_{x_2}/v_2 = 0$	Global output maximization and on the expansion path
$VMP_{x_1}/v_1 = 0, \; VMP_{x_2}/v_2 \neq 0$	On ridge line I for input x_1
$VMP_{x_1}/v_1 \neq 0, \; VMP_{x_2}/v_2 = 0$	On ridge line II for input x_2

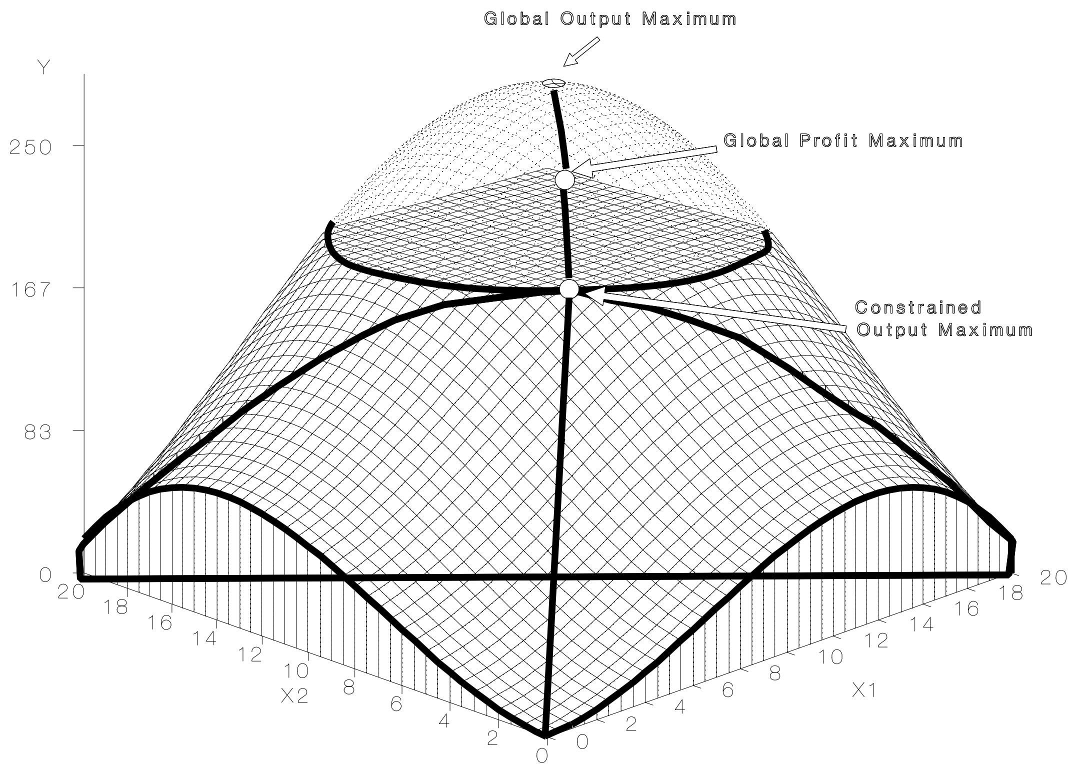

Figure 7.5 Constrained and Global Profit and Output Maxima along the Expansion path

If the farmer were given a choice, he or she would prefer to operate on the expansion path, for it is here that a given level of output can be produced at the lowest possible cost. A point on the expansion path is sometimes referred to as a point of least- cost combination. If the farmer were on the expansion path and a sufficient number of dollars were available for the purchase of x_1 and x_2, the farmer would prefer to be at the point of profit maximization where the pseudo scale lines intersect.

The farmer would never chose a level of input use on the expansion path beyond the point of profit maximization, for at any point on the expansion path beyond the point of profit maximization, the last dollar spent on inputs returns less than a dollar. The condition is analogous to using an input at a level beyond the point of profit maximization in the single-input factor-product model. In both instances, the last dollar spent on the input (or inputs) returns less than a dollar. Only if there is a limitation on the availability of dollars for the purchase of the two inputs would the farmer chose to operate on the expansion path but with an operation smaller than that needed to achieve global profit maximization.

Chapter 8 will develop the same set of decision rules for points of least-cost combination and profit maximization. However, rather using primarily graphics and algebra as a vehicle for presentation, Chapter 8 uses basic calculus and relies heavily on the maximization and minimization principles presented in Chapter 6.

Problems and Exercises

1. Consider the following table given in Problem 1, Chapter 5.

Combination	Units of x_1	Units of x_2	$MRS_{x_1x_2}$	$MRS_{x_2x_1}$
A	10	1		
			_______	_______
B	5	2		
			_______	_______
C	3	3		
			_______	_______
D	2	4		
			_______	_______
E	1.5	5		

a. Suppose that the price of x_1 and x_2 is each a dollar. What combination of x_1 and x_2 would be used to achieve the least-cost combination of inputs needed to produce 100 bushels of corn?

b. Suppose that the price of x_2 increased to \$2. What combination of x_1 and x_2 would be used to produce 100 bushels of corn?

c. If the farmer was capable of producing 100 bushels of corn when the price of x_1 and x_2 were both \$1, would he or she necessarily also be able to produce 100 bushels of corn when the price of x_2 increases to \$2? Explain.

2. Assume that a farmer has available \$200. What is the slope of the isocost line when

a. v_1 = \$1; v_2 = \$2.00?

b. v_1 = \$3; v_2 = \$1.75?

3. Assume that the following conditions hold. What action should the farmer take in each instance?

a. $VMP_{x_1}/v_1 = VMP_{x_2}/v_2 = 3$
b. $VMP_{x_1}/v_1 = VMP_{x_2}/v_2 = 5$
c. $VMP_{x_1}/v_1 = VMP_{x_2}/v_2 = 1$
d. $VMP_{x_1}/v_1 = VMP_{x_2}/v_2 = 0.2$
e. $VMP_{x_1}/v_1 = VMP_{x_2}/v_2 = 0$
f. $VMP_{x_1}/v_1 = VMP_{x_2}/v_2 = -0.15$
g. $VMP_{x_1}/v_1 = 9$; $VMP_{x_2}/v_2 = 5$
h. $VMP_{x_1}/v_1 = -2$; $VMP_{x_2}/v_2 = 5$
i. $VMP_{x_1}/v_1 = 2$; $VMP_{x_2}/v_2 = 1$
j. $VMP_{x_1}/v_1 = 1$; $VMP_{x_2}/v_2 = 0$
k. $VMP_{x_1}/v_1 = -1$; $VMP_{x_2}/v_2 = 1$