Beyond Propositional Logic

Łukasiewicz's System

Consider the following set of truth tables:

$\sim$	
1	0
0	1
#	#

$\wedge$	1	0	#
1	1	0	#
0	0	0	0
#	#	0	#

$\vee$	1	0	#
1	1	1	1
0	1	0	#
#	1	#	#

$\rightarrow$	1	0	#
1	1	0	#
0	1	1	1
#	1	#	1

Definition of Trivalent Interpretation: A trivalent interpretation is a function that assigns to each sentence letter exactly one of the values: 1, 0, #.

Łukasiewsicz Definitions of Validity and Consequence:

- ϕ is Łukasiewicz-valid ("$\vDash_{Ł}\ \phi$") iff for every trivalent interpretation $\mathscr{I}$, Ł$V_{\mathscr{I}}(\phi) = 1$
- ϕ is a Łukasiewicz-semantic consequence of Γ ("$\Gamma \vDash_{Ł}\ \phi$") iff for every trivalent interpretation, $\mathscr{I}$, if Ł$V_{\mathscr{I}}(\gamma) = 1$ for each $\gamma \in \Gamma$, then Ł$V_{\mathscr{I}}(\phi) = 1$.

So “valid” means *always true*; it doesn’t mean *never false*.

The point is this. If we have a set of possible truth values, for example, $\{1, 0\}$ or now $\{1, 0, \#\}$, we can talk of some subset as *designated* truth values: *designated* for our definitions of validity.

In the case of Łukasiewicz-validity, we have designated $\{1\}$ in our definition of validity and have so defined validity as “always true”. If we had wanted a definition of validity to mean *never false*, then we would have to designate $\{1, \#\}$.

Beyond Propositional Logic

Kleene's Tables

Stephen Cole Kleene also developed 3-valued truth tables. The tables for negation, conjunction, and disjunction are identical to those of Łukasiewicz. But the table for the $\rightarrow$ differs slightly:

Kleene's Table:

$\rightarrow$	1	0	#
1	1	0	#
0	1	1	1
#	1	#	#

Łukasiewicz's Table:

$\rightarrow$	1	0	#
1	1	0	#
0	1	1	1
#	1	#	1

As Sider says, the idea behind Kleene's "strong" truth table is that if there is enough classical information to determine the truth value of the conditional then we can put the classical truth value in the table, if not, then not — it must remain $\#$.

Notice, however, that there are two important differences between the systems of Kleene and Łukasiewicz. First, for Kleene, the formula $P \rightarrow P$ becomes *invalid* — for $\# \rightarrow \#$ becomes $\#$. Second, in Kleene's system $\sim P \vee Q$ has the same table as $P \rightarrow Q$, whereas this is not the case with the system of Łukasiewicz.

Consider Sider's Exercise 3.7: Show that there are no Kleene-valid wffs. How would you answer this?

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One way to think of the validity of a wff is if it is a *tautology* — that is, when all the truth values of a truth table are T (or 1). But if you think about Kleene's truth tables, you will see that whenever all of the atomic components of a compound formula have the value #, so does the compound formula. This means that for any formula, there is at least one truth value assignment on which it has the value #. Thus, no formula can be a tautology (or a contradiction).

Beyond Propositional Logic

Priest's Logic of Paradox

If we keep Kleene's truth tables but take $\{1, \#\}$ as the designated truth values, then we get Priest's logic of paradox (LP).

Definitions of Validity and Consequence for LP:

- ϕ is LP-valid ("$\vDash_{LP} \phi$") iff for every trivalent interpretation $\mathscr{I}$, $KV_{\mathscr{I}}(\phi) \neq 0$
- ϕ is a LP-semantic-consequence of Γ ("$\Gamma \vDash_{LP} \phi$") iff for every trivalent interpretation, $\mathscr{I}$, if $KV_{\mathscr{I}}(\gamma) \neq 0$ for each $\gamma \in \Gamma$, then $KV_{\mathscr{I}}(\phi) \neq 0$.

In LP two inferences of classical logic fail: ex falso quodlibet and modus ponens.

Concerning ex falso quodlibet: $P, \sim P \not\models_{LP} Q$ for suppose $KV_{\mathscr{I}}(P) = \#$ and thus $KV_{\mathscr{I}}(\sim P) = \#$ and $KV_{\mathscr{I}}(Q) = 0$; the premises are designated, the conclusion is not.

Concerning modus ponens: $P, P \rightarrow Q \not\models_{LP} Q$. This is an exercise for you.

But here's another inference that is valid in PL but *invalid* in LP: transitivity or hypothetical syllogism, that is,
$P \rightarrow Q, Q \rightarrow R \not\models_{LP} P \rightarrow R$.

Demonstration: By the definition of LP-valid, ϕ is LP valid iff $KV_{\mathscr{I}}(\phi) \neq 0$ for each trivalent interpretation, and ϕ is a semantic consequence of Γ iff for every trivalent interpretation, $\mathscr{I}$, if $KV_{\mathscr{I}}(\gamma) \neq 0$ for each $\gamma \in \Gamma$, then $KV_{\mathscr{I}}(\phi) \neq 0$. Thus, to show that $P \rightarrow Q, Q \rightarrow R \nvDash_{LP} P \rightarrow R$, we need to show that there is some interpretation in which $KV_{\mathscr{I}}(P \rightarrow Q) \neq 0$, $KV_{\mathscr{I}}(Q \rightarrow R) \neq 0$ and $KV_{\mathscr{I}}(P \rightarrow R) = 0$. If $KV_{\mathscr{I}}(P \rightarrow R) = 0$, then $KV_{\mathscr{I}}(P) = 1$ and $KV_{\mathscr{I}}(R) = 0$. But if $KV_{\mathscr{I}}(R) = 0$ and $KV_{\mathscr{I}}(Q \rightarrow R) \neq 0$, then $KV_{\mathscr{I}}(Q) \neq 1$. If $KV_{\mathscr{I}}(Q) \neq 1$ and $KV_{\mathscr{I}}(P) = 1$ and $KV_{\mathscr{I}}(P \rightarrow Q) \neq 0$, then $KV_{\mathscr{I}}(Q) \neq 0$, and thus $KV_{\mathscr{I}}(Q) = \#$. Therefore, if $KV_{\mathscr{I}}(P) = 1$, $KV_{\mathscr{I}}(Q) = \#$, and $KV_{\mathscr{I}}(R) = 0$, we have an interpretation that designates the premises but not the conclusion of the inference. It is invalid. Q.E.D.

Beyond Propositional Logic

Supervaluationism

In Kleene's system, we saw that if there was enough classical information to assign a truth-value, then we were justified in doing so even in a trivalent system.

The idea in *supervaluationism* is similar. Suppose ϕ contains some sentence letters $P_1, \ldots, P_n$ that are $\#$. If ϕ would be false no matter how we assign the classical truth values to $P_1, \ldots, P_n$, then ϕ is in fact false. Further, if ϕ would be true no matter how we precisified it, then ϕ is in fact true.

Beyond Propositional Logic

Intuitionistic Logic

In the philosophy of mathematics, intuitionism is the thesis that there are no mind-independent truths or mathematical facts. A proposition is to be considered true if and only if it can be proven or a proof can be constructed.

This has a parallel in logic, according to which certain classical laws are denied: the law of excluded middle, "ϕ or not-ϕ", and double negation elimination "$\sim\sim \phi \vDash \phi$".

Consider Sider's example: "either 0123456789 is contained in the expansion of π or it isn't".

For the intuitionist in mathematics, since there is no proof one way or another to show this, we are not justified in concluding that it it true.

Intuitionists reject "P or not-P". But they do not accept its denial either "not: P or not-P", for they accept the denial of this denial "not-not: P or not-P". (See the argument.)